Graceful and felicitous labeling of Stem-Lotus graph

A. ELUMALAI[*1]

[1] *Department of Mathematics, R.M.K Engineering College (An Autonomous Institution), Gummidipoondi-601206, Tamil Nadu, India.*

Abstract. A graph obtained from a shell graph $C(2n + 3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. In this paper, I have proved that a Stem-Lotus graph is graceful and also felicitous.

AMS Subject Classifications: 05C78, 05C38.

Keywords: Graceful labeling, Felicitous labeling, Stem-Lotus graph.

Contents

1. Introduction

We begin with simple, finite, undirected and connected graph $G = (p, q)$. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \ldots, q\}$ is called graceful, if the edge labels induced by $|f(x) - f(y)|$ for each edge xy are distinct. A graph which admits graceful labeling is called graceful graph. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \ldots, q\}$ is called felicitous, if the edge labels induced by $(f(x) + f(y))(\mod q)$ for each edge xy are distinct. A graph which admits felicitous labeling is called felicitous graph.

From the excellent survey of Gallian Dec 2, 2022, one can find many families of cycle related graphs, on which Shell graph is an important family. Deb & Limaye (2002) have defined a shell graph as a cycle C_n with $(n - 3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n - 3)$. In this paper, a new family of graph called a Stem-Lotus graph is introduced by adding a vertex in between each pair of adjacent vertices on the cycle of a shell graph. Here I have proved that Stem-Lotus graph is graceful and felicitous.

A graph obtained from a shell graph $C(2n + 3, 2n)$ where $n \geq 1$ by adding a vertex in between each pair of adjacent vertices on the cycle, adding an edge in apex and two more chords is called a Stem-Lotus graph. So clearly a Stem-Lotus graph must have a pollen grain (C_4), a stem (P_1), two leafs (each P_3) and some pairs of petals (each petal is P_3) as displayed in Figure 1.

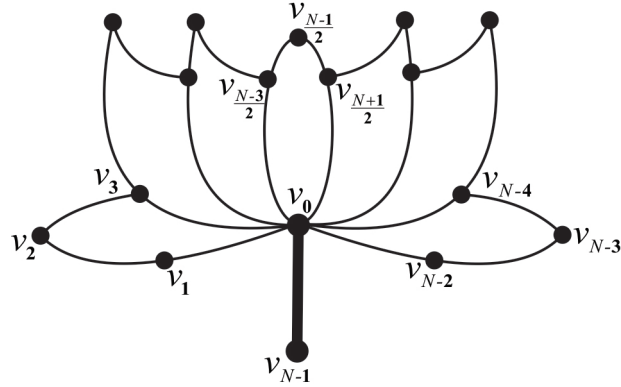*Corresponding author. Email address: **aeseven2000@gmail.com** (A. Elumalai)

Figure 1: Stem-Lotus graph

In this direction, I have proved that Stem-Lotus graph with n-pairs of petals is graceful and felicitous for all $n \geq 1$.

2. Stem-Lotus graph with n-pairs of petals is graceful and felicitous

In this section, I have proved that Stem-Lotus graph with n-pairs of petals is graceful and felicitous for all $n \geq 1$.

Theorem 2.1. *A Stem-Lotus graph with n-pairs of petals is graceful for all $n \geq 1$.*

Proof. Let G be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally $M = 6n + 11$ edges and $N = 4n + 9$ vertices for the graph G. Let $v_0, v_1, v_2, \ldots, v_{N-1}$ be the N vertices of G.

Define the vertex labeling of G as follows,

$$f(v_0) = 0$$
$$f(v_{N-1}) = \frac{3N - 5}{2}$$
$$f(v_{2i}) = \frac{N - 3 + 2i}{2}, 1 \leq i \leq \frac{N - 3}{2}$$
$$f(v_{2i-1}) = \frac{3N - 5 - 2i}{2}, 1 \leq i \leq \frac{N - 1}{2}$$

From the above vertex labeling, the set $\left\{ f(v_{2i})/1 \leq i \leq \frac{N-3}{2} \right\}$ form a monotonically increasing sequence and the sets $\{f(v_{N-1})\}$ and $\left\{ f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2} \right\}$ form a monotonically decreasing sequence. It is observed that,

$$\max \left\{ f(v_{2i})/1 \leq i \leq \frac{N-3}{2} \right\} < \min \left\{ \{f(v_{N-1})\} \cup \left\{ f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2} \right\} \right\}.$$

Therefore the labels of all vertices of G are distinct.

Now the edge values of G in the following manner,

- Let A_1 denote the edge $\{v_0 v_{N-1}\}$ of G.

- Let A_2 denote the set of $\frac{M+1}{3}$ edges $\{v_0 v_1, v_0 v_3, v_0 v_5, \ldots, v_0 v_{N-4}, v_0 v_{N-2}\}$ of G.

- Let A_3 denote the set of $\frac{2M-4}{3}$ edges $\{v_1 v_2, v_2 v_3, v_3 v_4, \ldots, v_{N-4} v_{N-3}, v_{N-3} v_{N-2}\}$ of G.

Let A_1', A_2' & A_3' be the sets of observed edge values of the sets A_1, A_2 & A_3.

$$A_1' = \{M\}$$

$$A_2' = \left\{M - 1, M - 2, \ldots, \frac{2M - 1}{3}\right\}$$

$$A_3' = \left\{\frac{2M - 4}{3}, \frac{2M - 7}{3}, \ldots, 3, 2, 1\right\}$$

It is observed that, $A_1' \cup A_2' \cup A_3' = \{M, M - 1, \ldots, 3, 2, 1\}$ and the values in the sets A_1', A_2' & A_3' are all distinct, hence G is graceful. ∎

Theorem 2.2. *A Stem-Lotus graph with n-pairs of petals is felicitous for all $n \geq 1$.*

Proof. Let G be a Stem-Lotus graph as displayed in Figure 1. It is observed that, there are totally $M = 6n + 11$ edges and $N = 4n + 9$ vertices for the graph G. Let $v_0, v_1, v_2, \ldots, v_{N-1}$ be the N vertices of G. Define the vertex labeling of G as follows,

$$f(v_0) = 0$$

$$f(v_{N-1}) = N - 2$$

$$f(v_{2i}) = \frac{N - 3 + 2i}{2}, 1 \leq i \leq \frac{N - 3}{2}$$

$$f(v_{2i-1}) = N - 2 + i, 1 \leq i \leq \frac{N - 1}{2}$$

From the above vertex labeling, the set $\left\{f(v_{2i})/1 \leq i \leq \frac{N-3}{2}\right\}$ form a monotonically increasing sequence and the sets $\{f(v_{N-1})\}$ and $\left\{f(v_{2i-1})/1 \leq i \leq \frac{N-1}{2}\right\}$ form a monotonically decreasing sequence.

It is observed that,

$$\max\left\{f(v_{2i})/1 \leq i \leq \frac{N - 3}{2}\right\} < \min\left\{\{f(v_{N-1})\} \cup \left\{f(v_{2i-1})/1 \leq i \leq \frac{N - 1}{2}\right\}\right\}.$$

Therefore the labels of all vertices of G are distinct.

Now the edge values of G in the following manner,

- Let A_1 denote the edge $\{v_0v_{N-1}\}$ of G.

- Let A_2 denote the set of $\frac{M+1}{3}$ edges $\{v_0v_1, v_0v_3, v_0v_5, \ldots, v_0v_{N-4}, v_0v_{N-2}\}$ of G.

- Let A_3 denote the set of $\frac{2M-4}{3}$ edges $\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2}\}$ of G.

Let A_1', A_2' & A_3' be the sets of observed edge values of the sets A_1, A_2 & A_3.

$$A_1' = \left\{\frac{2M - 1}{3}\right\}$$

$$A_2' = \left\{\frac{2M + 2}{3}, \frac{2M + 5}{3}, \ldots, M - 1, 0\right\}$$

$$A_3' = \left\{1, 2, 3, \ldots, \frac{2M - 7}{3}, \frac{2M - 4}{3}\right\}$$

It is observed that, $A_1' \cup A_2' \cup A_3' = \{M - 1, M - 2, \ldots, 2, 1, 0\}$ and the values in the sets A_1', A_2' & A_3' are all distinct, hence G is felicitous. ∎

Illustrative example of the labeling given in the proof of Theorem 2.1 is displayed in Figure 2.
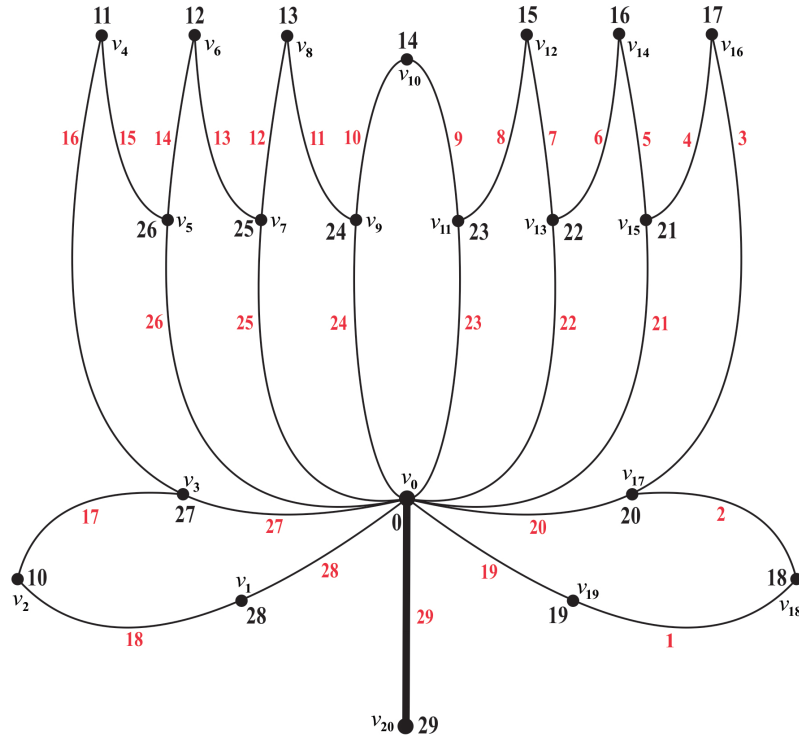
Figure 2: Graceful labeling of Stem-Lotus graph with 3-pairs of petals

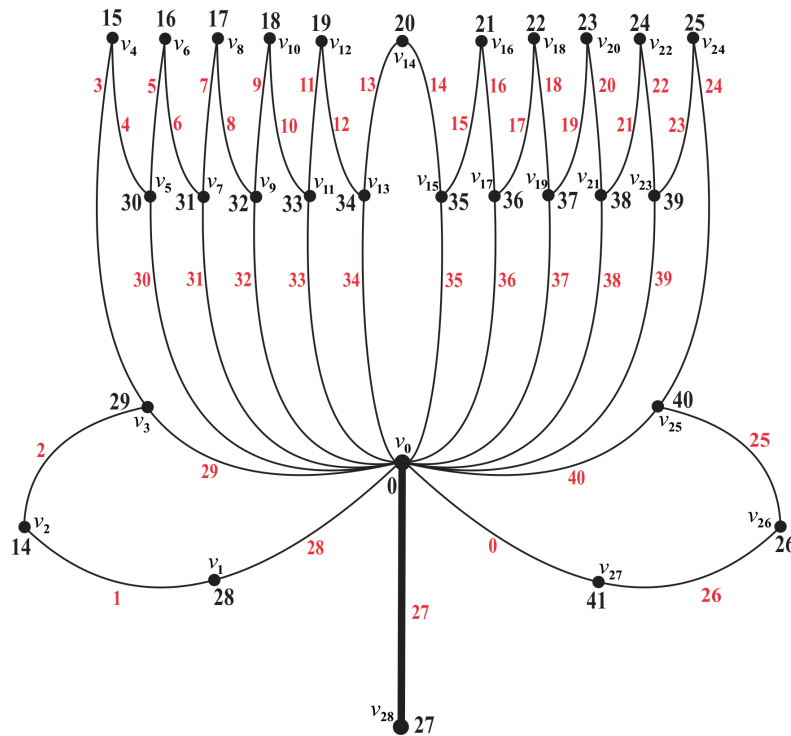Illustrative example of the labeling given in the proof of Theorem 2.2 is displayed in Figure 3.

Figure 3: Felicitous labeling of Stem-Lotus graph with 5-pairs of petals

References

[1] P. DEB AND N.B. LIMAYE, On elegant labeling of triangular snakes, *Journal of Combinatorics, Information and System Sciences*, **25**(2000), pp. 163–172.

[2] P. DEB AND N.B. LIMAYE, On Harmonious Labelling of some cycle related graphs, *Ars Combinatoria*, **65**(2002), pp. 177–197.

[3] C. DELORME, M. MAHEO, H. THUILIER, K.M. KOH AND H.K. TEO, Cycles with a chord are graceful, *Journal of Graph Theory*, **4**(1980), pp. 409–415.

[4] A. ELUMALAI AND G. SETHURAMAN, Gracefulness of a cycle with parallel P_k-chords, *Australasian Journal of Combinatorics*, **32**(2005), pp. 205–211.

[5] A. ELUMALAI AND G. SETHURAMAN, Cordialness of cycles with parallel P_k-chords and multiple subdivision graphs, *Ars Combinatoria*, **85**(2007), pp. 85–98.

[6] A. ELUMALAI AND G. SETHURAMAN, Gracefulness of union of cycle with parallel chords and complete bipartite graphs or paths, *Journal of Discrete Mathematical Sciences and Cryptography*, **12**(2009), pp. 245–252.

[7] A. ELUMALAI AND G. SETHURAMAN, Gracefulness of a cycle with parallel chords and parallel P_k-chords of different lengths, *Ars Combinatoria*, **104**(2012), pp. 143–148.

[8] R. FRUCHT, Graceful numbering of wheels and related graphs, *Annals of the New York Academy of Sciences*, **1**(1979), pp. 219–229.

[9] J.A. GALLIAN, Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, Dec. 2, 2022.