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Optimal strategy on inventory model under permissible delay in payments and return policy for deteriorating items with shortages

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Abstract. In this model, the manufacturer offers a trade credit policy to the retailer. Demand depends upon selling price and time for non-instantaneous deterioration items. The retailer offers the customer a returns policy. Customers can return the product to the retailer if the product is unsatisfactory for the customer. The retailer does not ultimately return the amount to its customer for the returned product. The manufacturer offers the retailer a trade credit policy. The retailer resale the returned products at the same selling price. A partially backlogged shortage is permitted and its rate is thought to depend on how long it takes for the following lot to arrive after a lot has been replenished. The main objective is to increase the retailers' overall profit by determining the optimal order quantity, optimal selling price, and optimal replenishment cycle. An EOQ is framed for analyzing the sample, which can obtain the optimal solution.

AMS Subject Classifications: 90-05, 90-06, 90-08, 90-10, 90-11.

Keywords: Non-instantaneous deteriorating items, Permissible delay in payments, resalable returns, Partial backlogging.

Contents

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1. Introduction

In the enterprise international, each production firm/provider usually continues the inventory so one can sell the product to their potential clients. There is a large question, how to keep the inventory degree and the way to sell the goods to their capacity clients? Basically, most profit or minimal loss relies upon the idea of these two questions. Here we are using some other factors such as controlling the deterioration rate and introducing some attractive offers that promote greater products. Deteriorating products are the greatest challenge to companies. Deterioration means damage, decay and spoilage of products from their condition initially. There is a fresh-product phase for some disintegrating products, during which they hold onto their original quality and worth before eventually degrading. These are non-instantaneous deteriorations. Permissible delay in payments is often used in most of business organizations. Trade credit is the arrangement to buy the goods on the account without making on the spot cash or cheque payments. Trade credit is a helpful device for developing companies. The retailer gets a trade credit policy from the manufacturer. The retailer has to pay the amount to the manufacturer by the next replenishment time. This helps the retailer to purchase products without paying immediately. Retailers must pay the price plus some interest if they don't pay within the allotted time. The retailer offered a return policy to the customers. This offer makes customers buy the products and return the product within a specific time period. For the returned product the retailer did not fully reimburse. Customer returns rise in proportion to both sales volume and product price. Duary et al. (2022) developed model for delay in payments and deteriorating items with partially backlogged shortages [1]. Geetha and Uthayakumar (2010) proposed the EOQ model for non-instantaneous deteriorating items with permissible delay in payments and partial backlogging [2]. Ghoreishi and Mirzazadeh (2013) studied the effect of inflation and customer returns on joint pricing and inventory control for deteriorating items [3]. Ghoreishi et al. (2015) developed an economic ordering policy model for non-instantaneous deteriorating items with selling price- and demand permissible delay in payments and customer returns [4]. Ghoreishi et al. (2013b) studied the optimal pricing and inventory control policy for non-instantaneously deteriorating items with the finite replenishment rate considering time- and price-dependent demand, customer returns and time value of money [5]. Jani et al. (2021) developed an EOQ model for customer returns and trade credit for deteriorating items with price sensitive demand [6]. Kumari and De investigated an EOQ model for deteriorating items analyzing retailer's optimal strategy under trade credit and return policy with nonlinear demand and resalable returns [7]. Maihami and Kamalabadi (2012) developed inventory control for non-instantaneous deteriorating items adopts a price and time dependent function with partially backlogged [8]. Mashud (2020) developed a deteriorating EOQ inventory model according to consideration of the price with shortage [9]. Musa and Sani (2010) developed a mathematical model on the inventory of deteriorating items that do not start deteriorating immediately they are stocked with permissible delay in payments [10]. Ouyang et al. (2006) investigated the inventory model for non-instantaneous deteriorating items considering permissible delay in payments [11]. Singh and Mishra (2022) developed an inventory model for deteriorating items [12]. Sundararajan et al. (2019) developed a deterministic inventory model for non-instantaneous deteriorating items with price and time-dependent demand with shortages [13] Yang et al. (2009) considered the optimal pricing and ordering strategies for non-instantaneous deteriorating items with partial backlogging and price dependent demand [14].

2. Assumptions

- The model includes a single non-instantaneous deteriorating item.
- Assume that the inventory system planning horizon is infinite.
- Demand rate is depends on time and selling price is given by:
  $D(y,t) = (\alpha - \beta y)e^{\eta t}$ where α is the demand scale, β represents price sensitivity, Demand is a linearly decreasing function of the price and decreases (increases) exponentially with time when $\eta < 0 (\eta > 0)$.

- Shortages are permitted. The unsatisfied demand is backlogged, and the fraction of shortage backordered is $\zeta(x) = K_0 e^{-\mu x}$ ($\mu > 0, 0 < K_0 \leq 1$), where x is a waiting time up to the upcoming replenishment and μ is a positive constant.
- It is plausible to say that buyer returns grow as more goods are sold. So,

$$\Lambda(y,t) = \nu D(y,t) \text{ where } 0 \leq \nu < 1.$$

The customers can return the products at any time in the replenishment cycle. But the retailer will not give the total amount of initial value, and the retailer will provide half the amount of initial value. The returned products can be resalable at the same selling price.

3. Notations

The terms used in the mathematical formulation are listed in the table 1.

Notation	Unit	Description
A	$\$$/ order	ordering cost
C_1	$\$$/ unit/unit time	holding cost
C_s	$\$$/ unit	shortage cost
C_p	$\$$/ unit	purchase price
Q		order quantity
θ		constant deterioration rate $0 < \theta < 1$
S	unit time	trade credit period offered by the manufacturer to the retailer
t_d	unit time	time at which deterioration starts
I_r	$\%$/unit time	interest earned by the retailer
I_m	$\%$/unit time	interest paid by the retailer to the manufacturer
R		maximum shortage level
Decision Variables		
y	$\$$/ unit	selling price
t_1	unit time	time at which inventory level reaches to zero
t_2	unit time	time at partially backlogged shortage
$TPC(y,t)$	$\$$/ unit time	Total profit

Table 1: Notations that are considered in the formulation of the inventory model

4. Model formulation

In this section, At the beginning of the cycle I_0 units of item arrive at the inventory system. In the time of interval, $(0, t_d)$, the inventory level depend upon demand and returns, at that time there is no deterioration. At $t = t_d$ the deterioration starts takes place. During the interval (t_d, t_1) the inventory level depends upon demand, returns and deterioration. At next stage, during the interval (t_1, t_2) shortage caused by partial backlogging and demand. In this research paper, it is assumed that the manufacturer offers permissible delay in payments to the retailer. The customers offered a product can return during the replenishment cycle to the retailer. The returned products can be sold again for the same price. And the retailer did not fully reimburse the amount of returned product to the customer. During the time interval $[0, t_d]$, the differential equation represents the inventory is given by

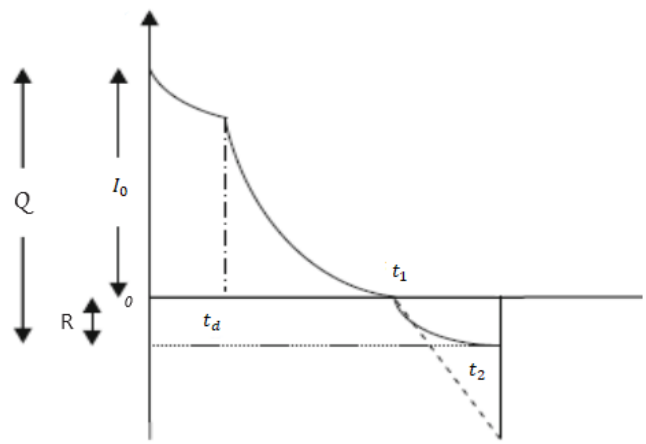

Figure 1:

$$\frac{dI_1(t)}{dt} = -(\alpha - \beta y)e^{\eta t} + \nu(\alpha - \beta y)e^{\eta t},\ 0 \leq t \leq t_d \tag{4.1}$$

When $t = 0$, put $I_1(0) = I_0$ in above equation we get

$$I_1(t) = I_0 + \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t} - 1)}{\eta}, 0 \leq t \leq t_d \tag{4.2}$$

The differential equation for the time interval $[t_d, t_1]$ is,

$$\frac{dI_2(t)}{dt} = -(\alpha - \beta y)e^{\eta t} + \nu(\alpha - \beta y)e^{\eta t} - \theta I(t),\ t_d \leq t \leq t_1 \tag{4.3}$$

When $t = t_1$, put $I_2(t_1) = 0$ in above equation we get

$$I_2(t) = \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t}}{(\eta + \theta)} \left[e^{(\eta+\theta)t} - e^{(\eta+\theta)t_1}\right], t_d \leq t \leq t_1 \tag{4.4}$$

At $t = t_d$, the equations (2) and (4) becomes

$$\begin{aligned} I_0 =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)} \left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] \\ &- \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} \end{aligned} \tag{4.5}$$

Substitute equation (5) in equation (2) we get,

$$\begin{aligned} I_1(t) =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)} \left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] \\ &- \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} + \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t} - 1)}{\eta}, 0 \leq t \leq t_d \end{aligned} \tag{4.6}$$

In the time interval $[t_1, t_2]$, partially backlogged shortage occurs according to a fraction $\zeta(t_2 - t_1)$. Then the differential equation for the inventory level is given by

$$\frac{dI_3(t)}{dt} = -(\alpha - \beta y)e^{\eta t}\zeta(t_2 - t),\ t_1 \le t \le t_2 \tag{4.7}$$

When $t = t_1$ put $I_3(t_1) = 0$ in above equation we get

$$I_3(t) = \frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t}\right],\ t_1 \le t \le t_2 \tag{4.8}$$

Put $t = t_2$ in eqn $I_3(t)$ where R is the maximum shortage level

$$R = -\frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right] \tag{4.9}$$

The sum of R and I_0 is the Order Quantity per cycle (Q) is

$$\begin{aligned} Q =& R + I_0 \\ =& \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta} \\ &- \frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right] \end{aligned} \tag{4.10}$$

This model's various costs are specified as follows.

(1) Ordering cost = A

(2) Purchasing cost

$$\begin{aligned} PC =& C_p Q \\ =& C_p\left[\frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t_d}}{(\eta + \theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu - 1)(\alpha - \beta y)(e^{\eta t_d} - 1)}{\eta}\right. \\ &\left. - \frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\right] \end{aligned} \tag{4.11}$$

(3) Sales revenue

$$\begin{aligned} SR &= y\left[\int_0^{t_1} D(y,t)dt - \int_0^{t_1} \frac{\Lambda(y,t)}{2}dt + R\right] \\ &= y\left[\int_0^{t_1} (\alpha - \beta y)e^{\eta t}dt - \int_0^{t_1} \frac{\nu(\alpha - \beta y)e^{\eta t}}{2}dt - \frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\right] \\ &= y\left[\left[\frac{(\alpha - \beta y)(e^{\eta t_1} - 1)}{\eta}\right]\left[1 - \frac{\nu}{2}\right] - \frac{(\alpha - \beta y)}{\eta + \mu} e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t_2}\right]\right] \end{aligned} \tag{4.12}$$

(4) Deterioration cost

$$\begin{aligned} DC &= C_p \int_{t_d}^{t_1} \theta I(t)dt \\ &= C_p \int_{t_d}^{t_1} \theta \frac{(\alpha - \beta y)(\nu - 1)e^{-\theta t}}{(\eta + \theta)}\left[e^{(\eta+\theta)t} - e^{(\eta+\theta)t_1}\right] dt \\ &= \left[\frac{C_p(\alpha - \beta y)(\nu - 1)\left[(\theta + \eta)e^{\eta t_1} - \theta e^{\eta t_d} - \eta e^{(\eta+\theta)t_1} e^{-\theta t_d}\right]}{\eta(\eta + \theta)}\right] \end{aligned} \tag{4.13}$$

(5) Holding cost

$$
\begin{aligned}
HC =&C_1 \left[\int_0^{t_d} I(t)dt + \int_{t_d}^{t_1} I(t)dt\right] \\
=&C_1 \left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\} t_d \right. \\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right] - \left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right] \\
&\left.\left[\theta e^{\eta t_d} - (\theta+\eta)e^{\eta t_1} + \eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]
\end{aligned} \tag{4.14}
$$

(6) Shortage cost

$$
\begin{aligned}
SC &= c_2\left[\int_{t_1}^{t_2} -I(t)dt\right] \\
&= c_2\left[\int_{t_1}^{t_2} \frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1} - e^{(\mu+\eta)t}\right]dt\right] \\
&= c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2} - e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]
\end{aligned} \tag{4.15}
$$

(7) Permissible delay in payments:
The retailer gets a trade credit policy from the manufacturer. The retailer has to pay the amount to the manufacturer by the delay period S. We suggest three subcases for the delay period based on the values of S, t_d, and t_1.

(i) $0 < S \leq t_d$

(ii) $t_d < S \leq t_1$

(iii) $S > t_1$

Case (i) : Payment delays occur previous to time deterioration: $0 \leq S \leq t_d$
In this subcase, the retailer has to pay the amount before the deterioration starts. Otherwise, he have to pay the interest to the manufacturer. Interest earns is estimated as follows:

$$
\begin{aligned}
IR_1 =&yI_r\left[\int_0^S\int_0^t(\alpha-\beta y)e^{\eta u}dudt - \int_0^S\int_0^t\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right] \\
=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\right\}
\end{aligned} \tag{4.16}
$$

Interest paid by the retailer to the manufacturer is estimated as follows:

$$
\begin{aligned}
IM_1 =&C_pI_m\left[\int_S^{t_d} I(t)dt + \int_{t_d}^{t_1} I(t)dt\right] \\
=&C_pI_m\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d} - e^{(\eta+\theta)t_1}\right] - \frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}(t_d-S)\right. \\
&+\left[\frac{(\nu-1)(\alpha-\beta y)}{\eta}\right]\left[S-t_d+\frac{(e^{\eta t_d}-e^{\eta S})}{\eta}\right] - \frac{e^{\eta t_d}}{\eta} - \frac{e^{(\eta+\theta)t_1}e^{-\theta t_d}}{\theta} \\
&\left.+\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)\right]\right]
\end{aligned} \tag{4.17}
$$

The total profit per unit time is estimated as follows:

$$
\begin{aligned}
&TPC_1(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&-C_pI_m\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}(t_d-S)\right.\\
&+\left[\frac{(\nu-1)(\alpha-\beta y)}{\eta}\right]\left[S-t_d+\frac{(e^{\eta t_d}-e^{\eta S})}{\eta}\right]-\frac{e^{\eta t_d}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta t_d}}{\theta}\\
&\left.+\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)\right]\right]+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\right\}.
\end{aligned}
\tag{4.18}
$$

Case (ii) :Payment delays occur between deterioration time and before the inventory cycle. $t_d < S \leq t_1$

Interest paid by the retailer to the manufacturer

$$
\begin{aligned}
IM_2&=C_pI_m\left[\int_S^{t_1}I(t)dt\right]\\
&=C_pI_m\left[\int_S^{t_1}\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t}}{(\eta+\theta)}\left[e^{(\eta+\theta)t}-e^{(\eta+\theta)t_1}\right]dt\right]\\
&=C_pI_m\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)-\frac{e^{\eta M}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta S}}{\theta}\right]
\end{aligned}
\tag{4.19}
$$

Interest earns is estimated as follows:

$$
\begin{aligned}
IR_2=&yI_r\left[\int_0^S\int_0^t(\alpha-\beta y)e^{\eta u}dudt-\int_0^S\int_0^t\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right]\\
=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta M-1\right]\left[1-\frac{\nu}{2}\right]\right\}
\end{aligned}
\tag{4.20}
$$

The total profit per unit time is estimated as follows:

$$\begin{aligned}
&TPC_2(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&-C_pI_m\left[\frac{(\nu-1)(\alpha-\beta y)}{(\eta+\theta)}\right]\left[\frac{e^{\eta t_1}}{\theta\eta}(\theta+\eta)-\frac{e^{\eta M}}{\eta}-\frac{e^{(\eta+\theta)t_1}e^{-\theta S}}{\theta}\right]\\
&+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta M-1\right]\left[1-\frac{\nu}{2}\right]\right\}.
\end{aligned}\tag{4.21}$$

Case (iii) : $S > t_1$

In this subcase, the delay period is greater than the time at which the amount of inventory reaches zero. During this time retailer pays completely all of his bills. Then

$$IM_3 = 0. \tag{4.22}$$

Interest earns is estimated as follows:

$$\begin{aligned}
IR_3 =&yI_r\left[\int_0^{t_1}\int_0^{t}(\alpha-\beta y)e^{\eta u}dudt-\int_0^{t_1}\int_0^{t}\frac{\nu(\alpha-\beta y)e^{\eta u}}{2}dudt\right.\\
&\left.+(S-t_1)\int_0^{t_1}(\alpha-\beta y)e^{\eta t}dt-(S-t_1)\int_0^{t_1}\frac{\nu(\alpha-\beta y)e^{\eta t}}{2}dt\right]\\
=&yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\left[\frac{(\alpha-\beta y)(S-t_2)(e^{\eta t_2}-1)}{\eta}\right]\right\}
\end{aligned}\tag{4.23}$$

The total profit per unit time is estimated as follows:

$$\begin{aligned}
&TPC_3(y,t_2)\\
=&\frac{SR-A-PC-DC-HC-SC-IM_1+IR_1}{t_2}\\
=&y\left[\left[\frac{(\alpha-\beta y)(e^{\eta t_1}-1)}{\eta}\right]\left[1-\frac{\nu}{2}\right]-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t_2}\right]\right]\\
&-A-C_p\left[\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right.\\
&\left.-\frac{(\alpha-\beta y)}{\eta+\mu}e^{-\mu t_2}\left[e^{(\mu+\eta)t_1}-e^{(\mu+\eta)t}\right]\right]\\
&-\left[\frac{C_p(\alpha-\beta y)(\nu-1)\left[(\theta+\eta)e^{\eta t_1}-\theta e^{\eta t_d}-\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]}{\eta(\eta+\theta)}\right]\\
&-C_1\left[\left\{\frac{(\alpha-\beta y)(\nu-1)e^{-\theta t_d}}{(\eta+\theta)}\left[e^{(\eta+\theta)t_d}-e^{(\eta+\theta)t_1}\right]-\frac{(\nu-1)(\alpha-\beta y)(e^{\eta t_d}-1)}{\eta}\right\}t_d\right.\\
&+\left[(\nu-1)(\alpha-\beta y)\frac{(e^{\eta t_d}-\eta t_d-1)}{\eta^2}\right]-\left[\frac{(\alpha-\beta y)(\nu-1)}{\theta\eta(\theta+\eta)}\right]\\
&\left.\left[\theta e^{\eta t_d}-(\theta+\eta)e^{\eta t_1}+\eta e^{(\eta+\theta)t_1}e^{-\theta t_d}\right]\right]\\
&-c_2\left[\frac{(\alpha-\beta y)}{(\eta+\mu)^2}e^{-\mu t_2}\left[e^{(\mu+\eta)t_2}-e^{(\mu+\eta)t_1}\left[(t_2-t_1)(\mu+\eta)+1\right]\right]\right]\\
&+yI_r\left\{\frac{(\alpha-\beta y)}{\eta^2}\left[e^{\eta S}-\eta S-1\right]\left[1-\frac{\nu}{2}\right]\left[\left[\frac{(\alpha-\beta y)(S-t_2)(e^{\eta t_2}-1)}{\eta}\right]\right]\right\}
\end{aligned} \tag{4.24}$$

5. Solution Procedure

The following method is used to resolve the aforementioned issue.

Step 1: Fill the equation with all of the values for the necessary parameters for the proposed model.

Step 2: Put $\frac{\partial TPC_i}{\partial y}=\frac{\partial TPC_i}{\partial t_1}=0$, where $i=1,2,3$

Step 3: Fix the optimization issue TPC_i for $i=1,2,3$ and hold the optimal values of y, t_1 and TPC

Step 4: Compare the values of TPC_1, TPC_2 and TPC_3

Step 5: Choose the highest value among TPC_1, TPC_2 and TPC_3.

Step 6: Stop.

6. Numerical Example

Consider a numerical example to demonstrate the model. The parameter values are as follows: $\alpha=290$; $\beta=4$; $t_d=1/12$; $t_1=0.765$; $C_s=2.5$; $I_r=10\%$ per year; $I_m=15\%$ per year;
$\nu=0.1$; $\mu=0.1$; $\theta=0.08$; $C_p=20$; $C_s=2.5$; $A=200$; $\eta=-0.98$; $C_1=1$;
If $S=0.08$;, then it is in the category of case (i), since $S<t_d$.
If $S=0.4$;, then it is in the category of case (ii), since $t_d<S\le t_1$.
If $S=0.91$;, then it is in the category of case (iii), since $S>t_1$.
Then we obtain the following results. $y=35.5357; t_2=0.9$.

Expressions	Case 1	Case 2	Case 3
S	0.08	0.4	0.91
Q	32.049	32.049	32.049
SR	1173.32	1173.32	1173.32
DC	11.77	11.77	11.77
HC	9.85	9.85	9.85
SC	0.5908	0.5908	0.5908
IR	0.609	13.769	60.435
IM	22.323	5.6123	0
TPC	320.435	353.626	411.713

Table 2: Results of numerical example

Parameter	% Changes in parameters	Case 1	Case 2	Case 3
I_r	-30	320.23	445.42	814.62
	-20	320.30	460.72	881.77
	-10	320.37	476.02	948.92
	10	320.50	355.16	418.43
	20	320.57	356.69	425.14
	30	320.64	358.22	431.86
I_m	-30	327.88	355.50	411.71
	-20	325.40	354.87	411.71
	-10	322.92	354.25	411.71
	10	317.96	353.00	411.71
	20	315.48	352.38	411.71
	30	312.99	351.76	411.71
C_p	-30	545.47	573.09	629.30
	-20	470.46	499.93	556.77
	-10	395.45	426.78	484.24
	10	245.43	280.47	339.18
	20	170.42	207.32	266.65
	30	95.40	134.17	194.12
A	-30	387.10	420.29	478.38
	-20	364.88	398.07	456.16
	-10	342.66	375.85	433.94
	10	298.21	331.40	389.49
	20	275.99	309.18	367.27
	30	253.77	286.96	345.05
C_1	-30	323.72	356.91	415.00
	-20	322.63	355.82	413.90
	-10	321.53	354.72	412.81
	10	319.34	352.53	410.62
	20	318.25	351.44	409.52
	30	317.15	350.34	408.43
α	-30	-242.32	-243.55	-245.70
	-20	-54.73	-44.49	-26.56
	-10	132.85	154.57	192.58
	10	508.02	552.68	630.85
	20	695.61	751.74	849.99
	30	883.19	950.80	1,069.13

Table 3: Results of sensitivity analysis

7. Sensitivity Analysis

Using the numerical example, we do sensitivity analyses for various parameters. In any circumstance requiring decision-making, uncertainty may cause parameter values to vary. Sensitivity analysis is given here for the three cases. The changes are made from -30 percent to $+30$ percent. The result of this analysis is in the following table 3 and table 4. The main conclusion from the sensitivity analysis are as follows:

- When α is increased (decreased), the total profit for the three cases increases(decreases).
- There is an increase (decrease) in the total profit for the three cases value when A, C_p, η and β are decreases(increases).
- I_r is less sensitive and ν, C_1 and θ are moderately sensitive.
- Other parameter modifications have minimal impact on the total profit.

Parameter	% Changes in parameters	Case 1	Case 2	Case 3
β	-30	720.39	778.05	878.95
	-20	587.08	636.57	723.20
	-10	453.76	495.10	567.46
	10	187.12	212.15	255.97
	20	53.80	70.68	100.22
	30	-79.52	-70.79	-55.52
θ	-30	328.57	361.65	419.72
	-20	325.87	358.98	417.06
	-10	323.16	356.31	414.39
	10	317.71	350.93	409.03
	20	314.97	348.23	406.33
	30	312.22	345.52	403.63
η	-30	384.32	420.86	484.94
	-20	361.85	397.22	459.21
	-10	340.58	374.83	434.83
	10	301.35	333.52	389.79
	20	283.25	314.45	368.98
	30	266.08	296.36	349.22
ν	-30	316.08	350.12	409.23
	-20	317.53	351.29	410.06
	-10	318.98	352.46	410.89
	10	321.89	354.80	412.54
	20	323.34	355.97	413.37
	30	324.80	357.14	414.20
μ	-30	320.56	353.75	411.83
	-20	320.52	353.71	411.79
	-10	320.48	353.67	411.75
	10	320.40	353.59	411.67
	20	320.36	353.55	411.63
	30	320.32	353.51	411.59

Table 4: Results of sensitivity analysis

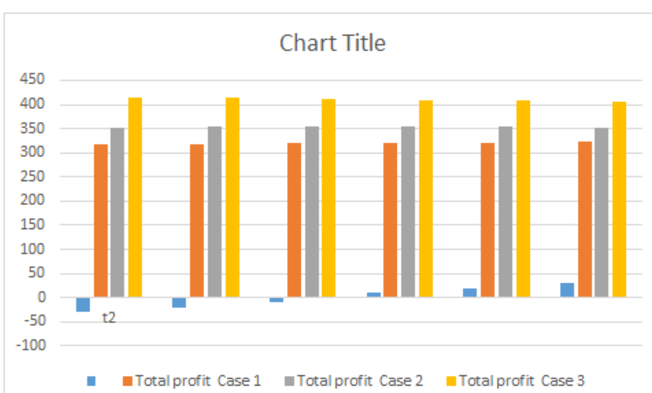

Figure 2:

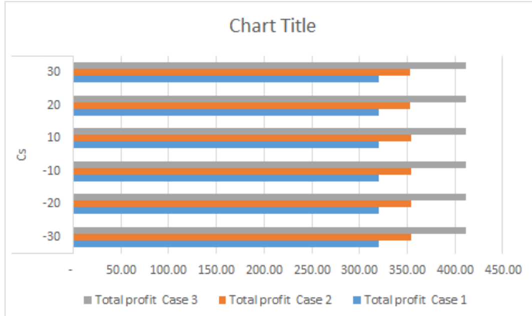

Figure 3:

Figure 4: Total profit by changing the parameters C_s and t_2

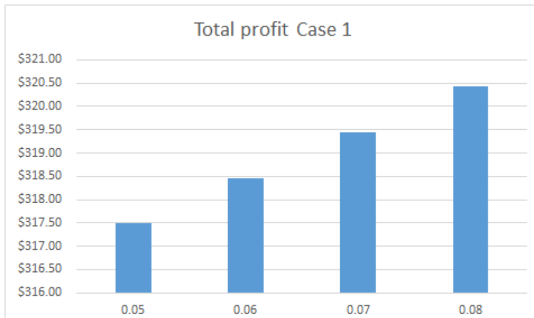

Figure 5:

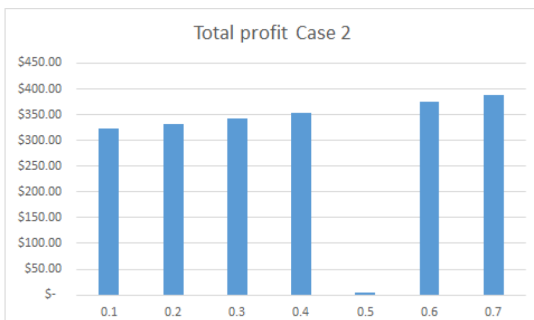

Figure 6:

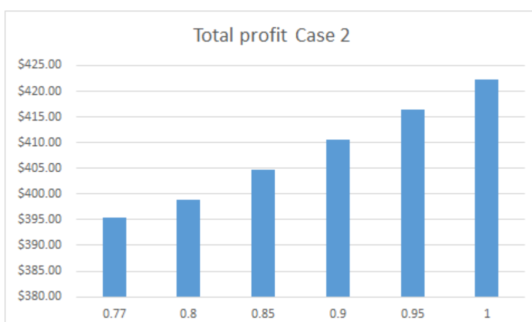

Figure 7:

8. Conclusion

In this work, an inventory model with a single item is developed for a non-instantaneous deterioration item with a return policy, allowable payment delays, and partial backlogging. Customers may return products at any time during the replenishment cycle. Products that have been returned may be resalable at the same selling price. During shortages partially backlogged is considered. This model maximizes the total profit by selling price and time. We investigated three cases. Solution procedure and numerical example are given. for future research, this model can be extended to advance payment with fully backlogged and instantly deteriorating items.

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References

[1] Duary, A., Das, S., Arif, M. G., Abualnaja, K. M., Khan, M. A. A., Zakarya, M., & Shaikh, A. A, Advance and delay in payments with the price-discount inventory model for deteriorating items under capacity constraint and partially backlogged shortages, *Alexandria Engineering Journal*, **61**(2)(2022), 1735–1745.

[2] Geetha, K. V and Uthayakumar, R, Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments, *Journal of Computational and Applied Mathematics*, **223**(2010), 2492–2505.

[3] Ghoreishi, M and Mirzazadeh, Joint optimal pricing and inventory control for deteriorating items under inflation and customer returns, *Journal of Industrial Engineering*, 2013, 1–10.

[4] Ghoreishi, M., Weber, G. W and Mirzazadeh, A, An inventory model for non-instantaneous deteriorating items with partial backlogging, permissible delay in payments, inflation-and selling price-dependent demand and customer returns, *Annals of Operations Research*, **226**(1)(2015), 221–238.

[5] Ghoreishi, M., Mirzazadeh, A and Weber, G. W, Optimal pricing and ordering policy for non-instantaneous deteriorating items under inflation and customer returns. Optimization, 2013, doi:10.1080/02331934.2013.853059.

[6] Jani, M. Y., Soni, V. M., & Betheja, M. R, Impact of customer returns and trade credit for deteriorating items with preservation technology investment under price-sensitive demand, *Investigation Operational*, 42(4)(2021), 540–551.

[7] Kumari, M and De, P. K, An EOQ model for deteriorating items analyzing retailer's optimal strategy under trade credit and return policy with nonlinear demand and resalable returns, *An International Journal of Optimization and Control: Theories & Applications*, 12(1)(2022), 47–55.

[8] Maihami, R and Kamalabadi, I. N, Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demand, *International Journal of Production Economics*, **136**(1)(2012), 116–122.

[9] Mashud, A. H. M, An EOQ deteriorating inventory model with different types of demand and fully backlogged shortages, *International Journal of Logistics Systems and Management*, 36(1)(2020), 16–45.

[10] MUSA, A AND SANI, B, Inventory ordering policies of delayed deteriorating items under permissible delay in payments, *International Journal of Production Economics*, 136(2010), 75–83.

[11] OUYANG, L. Y., WU, K. S AND YANG, C. T, A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments, *Computers and Industrial Engineering*, 51(2006), 637–651.

[12] SINGH, R AND MISHRA, V. K, Sustainable Integrated Inventory Model for Substitutable Deteriorating Items Considering Both Transport and Industry Carbon Emissions, *Journal of Systems Science and Systems Engineering*, (2022), 1–21.

[13] SUNDARARAJAN, R., PRABHA, M AND JAYA, R, An inventory model for non-instantaneous deteriorating items with multivariate demand and backlogging under inflation, *Journal of Management Analytics*, 6(3)(2019), 302–322.

[14] YANG, C. T., QUYANG, L. Y AND WU, H. H, Retailers optimal pricing and ordering policies for Noninstantaneous deteriorating items with price-dependent demand and partial backlogging, *Mathematical Problems in Engineering*, Volume 2009, Article ID 198305.