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On preserved properties for slant ruled surfaces under homothety in E^3

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Abstract. In mathematics, it is known that if $f : E^3 \to E^3$ represents a homothety and N denotes a surface in E^3, then $f(N) = \bar{N}$ is a surface in E^3. In this study, especially, the surface N is considered a slant ruled surface. Then, it is proved that the image surface $f(N) = \bar{N}$ is a slant ruled surface, too. Moreover, some significant properties are shown to be preserved under homothety in E^3.

AMS Subject Classifications: 53C05, 53B05, 53B15.

Keywords: Slant ruled surface, homothety, connection preserving map.

Contents

1. Introduction and Background

The notation of slant helix, where the normal lines of the curve form a fixed direction with a constant angle, was introduced in [4]. In literature, there are several studies about slant helices such as [10], [11] and [17]. In [10], the slant helices were investigated in E^3. In [11], a thorough analysis was conducted on the spherical images, tangent, and binormal indicatrices of a slant helix. In [17], a system of linear differential equations including an alternative frame was solved. Furthermore, the position vectors of slant helices by means of integration were determined in Minkowski 3-space. Considering the properties of the slant helix, the concept of the slant ruled surface was firstly expressed in [13] as follows: In mathematics, it is commonly known that the orthonormal vectors of a ruled surface are specified by the ruled surface's Frenet frame. In [13], the concept of special ruled surface which is called slant ruled surface was defined by regarding as the Frenet vectors of the ruled surface. Also, these vectors form some fixed directions with a constant angle in the space. Similar definition was adapted to Darboux vector of the ruled surface which is expressed as Darboux slant ruled surface in [14]. Then, in E^3, several substantial characterizations of the slant ruled surface were investigated in [8]. In [12], non-null slant ruled surface was defined and some fundamental theorems of being non-null slant ruled surface were proved. In [5], the concept of the slant ruled surface was defined by exploiting E. Study mapping and the isomorphism between the unit dual sphere and the subset of the tangent bundle of the unit 2-sphere.
The conformal and the properties of connection preserving maps in $n-$ dimensional C^∞ manifold were examined

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in [2]. Moreover, the condition of being connection preserving for conformal maps were proved in terms of homothety. Then, the connection preserving spray maps were investigated in [3]. Besides, the results for connection preserving conformal diffeomeorphisms of spheres were obtained without any restriction in [15]. By using the differential geometric concepts in [6], the normal curvatures of hypersurfaces were calculated under conformal, homothety and isometry maps in [1]. Furthermore, it was demonstrated that if the conformal map is an isometry, then the first and second fundamental forms of hypersurfaces are invariant.
Taking the geodesic Frenet trihedron defined in [16] and the condition of being a surface with the map $f : E^3 \to E^3$ into consideration, several properties for the ruled surfaces were denoted under the homothety in E^3. However, in literature, there is no research about the examination of some properties for the slant ruled surfaces under homothety in E^3. Therefore, we ask that question:"Which properties are preserved for the slant ruled surface under homothety in E^3?" In this study, we answer this question and obtain some results by using homothety in E^3. Thus, the structure of this study is as follows: In Section 2, some fundamental concepts about conformal map, the definition of slant ruled surface and its Frenet apparatus are represented. Moreover, the condition of a connection preserving is mentioned. In Section 3, some important properties are shown for the slant ruled surface under the homothety in E^3.

2. Preliminaries

Assume that M and $\bar{M}$ are two surfaces in E^3 and $f : M \to \bar{M}$ is a C^∞ map. If there exists a C^∞ real valued positive function F on M. Hence, $\forall P \in M$ and $\forall X_p, Y_p \in T_M P$

$$\langle f_*(X_p), f_*(Y_p)\rangle = F(P)\langle X_p, Y_p\rangle \tag{2.1}$$

is satisfied, f is called a conformal map. If F is constant, f is called a homothety of coefficient $F(P)$, where f_* is Jacobian map of f.
Assume that D and $\bar{D}$ are connections on M and $\bar{M}$, respectively. A C^∞ map $f : M \to \bar{M}$ is called connection preserving if

$$f_*(D_X Y) = \bar{D}_{f_*(X)} f_*(Y) \tag{2.2}$$

for all $X, Y \in \chi(M)$, see [2].

Theorem 2.1. *Assume that $f : M \to \bar{M}$ is a conformal map. Then f is a connection preserving iff f is a homothety, [9].*

Assume that I is an open interval in $\mathbb{R}$. The ruled surface N, parametrized as the following equation, is denoted by

$$\vec{r}(u, v) = \vec{\alpha}(u) + v\vec{q}(u). \tag{2.3}$$

Here $\vec{\alpha} = \vec{\alpha}(u)$ is base curve and $\vec{q} = \vec{q}(u)$ is rullings in E^3. Also, the distribution parameter is calculated by

$$P_q = \frac{det(\vec{\alpha}_u, \vec{q}, \vec{q}_u)}{\langle \vec{q}_u, \vec{q}_u\rangle}. \tag{2.4}$$

Here, $\vec{\alpha}_u = \frac{d\vec{\alpha}}{du}$ and $\vec{q}_u = \frac{d\vec{q}}{du}$, [6]. Additionally, the unit normal vector of N is

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} = \frac{(\vec{\alpha}_u + v\vec{q}_u) \times \vec{q}}{\sqrt{\langle \vec{\alpha}_u + v\vec{q}_u, \vec{\alpha}_u + v\vec{q}_u\rangle - \langle \vec{\alpha}_u, \vec{q}\rangle^2}}. \tag{2.5}$$

Along a ruling $u = u_1$, as v infinitely decreases, the surface's unit normal converges in a limiting direction. As the asymptotic normal direction, this direction is determined by Eq. (2.5), which is defined by

$$\vec{a} = lim_{v\to\pm\infty}\vec{m}(u_1, v) = \frac{\vec{q} \times \vec{q}_u}{||\vec{q}_u||}. \tag{2.6}$$

The point where $\vec{m}$ is orthogonal to $\vec{a}$ is defined as the striction point and represented by C. Thus, the striction curve of the surface is defined as the set of striction points of all rulings. The striction curve $\vec{c} = \vec{c}(u)$ is

$$\vec{c}(u) = \vec{\alpha}(u) + v_0 \vec{q}(u) = \vec{\alpha}(u) - \frac{\langle \vec{q}_u, \vec{\alpha}_u \rangle}{\langle \vec{q}_u, \vec{q}_u \rangle} \vec{q}(u). \tag{2.7}$$

Here, $v_0 = -\frac{\langle \vec{q}_u, \vec{\alpha}_u \rangle}{\langle \vec{q}_u, \vec{q}_u \rangle}$. Moreover, $\vec{h}$ computed by $\vec{h} = \vec{a} \times \vec{q}$ is defined as normal vector. Hence, the set $\{C; \vec{q}, \vec{h}, \vec{a}\}$ is Frenet frame of N. Here, C denotes the central point and $\vec{q}, \vec{h}$ and $\vec{a}$ also denote unit vectors of ruling, central normal and central tangent, respectively.
For the Frenet formulas of N in terms of the arc-length parameter s are

$$\begin{pmatrix} \frac{d\vec{q}}{ds} \\ \frac{d\vec{h}}{ds} \\ \frac{d\vec{a}}{ds} \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{pmatrix}.$$

Here, $k_1 = \frac{ds_1}{ds}$ and $k_2 = \frac{ds_3}{ds}$. s_1 and s_3 denote the arc lengths of the spherical curves bounded by $\vec{q}$ and $\vec{a}$, respectively. For more details, see [6].
Moreover, in [13], the characterizations of being $\vec{q}, \vec{h}$ and $\vec{a}$ slant ruled surfaces are classified as follows:
Assume that N is a ruled surface in E^3 denoted by the parametrization

$$\vec{r}(s, v) = \vec{c}(s) + v\vec{q}(s), \quad ||\vec{q}(s)|| = 1, \tag{2.8}$$

where $\vec{c}(s)$ denotes the striction curve of N and s represents arc length parameter of $\vec{c}(s)$. Let $\{\vec{q}, \vec{h}, \vec{a}, k_1, k_2\}$ be Frenet operators of N. If the rulling (or central normal, central tangent) form a fixed non-zero direction $\vec{u}$ with a constant angle θ, then N is called $\vec{q}-$(or $\vec{h}, \vec{a}$) slant ruled surface, respectively, see [13].
The curve which are drawn by $\vec{q}$ on the unit sphere S^2 is defined as the spherical indicatrix curve. Also, $\vec{q}$ is defined as the spherical indicatrix of N. For the Frenet vectors $\{\vec{q}, \vec{h}, \vec{a}\}$ given above, we can write

$$\begin{aligned} \vec{q}_{s_1} &= \vec{h}, \\ \vec{h}_{s_1} &= -\vec{q} + \frac{k_2}{k_1}\vec{a}, \\ \vec{a}_{s_1} &= -\frac{k_2}{k_1}\vec{h}, \end{aligned} \tag{2.9}$$

where s_1 denotes the arc-parameter of the spherical indicatrix curve $\vec{q}$. Also, $\vec{q}_{s_1} = \frac{d\vec{q}}{ds_1}, \vec{h}_{s_1} = \frac{d\vec{h}}{ds_1}$ and $\vec{a}_{s_1} = \frac{d\vec{a}}{ds_1}$, respectively.

3. Slant ruled surfaces under homothety

Assume that N is a slant ruled surface given in Eq. (2.8) and $f : E^3 \to E^3$ is a homothety of coefficient λ. For the point Z taken on each rulling d of N, we get

$$Z = \vec{c}(s) + v\vec{q}(s). \tag{3.1}$$

according to arc-parameter s. Then, we write

$$f_*(Z) = f_*(\vec{c}(s)) + v f_*(\vec{q}(s)). \tag{3.2}$$

Consequently, $f(d)$ is a striction line passing through the point $f(\vec{c}(s))$ of the image curve $f \circ c$. Hence, $f(N) = \bar{N}$ is a slant ruled surface with the striction curve $f \circ c$ in E^3.

Corollary 3.1. *Slant ruled surfaces transform to slant ruled surfaces under homothety.*

If we exploit the spherical indicatrix vector of $\bar{N}$ as $\vec{\bar{q}}$, we can acquire the parametric representation of $\bar{N}$ as follows:

$$\vec{\bar{r}}(s,v)=\vec{\bar{c}}(s)+v\vec{\bar{q}}(s)$$

or

$$\vec{\bar{r}}(s,v)=f_*(\vec{c}(s))+v\frac{1}{\lambda^{\frac{1}{2}}}f_*(\vec{q}(s)). \tag{3.3}$$

In this study, we will consider $f:E^3\rightarrow E^3$ as a homothety with the coefficient λ such that $f(N)=\bar{N}$. Moreover, the rulling (or central normal, central tangent) makes a constant angle θ with a fixed non-zero direction $\vec{u}$. Here, N and $\bar{N}$ are slant ruled surfaces in E^3.

Theorem 3.2. *For all X in $\chi(N)$, we have*

$$[f_*(X)]_s=f_*(X_s). \tag{3.4}$$

For proof of this theorem, see [7].

Let $\vec{\bar{h}}$ and $\vec{\bar{a}}$ be central normal and central tangent vectors of $\bar{N}$, respectively. Thus, we can simply write

$$\vec{\bar{h}}=\frac{\vec{\bar{q}}_s}{||\vec{\bar{q}}_s||},\quad \vec{\bar{a}}=\frac{\vec{\bar{q}}\times\vec{\bar{q}}_s}{||\vec{\bar{q}}_s||}. \tag{3.5}$$

From $\vec{\bar{q}}=\frac{f_*(\vec{q})}{\lambda^{\frac{1}{2}}}$ and Theorem 3.2, we get

$$\vec{\bar{h}}=\frac{f_*(\vec{q}_s)}{||f_*(\vec{q}_s)||} \tag{3.6}$$

and

$$\vec{\bar{a}}=\frac{1}{\lambda^{\frac{1}{2}}}\frac{f_*(\vec{q})\times f_*(\vec{q}_s)}{||f_*(\vec{q}_s)||}. \tag{3.7}$$

By using the following equation

$$f_*(\vec{q})\times f_*(\vec{q}_s)=det f_*.(\vec{q}\times\vec{q}_s), \tag{3.8}$$

we obtain

$$\vec{\bar{a}}=\frac{det f_*}{\lambda}\vec{a}. \tag{3.9}$$

The striction curve $\vec{c}$ is written by

$$\vec{c}(s)=\vec{\alpha}(s)-\frac{\langle\alpha_s,\vec{q}_s\rangle}{\langle\vec{q}_s,\vec{q}_s\rangle}\vec{q}(s). \tag{3.10}$$

Assume that $\vec{\bar{c}}$ is a striction curve of the slant ruled surface $\bar{N}$. Therefore, we get

$$\vec{\bar{c}}(s)=\vec{\bar{\alpha}}(s)+k\vec{\bar{q}}(s) \tag{3.11}$$

or

$$\vec{\bar{c}}(s)=f_*(\vec{\alpha}(s))+k\frac{1}{\lambda^{\frac{1}{2}}}f_*(\vec{q}(s)). \tag{3.12}$$

Due to the definition of the striction point, we acquire

$$\langle\vec{\bar{r}}_s\times\vec{\bar{r}}_v,f_*(\vec{q})\times f_*(\vec{q}_s)\rangle=0. \tag{3.13}$$

By using Eq. (3.3), we obtain

$$k=-\lambda^{\frac{1}{2}}\frac{\langle\vec{\alpha}_s,\vec{q}_s\rangle}{\langle\vec{q}_s,\vec{q}_s\rangle}. \tag{3.14}$$

Then, for the striction curve of $\bar{N}$, we calculate

$$\vec{\bar{c}}(s) = f_*(\vec{\alpha}(s)) - \frac{\langle \vec{\alpha}_s, \vec{q}_s \rangle}{\langle \vec{q}_s, \vec{q}_s \rangle} f_*(\vec{q}(s)). \tag{3.15}$$

Consequently, it is easily seen that $f_*(\vec{c}(s)) = \vec{\bar{c}}(s)$.

Corollary 3.3. *The property of being striction curve is preserved under the homothety.*

Assume that $\beta : I \subseteq \mathbb{R} \to E^3$ is the orthogonal trajectory for N and $\vec{T}$ is the tangent of β. For $s \in I$, we have

$$\langle \vec{T}, \vec{q} \rangle = 0. \tag{3.16}$$

Since f is homothety, we write

$$\langle f_*(\vec{T}), f_*(\vec{q}) \rangle = 0. \tag{3.17}$$

As a result, $f(\beta) = \bar{\beta}$ is a orthogonal trajectory for $\bar{N}$.

Corollary 3.4. *The property of being orthogonal trajectory is preserved under the homothety.*

If the slant ruled surface N is closed, a positive integer t exists such that

$$N(s+t, v) = N(s, v). \tag{3.18}$$

Hence, we write

$$\vec{c}(s+t) + v\vec{q}(s+t) = \vec{c}(s) + v\vec{q}(s). \tag{3.19}$$

Because of the linearity of f_*, we write

$$f_*(\vec{c}(s+t)) + v f_*(\vec{q}(s+t)) = f_*(\vec{c}(s)) + v f_*(\vec{q}(s)). \tag{3.20}$$

If we take the parametrization of $\bar{N}$,

$$\bar{N}(s+t, v) = \bar{N}(s, v). \tag{3.21}$$

Corollary 3.5. *The condition of being closed of the slant ruled surfaces is preserved under the homothety.*

For the distribution parameter P_q is

$$P_{\vec{q}} = \frac{det(\vec{c}_s, \vec{q}, \vec{q}_s)}{||\vec{q}_s||^2}. \tag{3.22}$$

Considering $\vec{q}_s = \frac{d\vec{q}}{ds_1}\frac{ds_1}{ds}$ and Frenet formulas given Eq. (2.9), we obtain

$$P_{\vec{q}} = \frac{1}{\vec{q}_s} \langle \vec{c}_s, \vec{a} \rangle. \tag{3.23}$$

Similarly, for the distribution parameter of $\bar{N}$, we have

$$P_{\vec{\bar{q}}} = \frac{det(\vec{\bar{c}}_s, \vec{\bar{q}}, \vec{\bar{q}}_s)}{||\vec{\bar{q}}_s||^2}. \tag{3.24}$$

Namely, we can write

$$P_{\vec{\bar{q}}} = \frac{det(f_*(\vec{c}(s)), f_*(\vec{q}), f_*(\vec{h}))}{\lambda^{\frac{1}{2}} \vec{q}_s ||f_*(\vec{h})||^2}. \tag{3.25}$$

Since $f_*(\vec{q}) \times f_*(\vec{h}) = \lambda^{\frac{1}{2}} f_*(\vec{a})$, we acquire

$$P_{\vec{q}} = P_{\vec{\bar{q}}}. \tag{3.26}$$

Corollary 3.6. *The property of being developable of the slant ruled surfaces is preserved under the homothety.*

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