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On existence of extremal integrable solutions and integral inequalities for nonlinear Volterra type integral equations

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Abstract. We prove the existence of maximal and minimal integrable solutions of nonlinear Volterra type integral equations. Two basic integral inequalities are obtained in the form of extremal integrable solutions which are further exploited for proving the boundedness and uniqueness of the integrable solutions of the considered integral equation.
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Contents

1. Introduction

The integral inequalities is an important topic discussed in the theory of differential and integral equations because they have nice applications for proving the boundedness and uniqueness of the solutions of such nonlinear equations. There exists a good amount of literature on integral inequalities and applications, see for example, Lakshmikantham and Leela [14] and references therein. The existence of maximal and minimal solutions of the integral equations play a significant role in the theory of integral inequalities and applications related to the integral equations. The most of the integral inequalities are about the continuous solutions of the continuous integral equations, but the study for integrable solutions for discontinuous integral equations is very rare. Therefore, it is interesting to prove some basic integral inequalities related to Volterra integral equations involving integrable solutions which is the main motivation of the present paper. The present paper deals with the extremal integrable solutions and integral inequalities involving integrable solutions related to nonlinear discontinuous Volterra type integral equations.

Given a closed and bounded interval $J = [0, T]$ in the real line $\mathbb{R}$, consider the nonlinear Volterra type integral equation (in short VIE),

$$x(t) = q(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds, \ \ t \in J, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$, the functions $q : J \to \mathbb{R}$ and $f : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain integrability and Chandrabhan type conditions to be specified later.

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Definition 1.1. *By an integrable solution of the VIE (1.1) we mean a function $x \in L^1(J,\mathbb{R})$ that satisfies the equation* (1.1) *on J, where $L^1(J,\mathbb{R})$ is the space of Lebesgue integrable real-valued functions defined on J.*

The VIE (1.1) is quite known and discussed sufficiently in the literature for different aspects of the solutions. The existence of an integrable solution has been discussed in Emmanule [11] and Banas [1] via measure of weak noncompactness whereas existence is discussed in Banas and El-Sayed [2] via Schauder fixed point principle. However, to the best of authors knowledge, no result is so far proved for maximal and minimal integrable solutions. Here we prove the existence of the extremal integrable solutions and integral inequalities for the VIE (1.1) under certain monotonicity condition along with applications. In the following section we present some preliminaries and notations needed in what follows.

2. Preliminaries

We place the problem of VIE (1.1) in the function space $L^1(J,\mathbb{R})$ of Lebesgue integrable real-valued functions defined on J. We define a standard norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt. \tag{2.1}$$

Clearly, $L^1(J,\mathbb{R})$ becomes a Banach space w.r.t. the norm $\|\cdot\|_{L^1}$ defined above. Next, we introduce an order relation $\preceq$ in $L^1(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in L^1(J,\mathbb{R}) \mid x(t) \geq 0 \text{ a.e. } t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K,$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \text{ a.e. } t \in J. \tag{2.3}$$

Lemma 2.1. *The partially ordered set $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a Banach lattice.*

Proof. Let K be a order cone in $L^1(J,\mathbb{R})$ and let $x, y \in K$ be such that $x \preceq y$, Then, we have

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt = \int_0^T x(t)\,dt \leq \int_0^T y(t)\,dt = \int_0^T |y(t)\,dt = \|y\|_{L^1}.$$

Hence $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a Banach lattice. □

Lemma 2.2. *The partially ordered set $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice.*

Proof. To finish, it is enough to show that $\big(L^1(J,\mathbb{R}),\preceq\big)$ is an abstract L-space. Let $x, y \in \big(L^1(J,\mathbb{R}),\preceq\big)$ be such that $x \succeq 0$ and $y \succeq 0$. Then by definition of the norm $\|\cdot\|_{L^1}$, we have

$$\begin{aligned}\|x+y\|_{L^1} &= \int_0^T |x(t)+y(t)|\,dt \\ &= \int_0^T \big[x(t)+y(t)\big]\,dt \\ &= \int_0^T x(t)\,dt + \int_0^T y(t)\,dt \\ &= \int_0^T |x(t)|\,dt + \int_0^T |y(t)|\,dt \\ &= \|x\|_{L^1} + \|y\|_{L^1}.\end{aligned} \tag{2.4}$$

This shows that $\big(L^1(J,\mathbb{R}),\preceq\big)$ is an abstract L-space. Hence by a theorem of uniformly monotone Banach lattice (Birkhoff [3, page 373]), $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice. □

As a consequence of Lemmas 2.1 and 2.2, we obtain the following useful result. See also Dhage [7] and Dhage and Patil [10] and references therein.

Lemma 2.3 (Birkhoff [3]). *A non-empty closed and bounded subset of the complete Banach lattice $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice.*

Now, the basic tool used in this paper is the algebraic fixed point theorem of Tarski [15]. Before stating this result, we mention a useful concept of isotone mapping on a lattice L into itself.

Definition 2.4. *A mapping on a lattice $(L,\preceq)$ is called isotone increasing if preserve the order relation $\preceq$, that is, if $x,y\in L$ with $x\preceq y$, then $\mathcal{T}x\preceq\mathcal{T}y$.*

Theorem 2.5 (Tarski [15]). *Let $(L,\preceq)$ be a partially ordered set and let $T:L\to L$ be a mapping. Suppose that*

(a) $\mathcal{T}$ is isotone increasing,

(b) $(L,\preceq)$ is a complete lattice, and

(c) $F_{\mathcal{T}}=\{u\in L\mid \mathcal{T}u=u\}$.

Then $F_{\mathcal{T}}\neq\emptyset$ and $\big(F_{\mathcal{T}},\preceq\big)$ is a complete lattice.

In the following section we prove the main results of this paper under suitable conditions.

3. Existence of Extremal Integrable Solutions

We consider the following definitions appeared in Dhage [6, 7] whcich is useful for dealing with the discontinuous differential and integral equations.

Definition 3.1. *A function $f:J\times\mathbb{R}\to\mathbb{R}$ is said to be* ***Chandrabhan*** *if*

(i) the map $t\mapsto f(t,x)$ is measurable for each $x\in\mathbb{R}$, and

(ii) the map $x\mapsto f(t,x)$ is nondecreasing for almost every $t\in J$.

*Furthermore, a Chandrabhan function $f(t,x)$ is called $L^1_{\mathbb{R}}$-**Chandrabhan** if*

(iii) there exists a function $h\in L^1(J,\mathbb{R})$ such that

$$|f(t,x)|\leq h(t)\ \ a.e.\ \ t\in J,$$

for all $x\in\mathbb{R}$.

Similarly, we have

Definition 3.2. *A function $k:J\times J\to\mathbb{R}$ is said to satisfy* ***integrability condition*** *if*

(i) the map $(t,s)\mapsto k(t,s)$ is jointly measurable, and

(iii) there exists a function $\gamma_k\in L^1(J,\mathbb{R})$ such that

$$|k(t,s)|\leq\gamma_k(s)\ \ a.e.\ t,s\in J.$$

Lemma 3.3 (Dhage [6, 7])**.** *If $f(t,x)$ is Chandrabhan, then the function $t \mapsto f(t,x(t))$ is measurable. Moreover, if $f(t,x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then $f(\cdot,x(\cdot))$ is Lebesgue integrable on J for each $x \in L^1(J,,\mathbb{R})$.*

Proof. The proof is similar to an analogous result for Carateódory functions $f(t,x)$ given in Granas and Dugundji [13]. We omit the details. □

Definition 3.4. *An integrable solution $x_M \in L^1(J,\mathbb{R})$ of the VIE* (1.1) *is said to be maximal if x is any other integrable solution, then $x(t) \leq x_M(t)$ for almost every $t \in J$. Similarly, a minimal integrable solution x_m of the VIE* (1.1) *is defined on J.*

We need the following hypotheses in what follows.

(H_1) The function $q : J \to \mathbb{R}$ is Lebesgue integrable.

(H_2) The function k is nonnegative and satisfy integrability condition on $J \times J$.

(H_3) The function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$.

Theorem 3.5. *Assume that hypotheses (H_1) through (H_3) hold. Then the VIE* (1.1) *has a maximal and a minimal integrable solution defined on J.*

Proof. Define a subset S of the complete lattice $\big(L^1(J,\mathbb{R}),\preceq\big)$ by

$$S = \big\{x \in \big(L^1(J,\mathbb{R}),\preceq\big) \mid \|x\|_{L^1} \leq r\big\} \tag{3.1}$$

where, $r = \|q\|_{L^1} + \lambda\,\|H\|_{L^1}T$ and $H(t) = \gamma(t)h(t)$, $t \in J$.

By Lemma 2.3, $(S,\preceq)$ is a complete lattice. Define an operator $\mathcal{T}$ on S by

$$\mathcal{T}x(t) = q(t) + \lambda \int_0^t k(t,s)f(s,x(s))\,ds, \;\; t \in J. \tag{3.2}$$

Then the VIE (1.1) is transformd into an operator equation

$$\mathcal{T}x(t) = x(t), \;\; t \in J. \tag{3.3}$$

We show that $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since the functions k and f are L^1_J-Carateódory and $L^1_{\mathbb{R}}$-Chandrabhan on $J \times J$ and $J \times \mathbb{R}$ respectively, the integral on right hand of the equation (3.2) exists. Moreover, the integral is continuous and hence Lebesgue integrable. Again the sum of two Lebesgue integrable functions is again Lebesgue integrable on J. Hence $\mathcal{T}x \in L^1(J,\mathbb{R})$. Moreover, we have

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq |q(t)| + \lambda \int_0^t k(t,s)|f(s,x(s))|\,ds \\
&\leq |q(t)| + \lambda \int_0^t \gamma(s)h(s)\,ds \\
&\leq |q(t)| + \lambda \int_0^t H(s)\,ds \\
&\leq |q(t)| + \lambda\,\|H\|_{L^1}.
\end{aligned} \tag{3.4}$$

Therefore, taking the integral on both sides from 0 to T, we obtain

$$\begin{aligned}
\|\mathcal{T}x\|_{L^1} &= \int_0^T |\mathcal{T}x(t)|\,dt \\
&\leq \int_0^T |q(t)|\,dt + \lambda \int_0^T \|H\|_{L^1}\,dt \\
&= \|q\|_{L^1} + \lambda\,\|H\|_{L^1}T,
\end{aligned}$$

which implies that $\mathcal{T}$ maps S into itself. Next we show that $\mathcal{T}$is isotone increasing operator on S into itself. Let $x, y \in S$ be such that $x \preceq y$. Then, in view of hypotheses (H$_2$) and (H$_3$),

$$\begin{aligned} \mathcal{T}x(t) &= q(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds \\ &\leq q(t) + \lambda \int_0^t k(t,s) f(s, y(s))\, ds \\ &= \mathcal{T}y(t) \end{aligned} \tag{3.5}$$

for almost every $t \in J$. This shows that $\mathcal{T}x \preceq \mathcal{T}y$ almost everywhere on J. As a result, $\mathcal{T}$ is isotone increasing operator on S. Now by application of Theorem 2.5 implies that $\mathcal{T}$ has a fixed point and the set $F_{\mathcal{T}}$ of all fixed points is a complete lattice. Thus, $F_{\mathcal{T}} \neq \emptyset$ and $\big(F_{\mathcal{T}}, \preceq\big)$ is a complete lattice. Consequently $x_m = \wedge F_{\mathcal{T}}$ and $x_M = \vee F_{\mathcal{T}}$ both exist and are respectively the minimal and maximal integrable solutions of the VIE (1.1) on J. This complete the proof. □

Example 3.6. *Let $J = [0,1] \subset \mathbb{R}$ and consider he nnlinear Volterra inegral equation,*

$$x(t) = t^2 + \int_0^t (t-s) \tanh x(s)\, ds, \quad t \in [0,1]. \tag{3.6}$$

Here, $q(t) = t^2$, $k(t,s) = t - s$ and $f(t,x) = \tanh x$ for $t \in [0,1]$ and $x \in \mathbb{R}$. Thus the above functions satisfy all the conditions of Theorem 3.5, whence the VIE (3.6) *has maximal and minimal integrable solutions defined on* $[0,1]$.

4. Integral Inequalities

Next, we prove two basic integral inequalities involving the integrable solutions related to the VIE (1.1) on J.

Theorem 4.1. *Assume that the hypotheses (H$_1$)- (H$_3$) hold. If there exists an element $u \in S$ such that*

$$u(t) \leq q(t) + \lambda \int_0^t k(t,s) f(s, u(s))\, ds, \tag{4.1}$$

for every $t \in J$, then there exists a maximal integrable solution x_M of the VIE (1.1) *such that*

$$u(t) \leq x_M(t) \ \text{ a. e. t} \in \text{J}. \tag{4.2}$$

Proof. Let $P = \sup S$ which does exist since $(S, \preceq)$ is a complete lattice. Now consider the lattice interval $[u, P]$ which is a closed set and hence a complete lattice. Define an operator $\mathcal{T}$ on $[u, P]$ by (3.2). Then from (4.1) we get $u \preceq \mathcal{T}u$ everywhere on J. Since $\mathcal{T}$ is isotone increasing, it maps the lattice integral $[u, P]$ into itself. Now, by an application of Theorem 2.5, $\mathcal{T}$ has a maximal fixed point x_M in $[u, P]$ which corresponds to the maximal integrable solution of the VIE (1.1) in $[u, P]$. By nature of x_M we have, $u(t) \leq x_M(t)$ a. e. t ∈ J.. This completes proof. □

Theorem 4.2. *Assume that the hypotheses (H$_1$)- (H$_3$) hold. If there exists an element $v \in S$ such that*

$$v(t) \geq q(t) + \lambda \int_0^t k(t,s) f(s, v(s))\, ds, \tag{4.3}$$

for every $t \in J$, then there exists a minimal integrable solution x_m of the VIE (1.1) *such that*

$$v(t) \geq x_m(t) \ \text{ a. e. t} \in \text{J}. \tag{4.4}$$

Proof. The proof is similar to Theorem 4.1 with appropriate modifications. We omit the details. □

Next, we apply the integral inequality stated in Theorem 4.1 to the VIE (1.1) for proving the boundedness and uniqueness of the integrable solutions on J. Now, consider the scaler VIE

$$r(t) = p(t) + \lambda \int_0^t k(t,s)F(s,r(s))\,ds, \ \ t \in J, \tag{4.5}$$

where $p : J \to \mathbb{R}_+$ is Lebesgue integrable and $F : J \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Chandrabhan function.

Theorem 4.3. *Assume that all hypotheses of Theorem 3.5 are satisfied with q and f replaced by p and F given in* (1.1) *and* (4.5) *respectively. Further suppose that the functions q, f and p, F satisfy the inequalities*

$$\left.\begin{aligned} |q(t)| &\leq p(t) \ \text{ a.e. t} \in \text{J}, \\ |f(t,x)| &\leq F(t,|x|) \ \text{ a.e. t} \in \text{J}, \end{aligned}\right\} \tag{4.6}$$

Then for any integrable solution u of the VIE (1.1)*, we obtain*

$$|u(t)| \leq r_M(t) \ \text{ a.e. t} \in \text{J}, \tag{4.7}$$

where r_M is a maximal integrable solution of the VIE (4.6) *on J.*

Proof. By Theorem 3.5, the scalar VIE (4.5) has a maximal integrable solution r_M on J. Let $u \in L^1(J,\mathbb{R})$ be any integrable solution of the VIE (1.1) on J. Then we have

$$u(t) = q(t) + \lambda \int_0^t k(t,s)f(s,u(s))\,ds, \ \ t \in J.$$

Therefore, by Theorem 4.1,

$$\begin{aligned} |u(t)| &\leq |q(t)| + \lambda \int_0^t k(t,s)|f(s,u(s))|\,ds \\ &\leq p(t) + \lambda \int_0^t k(t,s)F(s,|u(s)|)\,ds \\ &\leq r_M(t) \end{aligned}$$

for almost every $t \in J$. This completes the proof. □

Finally, we prove uniqueness result for the integrable solution of the VIE (1.1) on J.

Theorem 4.4. *Assume that all hypotheses of Theorem 3.5 are satisfied with q and f replaced by p and F given in* (1.1) *and* (4.5) *respectively. Further suppose that the functions f and F satisfy the inequality*

$$|f(t,x) - f(t,y)| \leq F(t,|x-y|) \ \text{a.e. t} \in \text{J}, \tag{4.8}$$

for all $x, y \in \mathbb{R}$. Further, if identically zero function is the only solution of the VIE (4.5) *with $p \equiv 0$ on J, then the VIE* (1.1) *has a unique integrable solution on J.*

Proof. Suppose that u and v are two integrable solutions of the VIE (1.1) on J. Then we have

$$u(t) = q(t) + \lambda \int_0^t k(t,s)f(s,u(s))\,ds, \ \ t \in J,$$

and

$$v(t) = q(t) + \lambda \int_0^t k(t,s)f(s,v(s))\,ds, \ \ t \in J.$$

Therefore, by inequality (4.8), we obtain

$$\begin{aligned}|u(t)-v(t)| &\leq \lambda \int_0^t k(t,s)|f(s,u(s)-f(s,v(s)|\,ds\\ &\leq \lambda \int_0^t k(t,s)F(s,|u(s)-v(s)|)\,ds,\end{aligned}$$

for almost every $t \in J$. Now, applying integral inequality given in Theorem 4.1 yields that $u(t) = v(t)$ a. e. t $\in$ J. This completes the proof. □

Remark 4.5. *Under the hypotheses (H_1)- (H_3), the results of this paper may be extended to nonlinear integral equation of Fredholm type*

$$x(t) = q(t) + \lambda \int_0^T k(t,s)f(s,x(s))\,ds, \quad t \in J, \tag{4.9}$$

using the similar arguments with appropriate modifications.

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