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Generalized Hyers-Ulam stability of a 3D additive-quadratic functional equation in Banach spaces: A study with counterexamples

G. Yagachitradevi*[1], S. Lakshminarayanan[2] and P. Ravindiran[3]

[1] *Department of Mathematics, Siga College of Management and Computer Science, Villupuram - 605601, Tamil Nadu, India.*
[2,3] *Department of Mathematics, Arignar Anna Government Arts College, Villupuram - 605 602, Tamil Nadu, India.*

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Abstract. In this research, we focus on solving a mixed type additive-quadratic functional equation expressed as:

$$h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) = 12\tilde{h}(s_1)+8\tilde{h}(s_2)+2\tilde{h}(s_3)+12h(s_1)$$

where $\tilde{h}(s_1)=h(s_1)+h(-s_1)$ is derived. We proceed to investigate the generalized Hyers-Ulam stability of this equation within the framework of Banach spaces, employing the Hyers direct method. Additionally, examples of non-stable cases are also provided.

AMS Subject Classifications: 39B52, 39B72, 39B82.

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Contents

1. Introduction

Ulam's seminal work on the stability of group homomorphisms [30] sparked a new line of inquiry into the stability of functional equations. Hyers gave a favorable answer to this topic in the context of Banach spaces, making significant progress [12]. Credit for extending this research to the broader topic of Generalized Hyers-Ulam stability inside functional equations belongs to Aoki [2] and Rassias [23]. Aoki expanded Rassias's original approach to incorporate additive mappings, which included employing an infinite Cauchy difference for linear mappings. In 1994, Gavruta [11] proposed the generalized control function $\phi(s_1, s_2)$ as a substantial alternative to the boundless Cauchy difference. In 2008, following this work, Ravi et al. [27] used the product and sum

*__Corresponding author.__ Email addresses: **yagachitradevi@gmail.com** (G. Yagachitradevi), **slnvpm29@gmail.com** (S. Lakshminarayanan), **p.ravindiran@gmail.com** (P. Ravindiran)

of two $p-$norms to prove a particular version of Gavruta's theorem. Many researchers have thoroughly studied the stability issues of several functional equations and there are numerous noteworthy outcomes related to this problem, as seen in [1, 3, 8–10, 13, 16, 18–20, 22, 24–26, 29] and other cited references.

The Cauchy equation, which has the form:

$$h(s_1 + s_2) = h(s_1) + h(s_2), \tag{1.1}$$

is one of the most well-known functional equations in mathematics. Functions that have this relationship are called *additive functions*.

The quadratic functional equation

$$h(s_1 + s_2) + h(s_1 - s_2) = 2h(s_1) + 2h(s_2) \tag{1.2}$$

is connected to a symmetric bi-additive function, as shown by the work of [1, 17]. Each of the solutions to this equation is a *quadratic function*. Skof [28] addressed a stability issue connected to the quadratic functional equation (1.2) by studying functions $h : K \to L$, where K is a normed space and L is a Banach space. An Abelian group may stand in for the domain K without affecting the validity of the argument, as noted by Cholewa [6], who elaborated on Skof's work. Czerwik [7] adds to the expanding body of evidence supporting the stability of the quadratic functional equation by demonstrating that it is Hyers-Ulam-Rassias stable.

The quadratic and additive functional equation

$$h(s_1 + ds_2) + dh(s_1 - s_2) = h(s_1 - ds_2) + dh(s_1 + s_2) \tag{1.3}$$

was studied by Jun and Kim [14], who examined the general solution and the generalized Hyers-Ulam stability for any positive integer d with $d \neq -1, 0, 1$. Additionally, Najati and Moghimi [21] investigated the quadratic and additive functional equation

$$h(2s_1 + s_2) + h(2s_1 - s_2) = 2h(s_1 + s_2) + 2h(s_1 - s_2) + 2h(2s_1) - 4h(s_1). \tag{1.4}$$

K. Balamurugan et al. [5] obtained the general solution to the cubic functional equation

$$\begin{aligned} &g(3s_1 + 2s_2 + s_3) + g(3s_1 + 2s_2 - s_3) + g(3s_1 - 2s_2 + s_3) + g(3s_1 - 2s_2 - c) \\ &= 24[g(s_1 + s_2) + g(s_1 - s_2)] + 6[g(s_1 + s_3) + g(s_1 - s_3)] + g(s_1) \end{aligned} \tag{1.5}$$

and investigated its generalized Hyers-Ulam stability.

M. Arunkumar et al. [4] have recently developed a general solution and generalized Hyers-Ulam stability for the three-dimensional additive-quadratic functional equation

$$\begin{aligned} &g(s_1 + 2s_2 + 3s_3) + g(s_1 + 2s_2 - 3s_3) + g(s_1 - 2s_2 + 3s_3) + g(-s_1 + 2s_2 + 3s_3) \\ &= g(s_1 + s_2 + s_3) + g(s_1 + s_2 - s_3) + g(s_1 - s_2 + s_3) + g(-s_1 + s_2 + s_3) \\ &\quad + 2g(s_2) + 4g(s_3) + 5[g(s_2) + g(-s_2)] + 14[g(s_3) + g(-s_3)] \end{aligned} \tag{1.6}$$

using a direct and fixed point approach in Banach space and non-Archimedean fuzzy Banach space.

In this study, we provide a general solution to the additive-quadratic functional equation

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12\tilde{h}(s_1) + 8\tilde{h}(s_2) + 2\tilde{h}(s_3) + 12h(s_1), \end{aligned} \tag{1.7}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$ and investigate the generalized Hyers - Ulam stability of this equation with the Hyers direct technique. In addition, unstable counterexamples are provided.

In Section 2, we provide a generic solution to (1.7). Using the direct method and the concept of generalized Hyers-Ulam, we demonstrate the stability of equation (1.7) for odd, even, and mixed mappings, with counterexamples provided in Sections 3, 4, and 5.

2. General Solution of (1.7)

This section examines the general solution of the functional equation (1.7) when $\mathcal{K}$ and $\mathcal{L}$ are treated as real vector spaces.

Theorem 2.1. *If an odd function $h : \mathcal{K} \to \mathcal{L}$ meets the requirements of the functional equation* (1.7) *for all $s_1, s_2, s_3 \in \mathcal{K}$, then it must also meet the functional equation* (1.1) *for all $s_1, s_2 \in \mathcal{K}$ and vice versa.*

Proof. Consider the odd function $h : \mathcal{K} \to \mathcal{L}$ to satisfy the functional equation (1.7). By inputting (s_1, s_2, s_3) as $(0, 0, 0)$ in (1.7), we determine that $h(0) = 0$. By setting s_3 to 0 in (1.7) and using the property that h is odd, we can deduce that

$$h(3s_1 + 2s_2) + h(3s_1 - 2s_2) = 6h(s_1), \tag{2.1}$$

for any $s_1, s_2 \in \mathcal{K}$. Additionally, by setting s_2 to 0 in this equation, we find that

$$h(3s_1) = 3h(s_1) \tag{2.2}$$

for any $s_1 \in \mathcal{K}$. By substituting $\frac{s_1}{3}$ for s_1 in this equation, we arrive

$$h\left(\frac{s_1}{3}\right) = \frac{1}{3}h(s_1) \tag{2.3}$$

for any $s_1 \in \mathcal{K}$. Finally, by replacing s_1 with $\frac{s_1}{3}$ and s_2 with $\frac{s_1}{2}$ in (2.1) and using (2.3), we can conclude that

$$h(2s_1) = 2h(s_1) \tag{2.4}$$

for any $s_1 \in \mathcal{K}$. Hence, for any positive whole number b,

$$h(bs_1) = bh(s_1) \tag{2.5}$$

for any $s_1 \in \mathcal{K}$. By entering (s_1, s_2) as $\left(\frac{s_1}{3}, \frac{s_2}{2}\right)$ into (2.1) and using (2.3), we infer that

$$h(s_1 + s_2) + h(s_1 - s_2) = 2h(s_1), \tag{2.6}$$

for any $s_1, s_2 \in \mathcal{K}$. By switching the positions of s_1 and s_2 and applying the characteristic of h being an odd function, we arrive

$$h(s_1 + s_2) - h(s_1 - s_2) = 2h(s_2), \tag{2.7}$$

for any $s_1, s_2 \in \mathcal{K}$. By merging equations (2.6) and (2.7), we reach the desired outcome of (1.1).

Let us suppose, on the other hand, that an atypical odd mapping $h : \mathcal{K} \to \mathcal{L}$ satisfies the conditions stated in functional equation (1.1). By plugging in $s_1 = 0$ and $s_2 = 0$ into equation (1.1), we find that $h(0) = 0$. By also plugging in s_1 for s_2 and $2s_1$ for s_2 into (1.1), we get two new equations:

$$h(2s_1) = 2h(s_1) \quad \text{and} \quad h(3s_1) = 3h(s_1) \tag{2.8}$$

for any $s_1 \in \mathcal{K}$. By induction, for any natural number c, we have

$$h(cs_1) = ch(s_1) \tag{2.9}$$

for any $s_1 \in \mathcal{K}$. We start with the equation (1.1) and replace the variable s_2 with $s_2 + s_3$ and use (1.1). This gives us

$$h(s_1 + s_2 + s_3) = h(s_1) + h(s_2) + h(s_3) \tag{2.10}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Next, substitute (s_1, s_2, s_3) with $(3s_1, 2s_2, s_3)$ in (2.10) and we use (2.8) to obtain

$$h(3s_1 + 2s_2 + s_3) = 3h(s_1) + 2h(s_2) + h(s_3) \tag{2.11}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then change the sign of s_2 in (2.11) to get

$$h(3s_1 - 2s_2 + s_3) = 3h(s_1) + 2h(-s_2) + h(s_3) \tag{2.12}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$ and repeat the process with s_3 to get

$$h(3s_1 + 2s_2 - s_3) = 3h(s_1) + 2h(s_2) + h(-s_3) \tag{2.13}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then substitute both s_2 and s_3 with their negative versions in (2.11) to obtain

$$h(3s_1 - 2s_2 - s_3) = 3h(s_1) + 2h(-s_2) + h(-s_3) \tag{2.14}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. By adding together equations (2.11), (2.12), (2.13) and (2.14), we arrive at

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) \end{aligned} \tag{2.15}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then add $12h(s_1) + 4h(s_2)$ to both sides of equation (2.15) to get

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) + 12h(s_1) + 4h(s_2) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) + 12h(s_1) + 4h(s_2) \end{aligned} \tag{2.16}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. From (2.15), we can conclude

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &= 12h(s_1) + 4h(s_2) + 4h(-s_2) + 2h(s_3) + 2h(-s_3) + 12h(s_1) + 4h(s_2) - 12h(s_1) - 4h(s_2) \end{aligned} \tag{2.17}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. The fact that h is an odd function has allowed us to convincingly demonstrate our conclusion. ■

Theorem 2.2. *If an even function $h : \mathcal{K} \to \mathcal{L}$ meets the requirements of the functional equation* (1.7) *for all $s_1, s_2, s_3 \in \mathcal{K}$, then it must also meet the functional equation* (1.2) *for all $s_1, s_2 \in \mathcal{K}$ and vice versa.*

Proof. Consider the even function $h : \mathcal{K} \to \mathcal{L}$ to satisfy the functional equation (1.7). By inputting (s_1, s_2, s_3) as $(0, 0, 0)$ in (1.7), we determine that $h(0) = 0$. By setting (s_1, s_2, s_3) as $(0, s_1, s_2)$ in (1.7) and using the property that h is even, we can deduce that

$$h(2s_1 + s_2) + h(2s_1 - s_2) = 8h(s_1) + 2h(s_2), \tag{2.18}$$

for any $s_1, s_2 \in \mathcal{K}$. Additionally, by setting s_2 to 0 in this equation, we find that

$$h(2s_1) = 4h(s_1) \tag{2.19}$$

for any $s_1 \in \mathcal{K}$. By substituting $\frac{s_1}{2}$ for s_1 in this equation, we arrive

$$h\left(\frac{s_1}{2}\right) = \frac{1}{4}h(s_1) \tag{2.20}$$

for any $s_1 \in \mathcal{K}$. Finally, by replacing s_2 with s_1 in (2.18), we can conclude that

$$h(3s_1) = 9h(s_1) \tag{2.21}$$

for any $s_1 \in \mathcal{K}$. Hence, for any positive whole number a,

$$h(as_1) = a^2 h(s_1) \tag{2.22}$$

for any $s_1 \in \mathcal{K}$. By substituting $\frac{s_1}{2}$ for s_1 in equation (2.18) and using (2.20), we arrive at equation(1.2), as intended.

Let us suppose, on the other hand, that an atypical even mapping $h : \mathcal{K} \to \mathcal{L}$ satisfies the conditions stated in functional equation (1.2). By substituting $s_1 = 0$ and $s_2 = 0$ into (1.2), we can determine that $h(0) = 0$. Additionally, by inputting s_1 for s_2 and $2s_1$ for s_2 into the same equation and taking into account that h is an even function, we obtain two additional equations:

$$h(2s_1) = 4h(s_1) \quad \text{and} \quad h(3s_1) = 9h(s_1) \tag{2.23}$$

for any $s_1 \in \mathcal{K}$. We can prove that for any natural number c through the method of induction, we have

$$h(cs_1) = c^2 h(s_1) \tag{2.24}$$

for any $s_1 \in \mathcal{K}$. By replacing s_1 with $3s_1$ and s_2 with $2s_2$ in (1.2) and using (2.24), we obtain

$$h(3s_1 + 2s_2) + h(3s_1 - 2s_2) = 18h(s_1) + 8h(s_2) \tag{2.25}$$

for any $s_1, s_2 \in \mathcal{K}$. Again replacing s_1 with $3s_1 + 2s_2$ and s_2 with s_3 in (1.2), we have

$$h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) = 2h(3s_1 + 2s_2) + 2h(s_3) \tag{2.26}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. We then change the sign of s_2 in (2.26) to get

$$h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) = 2h(3s_1 - 2s_2) + 2h(s_3) \tag{2.27}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Adding both (2.26) and (2.27), we obtain

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &\quad = 2[h(3s_1 + 2s_2) + h(3s_1 - 2s_2)] + 4h(s_3) \end{aligned} \tag{2.28}$$

for any $s_1, s_2, s_3 \in \mathcal{K}$. Using (2.25) in (2.28) and the property of h being even, we achieve

$$\begin{aligned} &h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\ &\quad = 36h(s_1) + 16h(s_2) + 4h(s_3) = 12\tilde{h}(s_1) + 8\tilde{h}(s_2) + 2\tilde{h}(s_3) + 12h(s_1), \end{aligned} \tag{2.29}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$, for any $s_1, s_2, s_3 \in \mathcal{K}$. ■

Hearafter, throughout this analysis, we will presume that $\mathcal{K}$ is a normed space and $\mathcal{L}$ is a Banach space, and we will introduce the mapping $Dh : \mathcal{K}^3 \to \mathcal{L}$ in the following manner:

$$\begin{aligned} Dh(s_1, s_2, s_3) =& h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) \\ &+ h(3s_1 - 2s_2 - s_3) - 12\tilde{h}(s_1) - 8\tilde{h}(s_2) - 2\tilde{h}(s_3) - 12h(s_1), \end{aligned}$$

where $\tilde{h}(s_1) = h(s_1) + h(-s_1)$, for all $s_1, s_2, s_3 \in \mathcal{K}$.

3. Stability of (1.7) for odd mappings

In this paper, we examine the generalized Hyers-Ulam stability of the functional equation (1.7), in particular for the case of an odd mapping.

Theorem 3.1. *Let $s = \pm 1$ and $\xi : \mathcal{K}^3 \to [0, \infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty} \frac{\xi\left(6^{si}s_1, 6^{si}s_2, 6^{si}s_3\right)}{6^{si}} < \infty \tag{3.1}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Let $h : \mathcal{K} \to \mathcal{L}$ be an odd mapping that satisfies

$$\|Dh(s_1, s_2, s_3)\| \leq \xi\left(s_1, s_2, s_3\right) \tag{3.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ satisfying (1.7) *and*

$$\|h(s_1) - \mathcal{A}(s_1)\| \leq \frac{1}{6} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si}s_1\right)}{6^{si}} \tag{3.3}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ and $\mathcal{A}(s_1)$ are given by

$$\psi\left(s_1\right) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{3.4}$$

and

$$\mathcal{A}(s_1) = \lim_{i\to\infty} \frac{h(6^{si}s_1)}{6^{si}} \tag{3.5}$$

for all $s_1 \in \mathcal{K}$, respectively.

Proof. Assuming that s is equal to 1. By substituting (s_1, s_2, s_3) with (s_1, s_1, s_1) in (3.2) and make use of the oddness of h, we arrive at the inequality

$$\|h(6s_1) + h(4s_1) + h(2s_1) - 12h(s_1)\| \leq \xi\left(s_1, s_1, s_1\right) \tag{3.6}$$

for all $s_1 \in \mathcal{K}$. Similarly, substituting (s_1, s_2, s_3) with $(s_1, 0, s_1)$ in (3.2) and using the oddness of h, we get

$$\|h(4s_1) + h(2s_1) - 6h(s_1)\| \leq \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{3.7}$$

for all $s_1 \in \mathcal{K}$. Combining these two inequalities, we find that

$$\begin{aligned}\|h(6s_1) - 6h(s_1)\| &\leq \|h(6s_1) + h(4s_1) + h(2s_1) - 12h(s_1)\| + \|h(4s_1) + h(2s_1) - 6h(s_1)\| \\ &\leq \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right)\end{aligned} \tag{3.8}$$

for all $s_1 \in \mathcal{K}$. Dividing the preceding inequality by 6 yields

$$\left\|\frac{h(6s_1)}{6} - h(s_1)\right\| \leq \frac{\psi(s_1)}{6} \tag{3.9}$$

where

$$\psi(s_1) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right)$$

for all $s_1 \in \mathcal{K}$. By plugging in $6s_1$ in place of s_1 and dividing by 6 in (3.9), we acquire

$$\left\|\frac{h(6^2s_1)}{6^2} - \frac{h(6s_1)}{6}\right\| \leq \frac{\xi(6s_1)}{6^2} \tag{3.10}$$

for all $s_1 \in \mathcal{K}$. From (3.9) and (3.10), we obtain

$$\begin{aligned}\left\|\frac{h(6^2 s_1)}{6^2} - h(s_1)\right\| &\leq \left\|\frac{h(6s_1)}{6} - h(s_1)\right\| + \left\|\frac{h(6^2 s_1)}{6^2} - \frac{h(6s_1)}{6}\right\| \\ &\leq \frac{1}{6}\left[\xi(s_1) + \frac{\xi(6s_1)}{6}\right] \end{aligned} \tag{3.11}$$

for all $s_1 \in \mathcal{K}$. Then, by using induction to a positive integer n, we have

$$\begin{aligned}\left\|\frac{h(6^n s_1)}{6^n} - h(s_1)\right\| &\leq \frac{1}{6}\sum_{i=0}^{n-1}\frac{\xi(6^i s_1)}{6^i} \\ &\leq \frac{1}{6}\sum_{i=0}^{\infty}\frac{\xi(6^i s_1)}{6^i}\end{aligned} \tag{3.12}$$

for all $s_1 \in \mathcal{K}$. Substituting $6^m s_1$ for s_1 and dividing by 6^m in (3.12), we see that the sequence $\left\{\frac{h(6^n s_1)}{6^n}\right\}$ converges. It follows that for any m and n in the positive integer range, we can conclude that

$$\begin{aligned}\left\|\frac{h(6^{n+m} s_1)}{6^{(n+m)}} - \frac{h(6^m s_1)}{6^m}\right\| &= \frac{1}{6^m}\left\|\frac{h(6^n \cdot 6^m s_1)}{6^n} - h(6^m s_1)\right\| \\ &\leq \frac{1}{6}\sum_{i=0}^{n-1}\frac{\xi(6^{i+m} s_1)}{6^{(i+m)}} \\ &\leq \frac{1}{6}\sum_{i=0}^{\infty}\frac{\xi(6^{i+m} s_1)}{6^{(i+m)}} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $s_1 \in \mathcal{K}$. Thus $\left\{\frac{h(6^n s_1)}{6^n}\right\}$ is Cauchy. For complete set $\mathcal{L}$, a mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ exists with

$$\mathcal{A}(s_1) = \lim_{n\to\infty}\frac{h(6^n s_1)}{6^n}, \quad \forall \; s_1 \in \mathcal{K}.$$

When we plug in (3.12), where n may go to infinity, we get that (3.3) is true for every $s_1 \in \mathcal{K}$. To show that $\mathcal{A}$ satisfies (1.7), we substitute $(6^n s_1, 6^n s_2, 6^n s_3)$ for (s_1, s_2, s_3) in (3.2) and divide by 6^n to get

$$\frac{1}{6^n}\left\|Dh(6^n s_1, 6^n s_2, 6^n s_3)\right\| \leq \frac{1}{6^n}\xi(6^n s_1, 6^n s_2, 6^n s_3)$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Using the definition of $\mathcal{A}(s_1)$ and the aforementioned inequality, we can show that $\mathcal{A}(s_1)$ as n goes to infinity. Hence For all $s_1, s_2, s_3 \in \mathcal{K}$, $\mathcal{A}$ fulfils (1.7). If $\mathcal{A}$ is not unique, we may show that $\mathcal{C}(s_1)$ is also an additive mapping fulfilling (1.7) and (3.3), as

$$\begin{aligned}\|\mathcal{A}(s_1) - \mathcal{C}(s_1)\| &= \frac{1}{6^n}\|\mathcal{A}(6^n s_1) - \mathcal{C}(6^n s_1)\| \\ &\leq \frac{1}{6^n}\left\{\|\mathcal{A}(6^n s_1) - h(6^n s_1)\| + \|h(6^n s_1) - \mathcal{C}(6^n s_1)\|\right\} \\ &\leq \frac{1}{3}\sum_{i=0}^{\infty}\frac{\xi(6^{(i+n)} s_1)}{6^{(i+n)}} \\ &\to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

for every $s_1 \in \mathcal{K}$. For this reason, $\mathcal{A}$ cannot be found anywhere else. This proves that the theory is correct when s equals 1. Substituting $\frac{s_1}{6}$ for s_1 in inequality (3.8) leads to the conclusion that

$$\left\|h(s_1) - 6h\left(\frac{s_1}{6}\right)\right\| \leq \xi\left(\frac{s_1}{6}, \frac{s_1}{6}, \frac{s_1}{6}\right) + \frac{1}{2}\xi\left(\frac{s_1}{6}, 0, \frac{s_1}{6}\right)$$

for every $s_1 \in \mathcal{K}$. The remainder of the proof for $s = -1$ is the same as it is for $s = 1$. Therefore, the theorem is valid for both $s = 1$ and $s = -1$. The theorem has been proven at this point. ■

The next Corollary is directly derived from Theorem 3.1 concerning the stability of Equation (1.7).

Corollary 3.2. *Let t be a positive real value, and assume $\nu \geq 0$. For any $s_1, s_2, s_3 \in \mathcal{K}$, let $h : \mathcal{K} \to \mathcal{L}$ be a function that fulfils the inequality*

$$\|Dh(s_1, s_2, s_3)\| \leq \begin{cases} \nu, & \\ \nu\left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 1; \\ \nu||s_1||^t||s_2||^t||s_3||^t, & 3t \neq 1; \\ \nu\left\{||s_1||^t||s_2||^t||s_3||^t + ||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}, & 3t \neq 1. \end{cases} \tag{3.13}$$

If so, then for every $s_1 \in \mathcal{K}$, there is a unique additive function $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ such that

$$\|h(s_1) - \mathcal{A}(s_1)\| \leq \begin{cases} \frac{3\nu}{10}, \\ \frac{4\nu||s_1||^t}{|6^t - 6|}, \ t \neq 1; \\ \frac{\nu||s_1||^t}{|6^{3t} - 6|}, \ 3t \neq 1; \\ \frac{5\nu||s_1||^t}{|6^{3t} - 6|}, \ 3t \neq 1. \end{cases} \tag{3.14}$$

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 3.2, we will now present an illustration.

Example 3.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1, & \textit{if } |s_1| < 1 \\ \nu, & \textit{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq 432\nu(|s_1| + |s_2| + |s_3|) \tag{3.15}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1)| \leq \kappa|s_1| \quad \textit{for all} \quad s_1 \in \mathbb{R}. \tag{3.16}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^n|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^n} = \frac{6\nu}{5}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (3.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1| + |s_2| + |s_3| \geq \frac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} \leq \frac{6\nu}{5} \times 60 \times 6 = 432\nu$$

and hence (3.15) is obvious. Take into account the scenario where

$$0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{3.17}$$

This implies that $6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$, and $6^{m-1}z < \frac{1}{6}$. As a result,

$$\begin{aligned} &6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3), \\ &\qquad 6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3) \end{aligned}$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned} &6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3), \\ &\qquad 6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3) \end{aligned}$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is linear over this range, we may conclude that

$$\begin{aligned} &\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) \\ &- 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0, \end{aligned}$$

for $n = 0, 1, \ldots, m-1$. Using (3.17) and the definition of h, we may calculate

$$\begin{aligned} \frac{|Dh(s_1, s_2, s_3)|}{(|s_1| + |s_2| + |s_3|)} &\leq \sum_{n=m}^{\infty} \frac{1}{6^n(|s_1| + |s_2| + |s_3|)} \Big| \xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) \\ &\quad + \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] \\ &\quad - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) \Big| \\ &\leq \sum_{k=0}^{\infty} \frac{60\nu}{6^k 6^m(|s_1| + |s_2| + |s_3|)} \leq \sum_{k=0}^{\infty} \frac{360\nu}{6^k} = 432\nu. \end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}$, h fulfills (3.15). According to Corollary 3.2, the additive functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there is an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa > 0$. Since h is bounded and continuous for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ is also bounded and continuous within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1) = cs_1$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\,|s_1|. \tag{3.18}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n} \geq \sum_{n=0}^{m-1} \frac{\nu \times 6^n s_1}{6^n} = m\nu s_1 > (\kappa + |c|)\, s_1$$

which defies (3.18). Based on the inequality (3.13), it may be concluded that the equation (1.7) is not stable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 3.2, the functional equation (1.7) is unstable for $t = \frac{1}{3}$.

Example 3.4. *Suppose t is such that $0 < t < \frac{1}{3}$. Then, there exists a function $h : \mathbb{R} \to \mathbb{R}$ and a constant $\nu > 0$ such that for all real numbers $s_1, s_2, s_3 \in \mathbb{R}$,*

$$|Dh(s_1, s_2, s_3)| \leq \nu |s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}} \tag{3.19}$$

and for all additive mappings $\mathcal{A} : \mathbb{R} \to \mathbb{R}$

$$\sup_{s_1 \neq 0} \frac{|h\,(s_1) - \mathcal{A}\,(s_1)|}{|s_1|} = +\infty. \tag{3.20}$$

Proof. If we set $h(s_1) = s_1 \ln |s_1|$, if $s_1, \neq 0$, and $h(0) = 0$, then we may deduce that

$$\begin{aligned}
\sup_{s_1 \neq 0} \frac{|h\,(s_1) - \mathcal{A}\,(s_1)|}{|x|} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|h\,(n) - \mathcal{A}\,(n)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|n \ln |n| - n\,\mathcal{A}\,(1)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \ |\ln |n| - \mathcal{A}\,(1)|\ = \infty.
\end{aligned}$$

We need to show that (3.19).
Case (i): If $s_1, s_2, s_3 > 0$ in (3.19) then,

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3) \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3) \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3) \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3) \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1 \ln |s_1| - s_1 \ln |-s_1|] - 8[s_2 \ln |s_2| - s_2 \ln |-y|] \\
&\quad -2[s_3 \ln |s_3| - s_3 \ln |-s_3|] - 12 s_1 \ln |s_1||
\end{aligned}$$

If we set $s_1 = j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= |(3s_1 + 2s_2 + s_3) \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3) \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3) \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3) \ln |3s_1 - 2s_2 - s_3|
\end{aligned}$$

$$
\begin{aligned}
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1|| \\
&= |(3j+2k+l)\ln|3j+2k+l| + (3j+2k-l)\ln|3j+2k-l| \\
&+ (3j-2k+l)\ln|3j-2k+l| + (3j-2k-l)\ln|3j-2k-l| \\
&- 12[j\ln|j| - j\ln|-j|] - 8[k\ln|k| - k\ln|-k|] \\
&-2[l\ln|l| - l\ln|-l|] - 12j\ln|j|| \\
&|h(3j+2k+l) + h(3j+2k-l) + h(3j-2k+l) + h(3j-2k-l) \\
&-12[h(j)+h(-j)] - 8[h(k)+h(-k)] - 2[h(l)+h(-l)] - 12h(j)| \\
&\leq \nu|j|^{\frac{t}{3}}\ |k|^{\frac{t}{3}}\ |c|^{\frac{1-2t}{3}} = \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}
$$

Case (ii): If $s_1, s_2, s_3 < 0$ in (3.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1||
\end{aligned}
$$

If we set $s_1 = -j, s_2 = -k$, and $s_3 = -l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1|| \\
&= |(-j-2k-l)\ln|-3j-2k-l| + (-3j-2k+l)\ln|-3j-2k+l| \\
&+ (-3j+2k-l)\ln|-3j+2k-l| + (-3j+2k+l)\ln|-3j+2k+l| \\
&- 12[-j\ln|-j| + j\ln|j|] - 8[-k\ln|-k| + k\ln|k|] \\
&-2[-l\ln|-l| + l\ln|l|] + 12j\ln|-j|| \\
&|h(-j-2k-l) + h(-3j-2k+l) + h(-3j+2k-l) + h(-3j+2k+l) \\
&-12[h(-j)+h(j)] - 8[h(-k)+h(k)] - 2[h(-l)+h(l)] - 12h(-j)| \\
&\leq \nu|-j|^{\frac{t}{3}}\ |-k|^{\frac{t}{3}}\ |-c|^{\frac{1-2t}{3}} = \nu|s_1|^{\frac{t}{3}}\ |s_2|^{\frac{t}{3}}\ |s_3|^{\frac{1-2t}{3}}
\end{aligned}
$$

Case (iii): If $s_1 > 0, s_2, s_3 < 0$ in (3.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= |(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3| \\
&- 12[s_1 \ln|s_1| - s_1 \ln|-s_1|] - 8[s_2 \ln|s_2| - s_2 \ln|-y|] \\
&-2[s_3 \ln|s_3| - s_3 \ln|-s_3|] - 12 s_1 \ln|s_1||
\end{aligned}
$$

If we set $s_1 = j, s_2 = -k$, and $s_3 = -l$, then we get

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&=|(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3|\\
&\quad+(3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3|\\
&\quad-12[s_1\ln|s_1|-s_1\ln|-s_1|]-8[s_2\ln|s_2|-s_2\ln|-y|]\\
&\quad-2[s_3\ln|s_3|-s_3\ln|-s_3|]-12s_1\ln|s_1||\\
&=|(3j-2k-l)\ln|3j-2k-l|+(3j-2k+l)\ln|3j-2k+l|\\
&\quad+(3j+2k-l)\ln|3j+2k-l|+(3j+2k+l)\ln|3j+2k+l|\\
&\quad-12[j\ln|j|-j\ln|-j|]-8[-k\ln|-k|+k\ln|k|]\\
&\quad-2[-l\ln|-l|+l\ln|l|]-12j\ln|j||\\
&|h(3j-2k-l)+h(3j-2k+l)+h(3j+2k-l)+h(3j+2k+l)\\
&\quad-12[h(j)+h(-j)]-8[h(-k)+h(k)]-2[h(-l)+h(l)]-12h(j)|\\
&\le \nu|j|^{\frac{t}{3}}\,|-k|^{\frac{t}{3}}\,|-c|^{\frac{1-2t}{3}}\\
&=\nu|s_1|^{\frac{t}{3}}\,|s_2|^{\frac{t}{3}}\,|s_3|^{\frac{1-2t}{3}}
\end{aligned}$$

Case (iv): If $s_1 < 0, s_2, s_3 > 0$ in (3.19) then,

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&=|(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3|\\
&\quad+(3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3|\\
&\quad-12[s_1\ln|s_1|-s_1\ln|-s_1|]-8[s_2\ln|s_2|-s_2\ln|-y|]\\
&\quad-2[s_3\ln|s_3|-s_3\ln|-s_3|]-12s_1\ln|s_1||
\end{aligned}$$

If we set $s_1 = -j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3)\\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)|\\
&=|(3s_1+2s_2+s_3)\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)\ln|3s_1+2s_2-s_3|\\
&\quad+(3s_1-2s_2+s_3)\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)\ln|3s_1-2s_2-s_3|\\
&\quad-12[s_1\ln|s_1|-s_1\ln|-s_1|]-8[s_2\ln|s_2|-s_2\ln|-y|]\\
&\quad-2[s_3\ln|s_3|-s_3\ln|-s_3|]-12s_1\ln|s_1||\\
&=|(-3j+2k+l)\ln|-3j+2k+l|+(-3j+2k-l)\ln|-3j+2k-l|\\
&\quad+(-3j-2k+l)\ln|-3j-2k+l|+(-3j-2k-l)\ln|-3j-2k-l|\\
&\quad-12[-j\ln|-j|+j\ln|-j|]-8[k\ln|k|-k\ln|-k|]\\
&\quad-2[l\ln|l|-l\ln|-l|]+12j\ln|-j||\\
&|h(-3j+2k+l)+h(-3j+2k-l)+h(-3j-2k+l)+h(-3j-2k-l)\\
&\quad-12[h(-j)+h(-j)]-8[h(k)+h(-k)]-2[h(l)+h(-l)]-12h(-j)|\\
&\le \nu|-j|^{\frac{t}{3}}\,|k|^{\frac{t}{3}}\,|l|^{\frac{1-2t}{3}}\\
&=\nu|s_1|^{\frac{t}{3}}\,|s_2|^{\frac{t}{3}}\,|s_3|^{\frac{1-2t}{3}}
\end{aligned}$$

Case (v): If $s_1 = s_2 = s_3 = 0$ in (3.19), then the statement is obvious. ■

Here we present an example to show that, as mentioned in Corollary 3.2, the functional equation (1.7) is unstable for $t = \frac{1}{3}$.

Example 3.5. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^n}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq 432\nu \left(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3|\right) \tag{3.21}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1)| \leq \kappa |s_1| \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{3.22}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^n|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^n} = \frac{6\nu}{5}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (3.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| \geq \frac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{\left(|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3|\right)} \leq \frac{6\nu}{5} \times 60 \times 6 = 432\nu$$

and hence (3.15) is obvious. Take into account the scenario where

$$0 < |s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}|s_3|^{\frac{1}{3}} + |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{3.23}$$

This implies that $6^{m-1}s_1^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} < \frac{1}{6}, 6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$ and $6^{m-1}z < \frac{1}{6}$. As a result,

$$\begin{gathered}6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3), \\ 6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)\end{gathered}$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),$$

$$6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)$$

are also within the range of $(-1,1)$. Due to the fact that ξ is linear over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))+\xi(6^n(3s_1-2s_2+s_3))\\&+\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))=0,\end{aligned}$$

for $n=0,1,\ldots,m-1$. Utilising (3.17) and the definition of h, we may calculate

$$\begin{aligned}&\frac{|Dh(s_1,s_2,s_3)|}{(|s_1|^{\frac13}|s_2|^{\frac13}|s_3|^{\frac13}+|s_1|+|s_2|+|s_3|)}\\&\le\sum_{n=m}^{\infty}\frac{1}{6^n(|s_1|^{\frac13}|s_2|^{\frac13}|s_3|^{\frac13}+|s_1|+|s_2|+|s_3|)}\Big|\xi(6^n(3s_1+2s_2+s_3))\\&+\xi(6^n(3s_1+2s_2-s_3))+\xi(6^n(3s_1-2s_2+s_3))+\xi(6^n(3s_1-2s_2-s_3))\\&-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))\Big|\\&\le\sum_{k=0}^{\infty}\frac{60\nu}{6^k6^m(|s_1|^{\frac13}|s_2|^{\frac13}|s_3|^{\frac13}+|s_1|+|s_2|+|s_3|)}\le\sum_{k=0}^{\infty}\frac{360\nu}{6^k}=432\nu.\end{aligned}$$

Consequently, for all $s_1,s_2,s_3\in\mathbb{R}$ with $0<|s_1|^{\frac13}|s_2|^{\frac13}|s_3|^{\frac13}+|s_1|+|s_2|+|s_3|<\dfrac{1}{6}$, h fulfills (3.15). According to Corollary 3.2, the additive functional equation (1.7) is unstable at $t=\dfrac{1}{3}$. Let us assume, however, that there is an additive mapping $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa>0$. Since h is bounded and continuous for every $s_1\in\mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ is also bounded and continuous within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1)=cs_1$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)|\le(\kappa+|c|)\,|s_1|. \tag{3.24}$$

being true. However, by choosing a positive integer m with $m\nu>\kappa+|c|$, we can find $s_1\in\left(0,\frac{1}{6^{m-1}}\right)$ such that $6^ns_1\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this s_1, we get

$$h(s_1)=\sum_{n=0}^{\infty}\frac{\xi(6^ns_1)}{6^n}\ge\sum_{n=0}^{m-1}\frac{\nu\times6^ns_1}{6^n}=m\nu s_1>(\kappa+|c|)\,s_1$$

which defies (3.18). Based on the inequality (3.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t=\dfrac{1}{3}$. ■

4. Stability of (1.7) for even mappings

In this paper, we examine the generalized Hyers-Ulam stability of the functional equation (1.7), in particular for the case of an even mapping.

Theorem 4.1. *Let $s=\pm1$ and $\xi:\mathcal{K}^3\to[0,\infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{2si}}<\infty \tag{4.1}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Let $h : \mathcal{K} \to \mathcal{L}$ be an even mapping fulfills

$$\|Dh(s_1, s_2, s_3)\| \leq \xi\left(s_1, s_2, s_3\right) \tag{4.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then there is only one quadratic mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ that fulfills (1.7) *and*

$$\|h(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{6^2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si}s_1\right)}{6^{2si}} \tag{4.3}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ and $\mathcal{B}(s_1)$ are defined by

$$\psi\left(s_1\right) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{4.4}$$

and

$$\mathcal{B}(s_1) = \lim_{i\to\infty} \frac{h(6^{si}s_1)}{6^{2si}} \tag{4.5}$$

for all $s_1 \in \mathcal{K}$, respectively.

Proof. Assuming that s is equal to 1. By substituting (s_1, s_2, s_3) with (s_1, s_1, s_1) in (4.2) and make use of the evenness of h, we arrive at the inequality

$$\|h(6s_1) + h(4s_1) + h(2s_1) - 56h(s_1)\| \leq \xi\left(s_1, s_1, s_1\right) \tag{4.6}$$

for all $s_1 \in \mathcal{K}$. Similarly, substituting (s_1, s_2, s_3) with $(s_1, 0, s_1)$ in (4.2) and using the even property of h, we get

$$\|h(4s_1) + h(2s_1) - 20h(s_1)\| \leq \frac{1}{2}\xi\left(s_1, 0, s_1\right) \tag{4.7}$$

for all $s_1 \in \mathcal{K}$. Combining these two inequalities, we find that

$$\begin{aligned} \|h(6s_1) - 36h(s_1)\| &= \|h(6s_1) + h(4s_1) + h(2s_1) - 56h(s_1)\| + \|h(4s_1) + h(2s_1) - 20h(s_1)\| \\ &\leq \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right) \end{aligned} \tag{4.8}$$

for all $s_1 \in \mathcal{K}$. Dividing the preceding inequality by 6^2 yields

$$\left\|\frac{h(6s_1)}{6^2} - h(s_1)\right\| \leq \frac{\psi(s_1)}{6^2} \tag{4.9}$$

where

$$\psi(s_1) = \xi\left(s_1, s_1, s_1\right) + \frac{1}{2}\xi\left(s_1, 0, s_1\right)$$

for all $s_1 \in \mathcal{K}$. By plugging in $6s_1$ in place of s_1 and dividing by 6^2 in (4.9), we acquire

$$\left\|\frac{h(6^2s_1)}{6^4} - \frac{h(6s_1)}{6^2}\right\| \leq \frac{\xi(6s_1)}{6^4} \tag{4.10}$$

for all $s_1 \in \mathcal{K}$. From (4.9) and (4.10), we obtain

$$\begin{aligned} \left\|\frac{h(6^2s_1)}{6^4} - h(s_1)\right\| &\leq \left\|\frac{h(6s_1)}{6^2} - h(s_1)\right\| + \left\|\frac{h(6^2s_1)}{6^4} - \frac{h(6s_1)}{6^2}\right\| \\ &\leq \frac{1}{6^2}\left[\xi(s_1) + \frac{\xi(6s_1)}{6^2}\right] \end{aligned} \tag{4.11}$$

for all $s_1 \in \mathcal{K}$. Then, by induction to a postive integer n, we have

$$\begin{aligned} \left\| \frac{h(6^n s_1)}{6^{2n}} - h(s_1) \right\| &\leq \frac{1}{6^2} \sum_{i=0}^{n-1} \frac{\xi(6^i s_1)}{6^{2i}} \\ &\leq \frac{1}{6^2} \sum_{i=0}^{\infty} \frac{\xi(6^i s_1)}{6^{2i}} \end{aligned} \tag{4.12}$$

for all $s_1 \in \mathcal{K}$. Substituting $6^m s_1$ for s_1 and dividing by 6^{2m} in (4.12), we see that the sequence $\left\{ \frac{h(6^n s_1)}{6^{2n}} \right\}$ converges. It follows that for any m and n in the positive integer range, we can conclude that

$$\begin{aligned} \left\| \frac{h(6^{n+m} s_1)}{6^{2(n+m)}} - \frac{h(6^m s_1)}{6^{2m}} \right\| &= \frac{1}{6^{2m}} \left\| \frac{h(6^n \cdot 6^m s_1)}{6^{2n}} - h(6^m s_1) \right\| \\ &\leq \frac{1}{6^2} \sum_{i=0}^{n-1} \frac{\xi(6^{i+m} s_1)}{6^{2(i+m)}} \leq \frac{1}{6^2} \sum_{i=0}^{\infty} \frac{\xi(6^{i+m} s_1)}{6^{2(i+m)}} \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned}$$

for all $s_1 \in \mathcal{K}$. Thus $\left\{ \frac{h(6^n s_1)}{6^{2n}} \right\}$ is Cauchy. For complete set $\mathcal{L}$, a mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exists with

$$\mathcal{B}(s_1) = \lim_{n \to \infty} \frac{h(6^n s_1)}{6^{2n}}, \quad \forall \ s_1 \in \mathcal{K}.$$

When we plug in (4.12), where n may go to infinity, we get that (4.3) is true for every $s_1 \in \mathcal{K}$. To show that $\mathcal{A}$ satisfies (1.7), we substitute $(6^n s_1, 6^n s_2, 6^n s_3)$ for (s_1, s_2, s_3) in (4.2) and divide by 6^{2n} to get

$$\frac{1}{6^{2n}} \left\| Dh(6^n s_1, 6^n s_2, 6^n s_3) \right\| \leq \frac{1}{6^{2n}} \xi(6^n s_1, 6^n s_2, 6^n s_3)$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Using the definition of $\mathcal{B}(s_1)$ and the aforementioned inequality, we can show that $\mathcal{B}(s_1)$ as n goes to infinity. Hence For all $s_1, s_2, s_3 \in \mathcal{K}$, $\mathcal{B}$ fulfils (1.7). If $\mathcal{B}$ is not unique, we may show that $\mathcal{D}(s_1)$ is also an additive mapping fulfilling (1.7) and (4.3), as

$$\begin{aligned} \left\| \mathcal{B}(s_1) - \mathcal{D}(s_1) \right\| &= \frac{1}{6^{2n}} \left\| \mathcal{B}(6^n s_1) - \mathcal{D}(6^n s_1) \right\| \\ &\leq \frac{1}{6^{2n}} \left\{ \left\| \mathcal{B}(6^n s_1) - h(6^n s_1) \right\| + \left\| h(6^n s_1) - \mathcal{D}(6^n s_1) \right\| \right\} \\ &\leq \frac{2}{6^2} \sum_{i=0}^{\infty} \frac{\xi(6^{(i+n)} s_1)}{6^{2(i+n)}} \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for all $s_1 \in \mathcal{K}$. For this reason, $\mathcal{B}$ cannot be found anywhere else. This proves that the theory is correct when s equals 1. Substituting $\frac{s_1}{6}$ for s_1 in inequality (4.8) leads to the conclusion that

$$\left\| h(s_1) - 36h\left(\frac{s_1}{6}\right) \right\| \leq \xi\left(\frac{s_1}{6}, \frac{s_1}{6}, \frac{s_1}{6}\right) + \frac{1}{2}\xi\left(\frac{s_1}{6}, 0, \frac{s_1}{6}\right)$$

for all $s_1 \in \mathcal{K}$. The remainder of the proof for $s = -1$ is the same as it is for $s = 1$. Therefore, the theorem is valid for both $s = 1$ and $s = -1$. The theorem has been proven at this point. ■

The next Corollary is directly derived from Theorem 4.1 concerning the stability of Equation (1.7).

Corollary 4.2. *Let t be a positive real value, and assume $\nu \geq 0$. For any $s_1, s_2, s_3 \in \mathcal{K}$, let $h : \mathcal{K} \to \mathcal{L}$ be a function that fulfills the inequality*

$$\|Dh(s_1, s_2, s_3)\| \leq \begin{cases} \nu, \\ \nu \left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 2; \\ \nu ||s_1||^t ||s_2||^t ||s_3||^t, & 3t \neq 2; \\ \nu \left\{||s_1||^t ||s_2||^t ||s_3||^t + \left\{||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}\right\}, & 3t \neq 2. \end{cases} \tag{4.13}$$

If so, then for every $s_1 \in \mathcal{K}$, there is a unique quadratic function $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ such that

$$\|h(s_1) - \mathcal{B}(s_1)\| \leq \begin{cases} \dfrac{3\nu}{70}, \\ \dfrac{4\nu ||s_1||^t}{|6^t - 6^2|}, & t \neq 2; \\ \dfrac{\nu ||s_1||^t}{|6^{3t} - 6^2|}, & 3t \neq 2; \\ \dfrac{5\nu ||s_1||^t}{|6^{3t} - 6^2|}, & 3t \neq 2. \end{cases} \tag{4.14}$$

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 4.2, we will now present an illustration.

Example 4.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1^2, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}}$$

for all $s_1 \in \mathbb{R}$ fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{15552\nu}{7} (|s_1|^2 + |s_2|^2 + |s_3|^2) \tag{4.15}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{B}(s_1)| \leq \kappa |s_1|^2 \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{4.16}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^{2n}|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^{2n}} = \frac{36\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (4.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \frac{36\nu}{35} \times 60 \times 6^2 = \frac{15552\nu}{7}$$

and hence (4.15) is obvious. Take into account the scenario where

$$0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{4.17}$$

This implies that $6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$, and $6^{m-1}z < \frac{1}{6}$. As a result,

$$\begin{aligned}
&6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),\\
&6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)
\end{aligned}$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned}
&6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3),\\
&6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)
\end{aligned}$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is quadratic over this range, we may conclude that

$$\begin{aligned}
&\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3))\\
&+ \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))]\\
&- 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,
\end{aligned}$$

for integers $n = 0 \to m-1$. Utilising (4.17) and the definition of h, we may calculate

$$\begin{aligned}
\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} &\leq \sum_{n=m}^{\infty} \frac{1}{36^n(|s_1|^2 + |s_2|^2 + |s_3|^2)} \Big| \xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3))\\
&+ \xi(6^n(3s_1 - 2s_2 + s_3)) + \xi(6^n(3s_1 - 2s_2 - s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))]\\
&- 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) \Big|\\
&\leq \sum_{k=0}^{\infty} \frac{60\nu}{36^k 36^m(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \sum_{k=0}^{\infty} \frac{2160\nu}{36^k} = \frac{15552\nu}{7}.
\end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}$, h fulfills (4.15). According to Corollary 4.2, the quadratic functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there is a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ obeying (4.16), where R is the set of all real numbers and $\kappa > 0$. Since h is continuous and bounded for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{B}$ is also continuous and bounded within the interval. $\mathcal{B}$ must have the form $\mathcal{B}(s_1) = cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\, s_1^2. \tag{4.18}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}} \geq \sum_{n=0}^{m-1} \frac{\nu \times 6^{2n} x^2}{6^{2n}} = m\nu s_1^2 > (\kappa + |c|)\, s_1^2$$

which defies (4.18). Based on the inequality (4.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 4.2, the functional equation (1.7) is unstable for $t = \frac{2}{3}$.

Example 4.4. *Suppose t is such that $0 < t < \frac{2}{3}$. Then, there exists a function $h : \mathbb{R} \to \mathbb{R}$ and a constant $\nu > 0$ such that for all real numbers $s_1, s_2, s_3 \in \mathbb{R}$,*

$$|Dh(s_1, s_2, s_3)| \leq \nu |s_1|^{\frac{t}{3}} \, |s_2|^{\frac{t}{3}} \, |s_3|^{\frac{2-2t}{3}} \tag{4.19}$$

and for all quadratic mappings $\mathcal{B} : \mathbb{R} \to \mathbb{R}$

$$\sup_{s_1 \neq 0} \frac{|h(s_1) - \mathcal{B}(s_1)|}{|s_1|} = +\infty. \tag{4.20}$$

Proof. If we set $h(s_1) = s_1^2 \ln |s_1|$, if $s_1 \neq 0$, and $h(0) = 0$, then we may deduce that

$$\begin{aligned}
\sup_{x \neq 0} \frac{|h(s_1) - \mathcal{B}(s_1)|}{|s_1|} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|h(n) - \mathcal{B}(n)|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^2 \ln |n| - n^2 \, \mathcal{B}(1)\right|}{|n|} \\
&= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \; |\ln |n| - \mathcal{B}(1)| \; = \infty.
\end{aligned}$$

We need to show that (4.19).
Case (i): If $s_1, s_2, s_3 > 0$ in (4.19) then,

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1 + 2s_2 + s_3)^2 \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)^2 \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)^2 \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)^2 \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1^2 \ln |s_1| + s_1^2 \ln |-s_1|] - 8[s_2^2 \ln |s_2| + s_2^2 \ln |-s_2|] \\
&\quad -2[s_3^2 \ln |s_3| + s_3^2 \ln |-s_3|] - 12 s_1^2 \ln |s_1|\big|
\end{aligned}$$

If we set $s_1 = j, s_2 = k$, and $s_3 = l$, then we get

$$\begin{aligned}
&|h(3s_1 + 2s_2 + s_3) + h(3s_1 + 2s_2 - s_3) + h(3s_1 - 2s_2 + s_3) + h(3s_1 - 2s_2 - s_3) \\
&\quad -12[h(s_1) + h(-s_1)] - 8[h(s_2) + h(-s_2)] - 2[h(s_3) + h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1 + 2s_2 + s_3)^2 \ln |3s_1 + 2s_2 + s_3| + (3s_1 + 2s_2 - s_3)^2 \ln |3s_1 + 2s_2 - s_3| \\
&\quad + (3s_1 - 2s_2 + s_3)^2 \ln |3s_1 - 2s_2 + s_3| + (3s_1 - 2s_2 - s_3)^2 \ln |3s_1 - 2s_2 - s_3| \\
&\quad - 12[s_1^2 \ln |s_1| + s_1^2 \ln |-s_1|] - 8[s_2^2 \ln |s_2| + s_2^2 \ln |-s_2|] \\
&\quad -2[s_3^2 \ln |s_3| + s_3^2 \ln |-s_3|] - 12 s_1^2 \ln |s_1|\big| \\
&= \big|(3j + 2k + l)^2 \ln |3j + 2k + l| + (3j + 2k - l)^2 \ln |3j + 2k - l| \\
&\quad + (3j - 2k + l)^2 \ln |3j - 2k + l| + (3j - 2k - l)^2 \ln |3j - 2k - l| \\
&\quad - 12[j^2 \ln |j| + j^2 \ln |-j|] - 8[k^2 \ln |k| + k^2 \ln |-k|] \\
&\quad -2[l^2 \ln |l| + l^2 \ln |-l|] - 12 j^2 \ln |j|\big| \\
&|h(3j + 2k + l) + h(3j + 2k - l) + h(3j - 2k + l) + h(3j - 2k - l) \\
&\quad -12[h(j) + h(-j)] - 8[h(k) + h(-k)] - 2[h(l) + h(-l)] - 12h(j)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \nu |j|^{\frac{t}{3}} \ |k|^{\frac{t}{3}} \ |l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \ |s_2|^{\frac{t}{3}} \ |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (ii): If $s_1, s_2, s_3 < 0$ in (4.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) \\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3| \\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3| \\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|] \\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big|
\end{aligned}
$$

If we set $s_1 = -j, s_2 = -k$, and $s_3 = -l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) \\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3| \\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3| \\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|] \\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big| \\
&= \big|(-3j-2k-l)^2\ln|-3j-2k-l|+(-3j-2k+l)^2\ln|-3j-2k+l| \\
&\quad +(-3j+2k-l)^2\ln|-3j+2k-l|+(-3j+2k+l)^2\ln|-3j+2k+l| \\
&\quad -12[j^2\ln|j|+j^2\ln|-j|]-8[k^2\ln|k|+k^2\ln|-k|] \\
&\quad -2[l^2\ln|l|+l^2\ln|-l|]-12j^2\ln|j|\big| \\
&|h(-3j-2k-l)+h(-3j-2k+l)+h(-3j+2k-l)+h(-3j+2k+l) \\
&\quad -12[h(-j)+h(j)]-8[h(-k)+h(k)]-2[h(-l)+h(l)]-12h(-j)| \\
&\leq \nu |j|^{\frac{t}{3}} \ |-k|^{\frac{t}{3}} \ |-l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \ |s_2|^{\frac{t}{3}} \ |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (iii): If $s_1 > 0, s_2, s_3 < 0$ in (4.19) then

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) \\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3| \\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3| \\
&\quad -12[s_1^2\ln|s_1|+s_1^2\ln|-s_1|]-8[s_2^2\ln|s_2|+s_2^2\ln|-s_2|] \\
&\quad -2[s_3^2\ln|s_3|+s_3^2\ln|-s_3|]-12s_1^2\ln|s_1|\big|
\end{aligned}
$$

If we set $s_1 = j, s_2 = -k$, and $s_3 = -l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3)+h(3s_1+2s_2-s_3)+h(3s_1-2s_2+s_3)+h(3s_1-2s_2-s_3) \\
&\quad -12[h(s_1)+h(-s_1)]-8[h(s_2)+h(-s_2)]-2[h(s_3)+h(-s_3)]-12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2\ln|3s_1+2s_2+s_3|+(3s_1+2s_2-s_3)^2\ln|3s_1+2s_2-s_3| \\
&\quad +(3s_1-2s_2+s_3)^2\ln|3s_1-2s_2+s_3|+(3s_1-2s_2-s_3)^2\ln|3s_1-2s_2-s_3|
\end{aligned}
$$

$$
\begin{aligned}
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-s_1|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big| \\
&= \big|(3j-2k-l)^2 \ln|3j-2k-l)| + (3j-2k+l)^2 \ln|j-2k+l| \\
&+ (3j+2k-l)^2 \ln|3j+2k-l| + (3j+2k+l)^2 \ln|3j+2k+l| \\
&- 12[j^2 \ln|j| + j^2 \ln|-j|] - 8[k^2 \ln|k| + k^2 \ln|-k|] \\
&-2[l^2 \ln|l| + l^2 \ln|-l|] - 12 j^2 \ln|j|\big| \\
&|h(3j-2k-l) + h(3j-2k+l) + h(3j+2k-l) + h(3j+2k+l) \\
&-12[h(j)+h(-j)] - 8[h(-k)+h(k)] - 2[h(-l)+h(l)] - 12h(j)| \\
&\leq \nu |j|^{\frac{t}{3}} \, |-k|^{\frac{t}{3}} \, |-l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \, |s_2|^{\frac{t}{3}} \, |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (iv): If $s_1 < 0, s_2, s_3 > 0$ in (4.19) then,

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2 \ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)^2 \ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)^2 \ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)^2 \ln|3s_1-2s_2-s_3| \\
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-s_1|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big|
\end{aligned}
$$

If we set $s_1 = -j, s_2 = k$, and $s_3 = l$, then we get

$$
\begin{aligned}
&|h(3s_1+2s_2+s_3) + h(3s_1+2s_2-s_3) + h(3s_1-2s_2+s_3) + h(3s_1-2s_2-s_3) \\
&-12[h(s_1)+h(-s_1)] - 8[h(s_2)+h(-s_2)] - 2[h(s_3)+h(-s_3)] - 12h(s_1)| \\
&= \big|(3s_1+2s_2+s_3)^2 \ln|3s_1+2s_2+s_3| + (3s_1+2s_2-s_3)^2 \ln|3s_1+2s_2-s_3| \\
&+ (3s_1-2s_2+s_3)^2 \ln|3s_1-2s_2+s_3| + (3s_1-2s_2-s_3)^2 \ln|3s_1-2s_2-s_3| \\
&- 12[s_1^2 \ln|s_1| + s_1^2 \ln|-x|] - 8[s_2^2 \ln|s_2| + s_2^2 \ln|-s_2|] \\
&-2[s_3^2 \ln|s_3| + s_3^2 \ln|-s_3|] - 12 s_1^2 \ln|s_1|\big| \\
&= \big|(-3j+2k+l)^2 \ln|-3j+2k+l| + (-3j+2k-l)^2 \ln|-3j+2k-l| \\
&+ (-3j-2k+l)^2 \ln|-3j-2k+l| + (-3j-2k-l)^2 \ln|-3j-2k-l| \\
&- 12[j^2 \ln|-j| + j^2 \ln|j|] - 8[k^2 \ln|k| + k^2 \ln|-k|] \\
&-2[l^2 \ln|l| + l^2 \ln|-l|] - 12 j^2 \ln|-j|\big| \\
&|h(-3j+2k+l) + h(-3j+2k-l) + h(-3j-2k+l) + h(-3j-2k-l) \\
&-12[h(-j)+h(j)] - 8[h(k)+h(-k)] - 2[h(l)+h(-l)] - 12h(-j)| \\
&\leq \nu |-j|^{\frac{t}{3}} \, |k|^{\frac{t}{3}} \, |l|^{\frac{2-2t}{3}} \\
&= \nu |s_1|^{\frac{t}{3}} \, |s_2|^{\frac{t}{3}} \, |s_3|^{\frac{2-2t}{3}}
\end{aligned}
$$

Case (v): If $s_1 = s_2 = s_3 = 0$ in (4.19), then the statement is obvious. ■

Here we present an example to show that, as mentioned in Corollary 4.2, the functional equation (1.7) is unstable for $t = \frac{2}{3}$.

Example 4.5. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \nu s_1^2, & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant; the function $h : \mathbb{R} \to \mathbb{R}$, defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{\xi(6^n s_1)}{6^{2n}}$$

for all $s_1 \in \mathbb{R}$, fulfills the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{15552}{7} \left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2\right) \tag{4.21}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. If this is the case, then there cannot be a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{B}(s_1)| \leq \kappa s_1^2 \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{4.22}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|\xi(6^n s_1)|}{|6^{2n}|} \leq \sum_{n=0}^{\infty} \frac{\nu}{6^{2n}} = \frac{36\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (4.15). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2\right)} \leq \frac{36\nu}{35} \times 60 \times 6^2 = \frac{15552}{7}\nu$$

and hence (4.15) is obvious. Take into account the scenario where

$$0 < |s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario, there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}} + |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{4.23}$$

This implies that $6^{m-1}x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} < \frac{1}{6}, 6^{m-1}x < \frac{1}{6}, 6^{m-1}y < \frac{1}{6}$ and $6^{m-1}z < \frac{1}{6}$. As a result,

$$\begin{aligned} &6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3), \\ &6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3) \end{aligned}$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned} &6^n(3s_1 + 2s_2 + s_3), 6^n(3s_1 - 2s_2 + s_3), 6^n(3s_1 + 2s_2 - s_3), 6^n(3s_1 - 2s_2 - s_3), \\ &6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3) \end{aligned}$$

are also within the range of $(-1, 1)$. Due to the fact that ξ is quadratic over this range, we may conclude that

$$\xi(6^n(3s_1 + 2s_2 + s_3)) + \xi(6^n(3s_1 + 2s_2 - s_3)) + \xi(6^n(3s_1 - 2s_2 + s_3))$$

$$+\,\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]$$
$$-\,2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))=0,$$

for $n=0,1,\dots,m-1$. Utilising (4.17) and the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1,s_2,s_3)|}{\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2\right)} &\le \sum_{n=m}^{\infty}\frac{1}{6^{2n}\left(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2\right)}\\ &\quad\Big|\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))\\ &\quad+\xi(6^n(3s_1-2s_2+s_3))+\xi(6^n(3s_1-2s_2-s_3))\\ &\quad-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\ &\quad-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))\Big|\\ &\le \sum_{k=0}^{\infty}\frac{60\nu}{36^k36^m(|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2)}\\ &\le \sum_{k=0}^{\infty}\frac{2160\nu}{36^k}=\frac{15552}{7}\nu.\end{aligned}$$

Consequently, for all $s_1,s_2,s_3\in\mathbb{R}$ with $0<|s_1|^{\frac{2}{3}}|s_2|^{\frac{2}{3}}|s_3|^{\frac{2}{3}}+|s_1|^2+|s_2|^2+|s_3|^2<\frac{1}{6^2}$, h fulfills (4.15). According to Corollary 4.2, the quadratic functional equation (1.7) is unstable at $t=\frac{2}{3}$. Let us assume, however, that there is a quadratic mapping $\mathcal{B}:\mathbb{R}\to\mathbb{R}$ obeying (3.16), where R is the set of all real numbers and $\kappa>0$. Since h is bounded and continuous for every $s_1\in\mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{B}$ is also bounded and continuous within the interval. $\mathcal{B}$ must have the form $\mathcal{B}(s_1)=cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)|\le(\kappa+|c|)\,s_1^2. \tag{4.24}$$

being true. However, by choosing a positive integer m with $m\nu>\kappa+|c|$, we can find $s_1\in\left(0,\frac{1}{6^{m-1}}\right)$ such that $6^ns_1\in(0,1)$ for all $n=0,1,\dots,m-1$. For this s_1, we get

$$h(s_1)=\sum_{n=0}^{\infty}\frac{\xi(6^ns_1)}{6^{2n}}\ge\sum_{n=0}^{m-1}\frac{\nu\times6^{2n}s_1^2}{6^{2n}}=m\nu s_1^2>(\kappa+|c|)\,s_1^2$$

which defies (4.18). Based on the inequality (4.13), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t=\frac{2}{3}$. ■

5. Stability of (1.7) for mixed mappings

In this section, we will examine the generalised Hyers-Ulam stability of the functional equation (1.7) in the case where the mapping is a mixture of odd and even mappings.

Theorem 5.1. *Let $s=\pm1$ and $\xi:\mathcal{K}^3\to[0,\infty)$ be a mapping such that*

$$\sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{si}}<\infty \text{ and } \sum_{i=0}^{\infty}\frac{\xi\left(6^{si}s_1,6^{si}s_2,6^{si}s_3\right)}{6^{2si}}<\infty \tag{5.1}$$

for all $s_1,s_2,s_3\in\mathcal{K}$. Let $h:\mathcal{K}\to\mathcal{L}$ be a mapping satisfying the inequality

$$\|Dh(s_1,s_2,s_3)\|\le\xi\left(s_1,s_2,s_3\right) \tag{5.2}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then, a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ and a unique quadratic mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exist with

$$\|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \left\{ \frac{1}{6^{(si+1)}} + \frac{1}{6^{2(si+1)}} \right\} \left[\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right) \right]$$

where $\psi(s_1), \mathcal{A}(s_1)$ and $\mathcal{B}(s_1)$ are defined as in (3.4), (3.5) *and* (4.5) *for all $s_1 \in \mathcal{K}$, respectively.*

Proof. Define a mapping $h_a : \mathcal{K} \to H$ by

$$h_a(s_1) = \frac{1}{2}[h(s_1) - h(-s_1)] \tag{5.3}$$

for all $s_1 \in \mathcal{K}$. Then $h_a(0) = 0$ and $h_a(-s_1) = -h_a(s_1)$ for all $s_1 \in \mathcal{K}$. Hence

$$\|Dh_a(s_1, s_2, s_3)\| \leq \frac{1}{2} \left[\xi\left(s_1, s_2, s_3\right) + \xi\left(-x, -s_2, -s_3\right) \right] \tag{5.4}$$

for all $s_1 \in \mathcal{K}$. By Theorem 3.1, we have

$$\|h_a(s_1) - \mathcal{A}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right)}{6^{(si+1)}} \tag{5.5}$$

where $\psi(s_1)$ and $\mathcal{A}(s_1)$ are defined as in (3.4) and (3.5) for all $s_1 \in \mathcal{K}$, respectively. Also, define a mapping $h_q : \mathcal{K} \to H$ by

$$h_q(s_1) = \frac{1}{2}[h(s_1) + h(-s_1)] \tag{5.6}$$

for all $s_1 \in \mathcal{K}$. Then $h_q(0) = 0$ and $h_q(-s_1) = h_q(s_1)$ for all $s_1 \in \mathcal{K}$. Hence

$$\|Dh_q(s_1, s_2, s_3)\| \leq \frac{1}{2} \left[\xi\left(s_1, s_2, s_3\right) + \xi\left(-s_1, -s_2, -s_3\right) \right] \tag{5.7}$$

for all $s_1 \in \mathcal{K}$. By Theorem 4.1, we have

$$\|h_q(s_1) - \mathcal{B}(s_1)\| \leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \frac{\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right)}{6^{2(si+1)}} \tag{5.8}$$

where $\psi : \mathcal{K} \to \mathcal{L}$ and $\mathcal{B}(s_1)$ are defined as in (3.4) and (4.5) for all $s_1 \in \mathcal{K}$, respectively. From (5.3) and (5.5), we have

$$h(s_1) = h_a(s_1) + h_q(s_1) \tag{5.9}$$

for all $s_1 \in \mathcal{K}$. Using (5.5), (5.8) and (5.9), we get

$$\begin{aligned}
\|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| &= \|h_a(s_1) + h_q(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \\
&\leq \|h_a(s_1) - \mathcal{A}(s_1)\| + \|h_q(s_1) - \mathcal{B}(s_1)\| \\
&\leq \frac{1}{2} \sum_{i=\frac{1-s}{2}}^{\infty} \left\{ \frac{1}{6^{(si+1)}} + \frac{1}{6^{2(si+1)}} \right\} \left[\psi\left(6^{si} s_1\right) + \psi\left(-6^{si} s_1\right) \right]
\end{aligned}$$

where $\psi(s_1), \mathcal{A}(s_1)$ and $\mathcal{B}(s_1)$ are defined as in (3.4), (3.5) and (4.5) for all $s_1 \in \mathcal{K}$, respectively. ■

The Corollary that follows is a direct result of Theorem 4.1, which pertains to the stability of equation (1.7).

Corollary 5.2. *Assume that t be a positive real value and $\nu \geq 0$. Let $h : \mathcal{K} \to \mathcal{L}$ be a function fulfills the inequality*

$$\|Dh(s_1,s_2,s_3)\| \leq \begin{cases} \nu, \\ \nu\left\{||s_1||^t + ||s_2||^t + ||s_3||^t\right\}, & t \neq 1,2; \\ \nu||s_1||^t||s_2||^t||s_3||^t, & 3t \neq 1,2; \\ \nu\left\{||s_1||^t||s_2||^t||s_3||^t + \left\{||s_1||^{3t} + ||s_2||^{3t} + ||s_3||^{3t}\right\}\right\}, & 3t \neq 1,2; \end{cases} \tag{5.10}$$

for all $s_1, s_2, s_3 \in \mathcal{K}$. Then, a unique additive mapping $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ and a unique quadratic mapping $\mathcal{B} : \mathcal{K} \to \mathcal{L}$ exist with

$$\|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)\| \leq \begin{cases} 3\nu\left(\dfrac{1}{10} + \dfrac{1}{70}\right), \\ 4\nu||s_1||^t\left(\dfrac{1}{|6^t - 6|} + \dfrac{1}{|6^t - 6^2|}\right), & t \neq 1,2; \\ \nu||s_1||^{3t}\left(\dfrac{1}{|6^{3t} - 6|} + \dfrac{1}{|6^{3t} - 6^2|}\right), & 3t \neq 1,2; \\ 5\nu||s_1||^{3t}\left(\dfrac{1}{|6^{3t} - 6|} + \dfrac{1}{|6^{3t} - 6^2|}\right), & 3t \neq 1,2; \end{cases} \tag{5.11}$$

for all $s_1 \in \mathcal{K}$.

To demonstrate that (1.7) is not stable at $t = 1$, as stated in Corollary 4.2, we will now present an illustration.

Example 5.3. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \dfrac{\nu}{2}(s_1 + s_1^2), & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^n + 1)}{6^{2n}} \xi(6^n s_1)$$

for all $s_1 \in \mathbb{R}$, satisfies the functional inequality

$$|Dh(s_1,s_2,s_3)| \leq \frac{936 \times 6\nu}{7}(|s_1| + |s_2| + |s_3|) \tag{5.12}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. Then there is no existence of an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)| \leq \kappa(|s_1| + |s_1|^2) \quad \text{for all} \quad s_1 \in \mathbb{R}. \tag{5.13}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq \nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|6^n + 1|}{|6^{2n}|}|\xi(6^n s_1)| \leq \sum_{n=0}^{\infty} \frac{\nu(6^n + 1)}{6^{2n}} = \frac{78\nu}{35}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (5.12). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1| + |s_2| + |s_3| \geq \dfrac{1}{6}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1,s_2,s_3)|}{(|s_1| + |s_2| + |s_3|)} \leq \frac{78\nu}{35} \times 60 \times 6 = \frac{936 \times 6\nu}{7}$$

and hence (5.12) is obvious. Take into account the scenario where

$$0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{(m+1)}} \leq |s_1| + |s_2| + |s_3| < \frac{1}{6^m}. \tag{5.14}$$

This implies that $6^{m-1}s_1 < \frac{1}{6}, 6^{m-1}s_2 < \frac{1}{6}$, and $6^{m-1}s_3 < \frac{1}{6}$. As a result,

$$\begin{aligned}&6^{m-1}(3s_1+2s_2+s_3), 6^{m-1}(3s_1-2s_2+s_3), 6^{m-1}(3s_1+2s_2-s_3), 6^{m-1}(3s_1-2s_2-s_3),\\ &6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)\end{aligned}$$

are all within the range of $(-1, 1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned}&6^n(3s_1+2s_2+s_3), 6^n(3s_1-2s_2+s_3), 6^n(3s_1+2s_2-s_3), 6^n(3s_1-2s_2-s_3),\\ &6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)\end{aligned}$$

Due to the fact that ξ is additive-quadratic over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1+2s_2+s_3)) + \xi(6^n(3s_1+2s_2-s_3)) + \xi(6^n(3s_1-2s_2+s_3))\\ &+\xi(6^n(3s_1-2s_2-s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))] - 8[\xi(6^n(s_2)) + \xi(6^n(-s_2))]\\ &-2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1)) = 0,\end{aligned}$$

for $n = 0, 1, \ldots, m-1$. Utilising (5.14) and from the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1,s_2,s_3)|}{(|s_1|+|s_2|+|s_3|)} &\leq \sum_{n=m}^{\infty} \frac{(6^n+1)}{6^{2n}(|s_1|+|s_2|+|s_3|)}\Big|\xi(6^n(3s_1+2s_2+s_3)) + \xi(6^n(3s_1+2s_2-s_3))\\ &+\xi(6^n(3s_1-2s_2+s_3)) + \xi(6^n(3s_1-2s_2-s_3)) - 12[\xi(6^n(s_1)) + \xi(6^n(-s_1))]\\ &-8[\xi(6^n(s_2)) + \xi(6^n(-s_2))] - 2[\xi(6^n(s_3)) + \xi(6^n(-s_3))] - 12\xi(6^n(s_1))\Big|\\ &\leq \sum_{n=m}^{\infty} \frac{(6^n+1)60\nu}{6^{2n}(|s_1|+|s_2|+|s_3|)} \leq \frac{78\nu}{35} \times \frac{60}{6^m(|s_1|+|s_2|+|s_3|)} \leq \frac{936 \times 6\nu}{7}.\end{aligned}$$

Consequently, for all $s_1, s_2, s_3 \in \mathbb{R}$ with $0 < |s_1| + |s_2| + |s_3| < \frac{1}{6}$, h fulfills (5.12). According to Corollary 5.2, the the additive-quadratic functional equation (1.7) is unstable at $t = 1$. Let us assume, however, that there exist an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ that satisfies (5.13). Since h is continuous and bounded for every $s_1 \in \mathbb{R}$, when s_1 is in an open interval containing the origin, $\mathcal{A}$ and $\mathcal{B}$ are also continuous and bounded within the interval. $\mathcal{A}$ must have the form $\mathcal{A}(s_1) = cs_1$ and $\mathcal{B}$ must have the form $\mathcal{B}(s_1) = cs_1^2$ for any s_1 in $\mathbb{R}$, according to Theorem 2.1. This leads to

$$|h(s_1)| \leq (\kappa + |c|)\,(|s_1| + |s_1|^2). \tag{5.15}$$

being true. However, by choosing a positive integer m with $m\nu > \kappa + |c|$, we can find $s_1 \in \left(0, \frac{1}{6^{m-1}}\right)$ such that $6^n s_1 \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this s_1, we get

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^n+1)}{6^{2n}}\xi(6^n s_1) \geq \sum_{n=0}^{m-1} \frac{\nu(6^n+1)}{6^{2n}}(6^n s_1 + 6^{2n} s_1^2)$$

$$= \sum_{n=0}^{m-1} \frac{\nu(6^n+1)}{6^n}(s_1 + 6^n s_1^2) \geq \sum_{n=0}^{m-1} \nu(s_1 + s_1^2)$$
$$= m\nu(s_1 + s_1^2) > (\kappa + |c|)\,(s_1 + s_1^2)$$

which defies (5.15). Based on the inequality (5.10), it may be concluded that the equation (1.7) is unstable in the Hyers-Ulam-Rassias sense while $t = 1$. ■

Here we present an example to show that, as mentioned in Corollary 5.2, the functional equation (1.7) is unstable for $t = 2$

Example 5.4. *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\xi(s_1) = \begin{cases} \frac{\nu}{2}(s_1 + s_1^2), & \text{if } |s_1| < 1 \\ \nu, & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a constant, and the function $h : \mathbb{R} \to \mathbb{R}$, which is defined as

$$h(s_1) = \sum_{n=0}^{\infty} \frac{(6^{2n}+1)}{6^{4n}} \xi(6^{2n} s_1)$$

for all $s_1 \in \mathbb{R}$, satisfies the functional inequality

$$|Dh(s_1, s_2, s_3)| \leq \frac{31536 \times 6^2 \nu}{259} \left(|s_1|^2 + |s_2|^2 + |s_3|^2\right) \tag{5.16}$$

for all $s_1, s_2, s_3 \in \mathbb{R}$. Then there is no existence of an additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, a quadratic mapping $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(s_1) - \mathcal{A}(s_1) - \mathcal{B}(s_1)| \leq \kappa(|s_1| + |s_1|^2) \quad \textit{for all} \quad s_1 \in \mathbb{R}. \tag{5.17}$$

Proof. It is clear that ξ is a continuous function and $|\xi(s_1)| \leq 2\nu$ for all $s_1 \in \mathbb{R}$. Now

$$|h(s_1)| \leq \sum_{n=0}^{\infty} \frac{|6^{2n}+1|}{|6^{4n}|} |\xi(6^{2n} s_1)| \leq \sum_{n=0}^{\infty} \frac{\nu(6^{2n}+1)}{6^{4n}} = \frac{2628\nu}{1295}.$$

As a result, it becomes clear that h is bounded. Here, we'll show that h does, in fact, satisfy (5.16). If $s_1 = s_2 = s_3 = 0$, or $s_1, s_2, s_3 \in \mathbb{R}$ such that $|s_1|^2 + |s_2|^2 + |s_3|^2 \geq \frac{1}{6^2}$, then merely by virtue of the boundedness of h we have

$$\frac{|Dh(s_1, s_2, s_3)|}{(|s_1|^2 + |s_2|^2 + |s_3|^2)} \leq \frac{2628\nu}{1295} \times 60 \times 6^2 = \frac{31536 \times 6^2 \nu}{259}$$

and hence (5.16) is obvious. Take into account the scenario where

$$0 < |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^2}.$$

In the above scenario , there is a positive whole number m such that

$$\frac{1}{6^{2(m+1)}} \leq |s_1|^2 + |s_2|^2 + |s_3|^2 < \frac{1}{6^{2m}}. \tag{5.18}$$

This implies that $6^{m-1} s_1 < \frac{1}{6}$, $6^{m-1} s_2 < \frac{1}{6}$, and $6^{m-1} s_3 < \frac{1}{6}$. As a result,

$$6^{m-1}(3s_1 + 2s_2 + s_3), 6^{m-1}(3s_1 - 2s_2 + s_3), 6^{m-1}(3s_1 + 2s_2 - s_3), 6^{m-1}(3s_1 - 2s_2 - s_3),$$

$$6^{m-1}(s_1), 6^{m-1}(-s_1), 6^{m-1}(s_2), 6^{m-1}(-s_2), 6^{m-1}(s_3), 6^{m-1}(-s_3)$$

are all within the range of $(-1,1)$. Therefore, for every whole number n that ranges from 0 to $m-1$, the values of

$$\begin{aligned}&6^n(3s_1+2s_2+s_3), 6^n(3s_1-2s_2+s_3), 6^n(3s_1+2s_2-s_3), 6^n(3s_1-2s_2-s_3),\\&6^n(s_1), 6^n(-s_1), 6^n(s_2), 6^n(-s_2), 6^n(s_3), 6^n(-s_3)\end{aligned}$$

are also within the range of $(-1,1)$. Due to the fact that ξ is additive-quadratic over this range, we may conclude that

$$\begin{aligned}&\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))+\xi(6^n(3s_1-2s_2+s_3))\\&+\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]\\&-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))=0,\end{aligned}$$

for $n=0,1,\ldots,m-1$. Utilising (5.18) and from the definition of h, we may calculate

$$\begin{aligned}\frac{|Dh(s_1,s_2,s_3)|}{(|s_1|^2+|s_2|^2+|s_3|^2)} &\leq \sum_{n=m}^{\infty}\frac{(6^{2n}+1)}{6^{4n}(|s_1|^2+|s_2|^2+|s_3|^2)}\Big|\xi(6^n(3s_1+2s_2+s_3))+\xi(6^n(3s_1+2s_2-s_3))\\&\quad+\xi(6^n(3s_1-2s_2+s_3))+\xi(6^n(3s_1-2s_2-s_3))-12[\xi(6^n(s_1))+\xi(6^n(-s_1))]\\&\quad-8[\xi(6^n(s_2))+\xi(6^n(-s_2))]-2[\xi(6^n(s_3))+\xi(6^n(-s_3))]-12\xi(6^n(s_1))\Big|\\&\leq \sum_{n=m}^{\infty}\frac{(6^{2n}+1)}{6^{4n}}\times\frac{60\nu}{(|s_1|^2+|s_2|^2+|s_3|^2)}\\&\leq \frac{2628\nu}{1295}\times\frac{60}{6^{2m}(|s_1|^2+|s_2|^2+|s_3|^2)}=\frac{31536\times 6^2\nu}{259}.\end{aligned}$$

Thus h satisfies (5.16) with $0<|s_1|^2+|s_2|^2+|s_3|^2<\frac{1}{6^2}$ for all $s_1,s_2,s_3\in\mathbb{R}$. We assert that the additive-quadratic functional equation (1.7) is not stable when $t=2$ as stated in Corollary 5.2. To contradict this, let's assume that there exists an additive mapping $\mathcal{A}:\mathbb{R}\to\mathbb{R}$, a quadratic mapping $\mathcal{B}:\mathbb{R}\to\mathbb{R}$ and a constant $\kappa>0$ that satisfies (5.17). Since h is bounded and continuous for all $s_1\in\mathbb{R}$, $\mathcal{A}$ and $\mathcal{B}$are bounded within a range and continuous at the origin when s_1 is in an open interval containing the origin. In light of Theorem 5.1, $\mathcal{A}$ must have the form $\mathcal{A}(s_1)=cs_1$ and $\mathcal{B}(s_1)=cs_1^2$ for any s_1 in $\mathbb{R}$. This leads to

$$|h(s_1)|\leq(\kappa+|c|)\,(|s_1|+|s_1|^2). \tag{5.19}$$

being true. However, by choosing a positive integer m with $m\nu>\kappa+|c|$, we can find $s_1\in\left(0,\frac{1}{6^{m-1}}\right)$ such that $6^ns_1\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this s_1, we get

$$\begin{aligned}h(s_1)&=\sum_{n=0}^{\infty}\frac{(6^{2n}+1)}{6^{4n}}\xi(6^{2n}s_1)\geq\sum_{n=0}^{m-1}\frac{\nu(6^{2n}+1)}{6^{4n}}(6^{2n}s_1+6^{4n}s_1^2)\\&=\sum_{n=0}^{m-1}\frac{\nu(6^{2n}+1)}{6^{2n}}(s_1+6^{2n}s_1^2)\geq\sum_{n=0}^{m-1}\nu(s_1+s_1^2)\\&=m\nu(s_1+s_1^2)>(\kappa+|c|)\,(s_1+s_1^2)\end{aligned}$$

which contradicts (5.19). Therefore, the additive-quadratic functional equation (1.7) is not stable in the sense of Hyers-Ulam-Rassias when $t=1$, as stated in the inequality (5.10). ■

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