MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(02)**(2024), 122–155.
http://doi.org/10.26637/mjm1202/001

Existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions with nonlocal integral boundary conditions

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***R**eceived 30 January 2024; **A**ccepted 20 March 2024*

Abstract. A coupled system of nonlinear self-adjoint second-order ordinary differential inclusions supplemented with nonlocal non-separated coupled integral boundary conditions on an arbitrary domain is studied. The existence results for convex and non-convex valued maps involved in the given problem are proved by applying nonlinear alternative of Leray-Schauder type for multi-valued maps, and Covitz-Nadler's fixed point theorem for contractive multi-valued maps, respectively. Illustrative examples for the obtained results are presented. The paper concludes with some interesting observations.

AMS Subject Classifications: 34A60, 34B10, 34B15.

Keywords: Self-adjoint ordinary differential inclusions; coupled; nonlocal integral boundary conditions; existence; fixed point.

Contents

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1. Introduction

Inspired by the work of Bitsadze and Samarskii [6] on nonlocal elliptic boundary value problems, Il'in and Moiseev [19, 20] initiated the study of nonlocal boundary value problems for second order ordinary differential equations. Nonlocal boundary value problems constitute an important area of research as such problems find their applications in chemical engineering, thermo-elasticity, underground water flow and population dynamics, for details and examples, see [5, 30]. For a variety of interesting results on nonlocal boundary value problems, we refer the reader to the works [1–3, 8, 12–14, 16, 17, 23, 26, 28, 29] and the references cited therein. Self-adjoint differential equations are found to be of great interest in certain disciplines, for example, see [7, 11, 25, 27]. In [24], a self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal multi-point boundary conditions was studied. In a recent article [4], the authors established existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential equations complemented with nonlocal non-separated integral boundary conditions.

The aim of the present paper is to consider and investigate the existence of solutions for the multi-valued case of the problem discussed in [4]. In precise terms, we consider a self-adjoint coupled system of second-order ordinary differential inclusions on an arbitrary domain, subject to the nonlocal non-separated integral coupled boundary conditions given by

$$\begin{cases} \big(p(t)u'(t)\big)' \in \mu_1\, F(t,u(t),v(t)),\ t\in[a,b], \\ \big(q(t)v'(t)\big)' \in \mu_2\, G(t,u(t),v(t)),\ t\in[a,b], \\ \alpha_1 u(a)+\alpha_2 u(b)=\lambda_1\displaystyle\int_a^\eta v(s)ds,\ \ \alpha_3 u'(a)+\alpha_4 u'(b)=\lambda_2\int_a^\eta v'(s)ds, \\ \beta_1 v(a)+\beta_2 v(b)=\lambda_3\displaystyle\int_\xi^b u(s)ds,\ \ \beta_3 v'(a)+\beta_4 v'(b)=\lambda_4\int_\xi^b u'(s)ds, \end{cases} \tag{1.1}$$

where, $a<\eta<\xi<b$, $p,q\in C([a,b],\mathbb{R}^+)$, $\alpha_i,\beta_i,\lambda_i\in\mathbb{R}^+, i=1,2,3,4, \mu_j\in\mathbb{R}^+, j=1,2.$ and $F,G:[a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ are given multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We establish existence criteria for solutions of the problem (1.1) for convex and non-convex valued multivalued maps F and G by applying the nonlinear alternative of Leray-Schauder type for multi-valued maps in the convex case and Covitz and Nadler's fixed point theorem for contractive multi-valued maps in the non-convex case, respectively. The tools of the fixed point theory employed in our analysis are standard, however their application to the problem (1.1) is new. We emphasize that the results derived in this paper are new and enrich the literature on self-adjoint multivalued nonlocal boundary value problems.

The rest of the paper is organized as follows. We present background material about multivalued analysis in Section 2, while the main results are presented in Section 3. Numerical examples illustrating the obtained results are constructed in Section 4.

2. Preliminaries.

We begin this section by reviewing some basic definitions, lemmas, and theorems on multivalued maps from [10, 18] which are related to study of the problem (1.1).

For a normed space $(\mathcal{X},\|\cdot\|)$, we define the following:

(**i**) $P_{cl}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is closed}\}$,

(**ii**) $P_b(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is bounded}\}$,

(**iii**) $P_{cp}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is compact}\}$,

(**iv**) $P_{cp,c}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is compact and convex}\}$.

A multi-valued map $F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is:

(**a**) convex (closed) valued if $F(x)$ is convex (closed) for all $x \in \mathcal{X}$.

(**b**) F is called upper semi-continuous (u.s.c.) on $\mathcal{X}$ if for each $x_0 \in \mathcal{X}$, the set $F(x_0)$ is a nonempty closed subset of $\mathcal{X}$, and if for each open set $\mathcal{N}$ of $\mathcal{X}$ containing $F(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $F(\mathcal{N}_0) \subseteq \mathcal{N}$.

(**c**) The map F is bounded on bounded sets if $F(\mathbb{B}) = \cup_{x\in\mathbb{B}} F(x)$ is bounded in $\mathcal{X}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in F(x)\}\} < \infty$).

(**d**) F is said to be completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$. F has a fixed point if there is $x \in \mathcal{X}$ such that $x \in F(x)$.

Remark 2.1. *A multivalued map $F : W \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $b \in \mathbb{R}$, the function $t \longmapsto d(b, F(t)) = \inf\{|b - c| : c \in F(t)\}$ is measurable.We define the graph of F to be the set $Gr(F) = \{(x, y) \in X \times Y, y \in F(x)\}$.The fixed point set of the multivalued operator F will be denoted by $FixF$.*

Remark 2.2. *We recall the relationship between closed graphs and upper-semicontinuity ([10]): If $F : \mathcal{X} \to \mathcal{P}_{cl}(\mathcal{X})$ is u.s.c., then $Gr(F)$ is a closed subset of $X \times Y$, i.e. for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is upper semi-continuous.*

Definition 2.3. *A multivalued map $F : [a, b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(**i**) *$t \longmapsto F(t, u, v)$ is measurable for each $u, v \in \mathbb{R}$;*

(**ii**) *$(u, v) \longmapsto F(t, u, v)$ is upper semicontinuous for almost all $t \in [a, b]$;*

Further a Carathéodory function F is called L^1-Carathéodory if

(**iii**) *for each $\rho > 0$, there exists $\Omega_\rho \in L^1([a, b], \mathbb{R}^+)$ such that*

$$\|F(t, u, v)\| = \sup\{|x| : x \in F(t, u, v)\} \leq \Omega_\rho(t)$$

for all $\|u\|, \|v\| \leq \rho$ and for a.e. $t \in [a, b]$.

Definition 2.4. *A function $(u, v) \in \mathcal{F} \times \mathcal{F}$, where $\mathcal{F} = C^2([a, b], \mathbb{R})$ is a solution of the self-adjoint coupled system in (1.1) if it satisfies the coupled boundary conditions of (1.1) and there exist functions $\hat{f}, \hat{g} \in L^1([a, b], \mathbb{R})$ such that $\hat{f}(t) \in F(t, u(t), v(t)), \hat{g}(t) \in G(t, u(t), v(t))$ a.e on $[a, b]$.*

Let us now recall the following lemma from [4].

Lemma 2.5. *For $f_1, g_1 \in C([a,b],\mathbb{R})$ and $R \neq 0$, $E \neq 0$, the solution of the linear system*

$$\begin{cases} \big(p(t)u'(t)\big)' = \mu_1 f_1(t), \quad t \in [a,b], \\ \big(q(t)v'(t)\big)' = \mu_2 g_1(t), \quad t \in [a,b], \\ \alpha_1 u(a) + \alpha_2 u(b) = \lambda_1 \int_a^\eta v(s)ds, \;\; \alpha_3 u'(a) + \alpha_4 u'(b) = \lambda_2 \int_a^\eta v'(s)ds, \\ \beta_1 v(a) + \beta_2 v(b) = \lambda_3 \int_\xi^b u(s)ds, \;\; \beta_3 v'(a) + \beta_4 v'(b) = \lambda_4 \int_\xi^b u'(s)ds, \end{cases} \tag{2.1}$$

can be expressed by the formulas:

$$\begin{aligned} u(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u f_1(z)dz\Big)du + \frac{1}{R}\Bigg[- \alpha_2(\beta_1+\beta_2) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u f_1(z)dz\Big)du \\ & + \lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u g_1(z)dz\Big)du\, ds - \lambda_1\beta_2(\eta-a) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u g_1(z)dz\Big)du \\ & + \lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u f_1(z)dz\Big)du\, ds\Bigg] \\ & + \frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\ & + E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\ & - RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b f_1(z)dz\Big) + \Big(- E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\ & + E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\ & + E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s g_1(z)dz\, ds\Big) \\ & + \Big(E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\ & + E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\ & - RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)} \int_a^b g_1(z)dz\Big) + \Big(- E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\ & + E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\ & + E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s f_1(z)dz\, ds\Big)\Bigg], \end{aligned}$$

and

$$\begin{aligned} v(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)} \int_a^u g_1(z)dz\Big)du + \frac{1}{R}\Bigg[- \alpha_2\lambda_3(b-\xi) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u f_1(z)dz\Big)du \\ & + \lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u g_1(z)dz\Big)du\, ds - \beta_2(\alpha_1+\alpha_2) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u g_1(z)dz\Big)du \\ & + \lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u f_1(z)dz\Big)du\, ds\Bigg] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{ER}\bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b f_1(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s g_1(z)dz\,ds\Big) \\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b g_1(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s f_1(z)dz\,ds\Big)\bigg].
\end{aligned}
$$

where

$$
\begin{aligned}
R &= (\alpha_1+\alpha_2)(\beta_1+\beta_2) - \lambda_1\lambda_3(\eta-a)(b-\xi), \\
E &= E_1E_4 - E_2E_3, \\
E_1 &= \frac{\alpha_3}{p(a)} + \frac{\alpha_4}{p(b)},\ E_2 = \int_a^\eta \frac{\lambda_2}{q(s)}ds,\ E_3 = \int_\xi^b \frac{\lambda_4}{p(s)}ds,\ E_4 = \frac{\beta_3}{q(a)} + \frac{\beta_4}{q(b)}.
\end{aligned}
$$

Let $(\mathcal{F}, \|\cdot\|)$ denote the Banach space of all continuous real valued functions where $\mathcal{F} = \{u(t)|u(t) \in C([a,b],\mathbb{R})\}$ and $\|u\| = \sup\{|u(t)|, \quad t \in [a,b]\}$. Evidently the product space $(\mathcal{F}\times\mathcal{F}, \|(u,v)\|)$ is a Banach space with the norm given by $\|(u,v)\| = \|u\| + \|v\|$ for any $(u,v) \in \mathcal{F}\times\mathcal{F}$.

Let us consider the set of selections functions F and G for each $(u,v) \in \mathcal{F}\times\mathcal{F}$ defined by

$$S_{F,(u,v)} := \{\hat{f} \in L^1([a,b],\mathbb{R}) : \hat{f}(t) \in F(t,u(t),v(t)) \text{ for a.e. } t \in [a,b]\},$$

and

$$S_{G,(u,v)} := \{\hat{g} \in L^1([a,b],\mathbb{R}) : \hat{g}(t) \in G(t,u(t),v(t)) \text{ for a.e. } t \in [a,b]\}.$$

Define the operators $\Theta_1, \Theta_2 : \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ by

$$
\begin{aligned}
\Theta_1(u,v) = \{h_1 \in \mathcal{F}\times\mathcal{F} :\ &\text{there exists } \hat{f} \in S_{F,(u,v)}, \hat{g} \in S_{G,(u,v)} \text{ such that} \\
&h_1(u,v)(t) = \mathcal{Z}_1(t,u,v), \forall t \in [a,b]\},
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
\Theta_2(u,v) = \{h_2 \in \mathcal{F}\times\mathcal{F} :\ &\text{there exists } \hat{f} \in S_{F,(u,v)}, \hat{g} \in S_{G,(u,v)} \text{ such that} \\
&h_2(u,v)(t) = \mathcal{Z}_2(t,u,v), \forall t \in [a,b]\},
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
\mathcal{Z}_1(u,v)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\, ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\, ds\Big] \\
& +\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds \\
& -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\, ds\Big) \\
& +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds \\
& +E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds \\
& -RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\, ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Z}_2(u,v)(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du \\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\, ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du \\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\, ds\Big] \\
& +\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds \\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds \\
& -RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\, ds\Big)
\end{aligned}
$$

$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\, ds\Big)\Big].$$

Next we introduce an operator $\Theta : \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ as

$$\Theta(u,v)(t) = \begin{pmatrix}\Theta_1(u,v)(t)\\ \Theta_2(u,v)(t)\end{pmatrix},$$

where Θ_1 and Θ_2 are defined by (2.2) and (2.3) respectively.

For the sake of computational convenience, we set the notation:

$$\mathcal{E}_1 = \mathcal{D}_1 + \mathcal{D}_3,\ \ \mathcal{E}_2 = \mathcal{D}_2 + \mathcal{D}_4, \tag{2.4}$$

where

$$\begin{aligned}
\mathcal{D}_1 = {} & \frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R| + \alpha_2(\beta_1+\beta_2)\Big) + \frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big]\\
& + \frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
& + \frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big)\\
& + \Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
& + \frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big],\\
\mathcal{D}_2 = {} & \frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3} + \lambda_1\beta_2(\eta-a)(b-a)^2\Big]\\
& + \frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
& + \frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big)\\
& + \Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
& + \frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big],\\
\mathcal{D}_3 = {} & \frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(\alpha_2\lambda_3(b-\xi)\Big) + \frac{\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big]\\
& + \frac{1}{RE}\Big[\Big(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}} + \frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}} + \frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}\\
& + \frac{E_4\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_3(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big)
\end{aligned}$$

$$+\Big(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_2\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big],$$

$$\mathcal{D}_4=\frac{\mu_2}{|R\bar{q}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\beta_2(\alpha_1+\alpha_2)\Big)+\frac{\lambda_1\lambda_3(b-\xi)(\eta-a)^3}{6}\Big]$$
$$+\frac{1}{RE}\Big[\Big(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_4\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_3(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big)$$
$$+\Big(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_2\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big], \tag{2.5}$$

$$\bar{p}=\inf_{z\in[a,b]}|p(z)|,\quad \bar{q}=\inf_{z\in[a,b]}|q(z)|. \tag{2.6}$$

3. The Carathéodory case

To prove our first existence result for the multivalued problem (1.1), we need the following known results.

Lemma 3.1. *([22]) Let X be a Banach space. Let $F:[a,b]\times\mathbb{R}^2\to\mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1- Carathéodory multivalued map and let φ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then the operator*

$$\varphi\circ S_{F,u}:C([a,b],\mathbb{R})\to P_{cp,c}(C([a,b],\mathbb{R})),\quad u\mapsto(\varphi\circ S_{F,u})(u)=\varphi(S_{F,u})$$

is a closed graph operator in $C([a,b],\mathbb{R})\times C([a,b],\mathbb{R})$.

Lemma 3.2. *(Nonlinear alternative of Leray-Schauder type [15]). Let $\mathcal{S}$ be a Banach space, $\mathcal{S}_1$ a closed convex subset of $\mathcal{S}$, U an open subset of $\mathcal{S}_1$ and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{c,cv}(\mathcal{S}_1)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(\mathcal{S}_1)$ denotes the family of nonempty, compact convex subsets of $\mathcal{S}_1$. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.

Now we are in a position to present our first main result.

Theorem 3.3. *Assume that*

(H_1) *$F,G:[a,b]\times\mathbb{R}^2\longrightarrow\mathcal{P}(\mathbb{R})$ are L^1-Carathéodory possessing compact and convex values;*

(H_2) *There exist continuous nondecreasing functions $\psi_1,\psi_2,\phi_1,\phi_2:[0,\infty)\longrightarrow(0,\infty)$ such that*

$$\|F(t,u,v)\|_{\mathcal{P}}:=\sup\{|\hat{f}|:\hat{f}\in F(t,u,v)\}\leqslant p_1(t)[\psi_1(\|u\|)+\phi_1(\|v\|)],$$

and

$$\|G(t,u,v)\|_{\mathcal{P}}:=\sup\{|\hat{g}|:\hat{g}\in G(t,u,v)\}\leqslant p_2(t)[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

for each $(t,u,v)\in[a,b]\times\mathbb{R}^2$, where $p_1,p_2\in C([a,b],\mathbb{R}^+)$;

(H_3) *There exists a constant $N>0$ such that*

$$\frac{N}{\mathcal{E}_1\|p_1\|[\psi_1(N)+\phi_1(N)]+\mathcal{E}_2\|p_2\|[\psi_2(N)+\phi_2(N)]}>1,$$

where $\mathcal{E}_i$ $(i=1,2)$ are given in (2.4).

Then self-adjoint coupled multi-valued system (1.1) *has at least one solution on* $[a, b]$.

Proof. Consider the operators $\Theta_1, \Theta_2 : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ defined by (2.2) and (2.3) respectively. It follows from the assumption (H_1) that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty for each $(u, v) \in \mathcal{F} \times \mathcal{F}$. Then, for $\hat{f} \in S_{F,(u,v)}, \hat{g} \in S_{G,(u,v)}$ and $\forall\ (u, v) \in \mathcal{F} \times \mathcal{F}$, we have

$$\begin{aligned}
h_1(u,v)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du + \frac{1}{R}\Bigg[- \alpha_2(\beta_1+\beta_2) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du \\
& + \lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)dz\Big)du\, ds - \lambda_1\beta_2(\eta-a) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)dz\Big)du \\
& + \lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du\, ds\Bigg] \\
& + \frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\
& - RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}(z)dz\Big) + \Big(- E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\
& + E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\
& + E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}(z)dz\, ds\Big) \\
& + \Big(E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\
& - RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}(z)dz\Big) + \Big(- E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\
& + E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\
& + E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}(z)dz\, ds\Big)\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
h_2(u,v)(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)dz\Big)du + \frac{1}{R}\Bigg[- \alpha_2\lambda_3(b-\xi) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du \\
& + \lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)dz\Big)du\, ds - \beta_2(\alpha_1+\alpha_2) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)dz\Big)du \\
& + \lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du\, ds\Bigg] \\
& + \frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_3\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\
& - RE_3 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}(z)dz\Big) + \Big(- E_4\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}dz
\end{aligned}$$

$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Bigg],$$

where $h_1 \in \Theta_1(u,v), h_2 \in \Theta_2(u,v)$ and hence $(h_1,h_2) \in \Theta(u,v)$.

Now, we will verify the operator Θ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. In the first step, we show that $\Theta(u,v)$ is convex valued for each $(u,v) \in \mathcal{F}\times\mathcal{F}$. Let $(h_i,\ \tilde{h_i}) \in (\Theta_1,\Theta_2), i=1,2$. Then there exist $\hat{f}_i \in S_{F,(u,v)}, \hat{g}_i \in S_{G,(u,v)}, i=1,2$, such that, for each $t\in[a,b]$, we have

$$h_i(t) = \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du\,ds\Bigg]$$
$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)dz\,ds\Big)\Bigg],$$

and

$$
\begin{aligned}
\tilde{h_i}(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du \\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du \\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du\,ds\Bigg] \\
& +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)dz\,ds\Big) \\
& +\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
& +E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
& +E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)dz\,ds\Big)\Bigg].
\end{aligned}
$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,1]$, we have

$$
\begin{aligned}
[\omega h_1 + (1-\omega)h_2](t) = & \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du \\
& +\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u [\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)]dz\Big)du\,ds \\
& -\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u [\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)]dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du\,ds\Bigg] \\
& +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&+RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)\\
&+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&-E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&+RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
[\omega\tilde{h_1}+(1-\omega)\tilde{h_2}](t) = &\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\\
&+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\,ds\\
&-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)\\
&+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz + E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz + E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_3\int_a^t\frac{1}{p(z)}dz\Big)\\
&\times\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)\\
&+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\,ds\Big)\Big].
\end{aligned}
$$

Since $S_{F,(u,v)}, S_{G,(u,v)}$ are convex valued as F and G are convex valued maps, therefore, $\omega h_1+(1-\omega)h_2\in\Theta_1, \omega\tilde{h_1}+(1-\omega)\tilde{h_2}\in\Theta_2$ and hence $\omega(h_1,\tilde{h_1})+(1-\omega)(h_2,\tilde{h_2})\in\Theta$.

Now, we show that Θ maps bounded sets into bounded sets in $\mathcal{F}\times\mathcal{F}$. For a positive number ν^*, let $B_{\nu^*}=\{(u,v)\in\mathcal{F}\times\mathcal{F}:\|(u,v)\|\le\nu^*\}$ be a bounded set in $\mathcal{F}\times\mathcal{F}$. Then, for each $h_i\in\Theta_i, (i=1,2),\ (u,v)\in B_{\nu^*}$, there exist $\hat{f}\in S_{F,(u,v)}, \hat{g}\in S_{G,(u,v)}$ such that

$$
\begin{aligned}
h_1(u,v)(t) =& \int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du+\frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(u,v)(t) =& \int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Bigg].
\end{aligned}
$$

Then, for $t\in[a,b]$, we have

$$
\begin{aligned}
|h_1(u,v)(t)|\le&\int_a^t\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du+\frac{1}{|R|}\Bigg[|\alpha_2(\beta_1+\beta_2)|\int_a^b\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du\\
&+|\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\Big(\frac{|\mu_2|}{|q(u)|}\int_a^u|\hat{g}(z)|dz\Big)du\,ds+|\lambda_1\beta_2(\eta-a)|\int_a^b\Big(\frac{|\mu_2|}{|q(u)|}\int_a^u|\hat{g}(z)|dz\Big)du\\
&+|\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du\,ds\Bigg]\\
&+\frac{1}{|ER|}\Bigg[\Big(|E_4\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz+|E_3\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds\\
&+|E_3\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz+|E_4\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds\\
&+|RE_4|\int_a^t\frac{1}{|p(z)|}dz\Big)\Big(\frac{|\alpha_4\mu_1|}{|p(b)|}\int_a^b|\hat{f}(z)|dz\Big)+\Big(|E_4\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz\\
&+|E_3\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds+|E_3\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz\\
&+|E_4\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds+|RE_4|\int_a^t\frac{1}{|p(z)|}dz\Big)\Big(\int_a^\eta\frac{|\lambda_2\mu_2|}{|q(s)|}\int_a^s|\hat{g}(z)|dz\,ds\Big)\\
&+\Big(|E_2\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz+|E_1\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds\\
&+|E_1\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz+|E_2\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+|RE_2|\int_a^t \frac{1}{|p(z)|}dz\Big)\Big(\frac{|\beta_4\mu_2|}{|q(b)|}\int_a^b|\hat{g}(z)|dz\Big)+\Big(|E_2\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz\\
&+|E_1\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds+|E_1\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz\\
&+|E_2\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds+|RE_2|\int_a^t\frac{1}{|p(z)|}dz\Big)\Big(\int_\xi^b\frac{|\lambda_4\mu_1|}{|p(s)|}\int_a^s|\hat{f}(z)|dz\,ds\Big)\Big]\\
\le{}&\Big\{\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\alpha_2(\beta_1+\beta_2)\Big)+\frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big]\\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big)\\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big]\Big\}\\
&\times\|p_1\|\Big[\psi_1(\nu^*)+\phi_1(\nu^*)\Big]\\
&+\Big\{\frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3}+\lambda_1\beta_2(\eta-a)(b-a)^2\Big]\\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big)\\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big]\Big\}\\
&\times\|p_2\|\Big[\psi_2(\nu^*)+\phi_2(\nu^*)\Big]\\
={}&\mathcal{D}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+\mathcal{D}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)].
\end{aligned}
$$

Similarly, we can obtain that

$$|h_2(u,v)(t)|\le\mathcal{D}_3\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+\mathcal{D}_4\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)].$$

Thus, we get

$$
\begin{aligned}
\|h_1(u,v)\|&\le\mathcal{D}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+\mathcal{D}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)],\\
\|h_2(u,v)\|&\le\mathcal{D}_3\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+\mathcal{D}_4\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)],
\end{aligned}
$$

where $\mathcal{D}_i, (i=1,2,3,4)$ are defined by (2.5). In consequence, we have

$$
\begin{aligned}
\|(h_1,h_2)\|&=\|h_1(u,v)\|+\|h_2(u,v)\|\\
&\le(\mathcal{D}_1+\mathcal{D}_3)\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+(\mathcal{D}_2+\mathcal{D}_4)\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)]\\
&=\mathcal{E}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)]+\mathcal{E}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)]\\
&=\ell\text{ (constant)},
\end{aligned}
$$

where $\mathcal{E}_i, i=1,2$, are defined in (2.4).

Next, we verify that $\Theta(u,v)$ is equicontinuous. Let $t_1,\ t_2\in[a,b]$ with $t_1<t_2$. Then, for $\hat{f}\in S_{F,(u,v)}, \hat{g}\in S_{G,(u,v)}$, we get

$$\Big|h_1(u,v)(t_2)-h_1(u,v)(t_1)\Big|$$

$$
\begin{aligned}
&= \Bigg| \int_a^{t_2} \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(\tau) dz \Big) du - \int_a^{t_1} \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(\tau) dz \Big) du \\
&\quad + \Big(\frac{E_4}{E} \Big(\int_a^{t_2} \frac{1}{p(z)} dz - \int_a^{t_1} \frac{1}{p(z)} dz \Big) \Big(\frac{\alpha_4 \mu_1}{p(b)} \int_a^b \hat{f}(\tau) dz \Big) \Big) \\
&\quad + \Big(\frac{E_4}{E} \Big(\int_a^{t_2} \frac{1}{p(z)} dz - \int_a^{t_1} \frac{1}{p(z)} dz \Big) \Big(\int_a^{\eta} \frac{\lambda_2 \mu_2}{q(s)} \int_a^s \hat{g}(\tau) dz\, ds \Big) \Big) \\
&\quad + \Big(\frac{E_2}{E} \int_a^{t_2} \frac{1}{p(z)} dz - \int_a^{t_1} \frac{1}{p(z)} dz \Big) \Big(\frac{\beta_4 \mu_2}{q(b)} \int_a^b \hat{g}(\tau) dz \Big) \Big) \\
&\quad + \Big(\frac{E_2}{E} \Big(\int_a^{t_2} \frac{1}{p(z)} dz - \int_a^{t_1} \frac{1}{p(z)} dz \Big) \Big(\int_{\xi}^{b} \frac{\lambda_4 \mu_1}{p(s)} \int_a^s \hat{f}(\tau) dz\, ds \Big) \Big) \Bigg| \\
&\leq \Big[\Big(\frac{\mu_1}{|\bar{p}|} \Big) \frac{(t_2 - a)^2 - (t_1 - a)^2}{2} + \frac{E_4}{E|\bar{p}|} \Big(\frac{\alpha_4 \mu_1}{|p(b)|} \Big) (t_2 - t_1)(b - a) \\
&\quad + \frac{E_2}{E|\bar{p}|} \frac{\big(\lambda_4 \mu_1 \big)(t_2 - t_1) \big[(b-a)^2 - (\xi - a)^2 \big]}{2} \Big] \|p_1\| [\psi_1(\nu^*) + \phi_1(\nu^*)] \\
&\quad + \Big[\frac{E_4}{E|\bar{p}|} \frac{\big(\lambda_2 \mu_2 \big)(t_2 - t_1)(\eta - a)^2}{2\bar{q}} + \frac{E_2}{E|\bar{p}|} \Big(\frac{\beta_4 \mu_2}{|q(b)|} \Big) (t_2 - t_1)(b - a) \Big] \\
&\quad \times \|p_2\| [\psi_2(\nu^*) + \phi_2(\nu^*)] \to 0 \text{ as } t_2 \to t_1 \text{ independent of } (u, v).
\end{aligned}
$$

Analogously, it can be shown that

$$
|h_2(u,v)(t_2) - h_2(u,v)(t_1)| \to 0 \text{ as } t_2 \to t_1 \text{ independent of } (u,v).
$$

Therefore, the operator $\Theta(u,v)$ is equicontinuous and hence we deduce that $\Theta(u,v) : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ is completely continuous by the Arzelá-Ascoli Theorem.

In the next step, we show that $\Theta(u,v)$ is upper semicontinuous. Instead it will be established that $\Theta(u,v)$ has a closed graph in view of the fact that a completely continuous operator is upper semicontinuous if it has a closed graph. Let $(u_k, v_k) \longrightarrow (u_*, v_*)$ and $(h_k, \tilde{h_k}) \in \Theta(u_k, v_k)$ and $(h_k, \tilde{h_k}) \longrightarrow (h_*, \tilde{h_*})$. Then we have to show that $(h_*, \tilde{h_*}) \in \Theta(u_*, v_*)$. Associated with $(h_k, \tilde{h_k}) \in \Theta(u_k, v_k)$ and $\hat{f}_k \in S_{F,(u,v)}, \hat{g}_k \in S_{G,(u,v)}$, for each $t \in [a,b]$, we have

$$
\begin{aligned}
h_k(u_k, v_k)(t) &= \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z) dz \Big) du + \frac{1}{R} \Bigg[- \alpha_2(\beta_1 + \beta_2) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z) dz \Big) du \\
&\quad + \lambda_1(\beta_1 + \beta_2) \int_a^{\eta} \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z) dz \Big) du\, ds - \lambda_1 \beta_2(\eta - a) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z) dz \Big) du \\
&\quad + \lambda_1 \lambda_3(\eta - a) \int_{\xi}^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z) dz \Big) du\, ds \Bigg] \\
&\quad + \frac{1}{ER} \Bigg[\Big(E_4 \alpha_2(\beta_1 + \beta_2) \int_a^b \frac{1}{p(z)} dz - E_3 \lambda_1(\beta_1 + \beta_2) \int_a^{\eta} \int_a^s \frac{1}{q(z)} dz\, ds \\
&\quad + E_3 \lambda_1 \beta_2(\eta - a) \int_a^b \frac{1}{q(z)} dz - E_4 \lambda_1 \lambda_3(\eta - a) \int_{\xi}^b \int_a^s \frac{1}{p(z)} dz\, ds \\
&\quad - R E_4 \int_a^t \frac{1}{p(z)} dz \Big) \Big(\frac{\alpha_4 \mu_1}{p(b)} \int_a^b \hat{f}_k(z) dz \Big) + \Big(- E_4 \alpha_2(\beta_1 + \beta_2) \int_a^b \frac{1}{p(z)} dz \\
&\quad + E_3 \lambda_1(\beta_1 + \beta_2) \int_a^{\eta} \int_a^s \frac{1}{q(z)} dz\, ds - E_3 \lambda_1 \beta_2(\eta - a) \int_a^b \frac{1}{q(z)} dz \\
&\quad + E_4 \lambda_1 \lambda_3(\eta - a) \int_{\xi}^b \int_a^s \frac{1}{p(z)} dz\, ds + R E_4 \int_a^t \frac{1}{p(z)} dz \Big) \Big(\int_a^{\eta} \frac{\lambda_2 \mu_2}{q(s)} \int_a^s \hat{g}_k(z) dz\, ds \Big) \\
&\quad + \Big(E_2 \alpha_2(\beta_1 + \beta_2) \int_a^b \frac{1}{p(z)} dz - E_1 \lambda_1(\beta_1 + \beta_2) \int_a^{\eta} \int_a^s \frac{1}{q(z)} dz\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h_k}(u_k,v_k)(t) = &\int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du \\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du \\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\,ds\Big] \\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)dz\,ds\Big) \\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big].
\end{aligned}
$$

Consider the continuous linear operators $\Psi_1, \Psi_2 : L^1([a,b],\mathcal{F}\times\mathcal{F}) \longrightarrow C([a,b],\mathcal{F}\times\mathcal{F})$ given by

$$
\begin{aligned}
\Psi_1(u,v)(t) = &\int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du \\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du \\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{ER}\bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_2(u,v)(t)=&\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du+\frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\bigg]\\
&+\frac{1}{ER}\bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz
\end{aligned}
$$

$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big].$$

From Lemma 3.1, we know that $(\Psi_1,\Psi_2)\circ(S_F,S_G)$ are closed graph operators. Moreover, we have $(h_k,\tilde{h_k})\in(\Psi_1,\Psi_2)\circ(S_{F,(u_k,v_k)},S_{G,(u_k,v_k)})$ for all k. Since $(u_k,v_k)\longrightarrow(u_*,v_*)$, $(h_k,\tilde{h_k})\longrightarrow(h_*,\tilde{h_*})$, it follows that $\hat{f}_*\in S_{F,(u,v)},\hat{g}_*\in S_{G,(u,v)}$ such that

$$\begin{aligned}
h_*(u_*,v_*)(t) = &\int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_*(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_*(z)dz\,ds\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h_*}(u_*,v_*)(t) = &\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds
\end{aligned}$$

$$+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_*(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_*(z)dz\,ds\Big)\Big],$$

which lead to the conclusion that $(h_k, \tilde{h_k}) \in \Theta(u_*, v_*)$.

Finally, we show that there exists an open set $U \subseteq \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ with $(u,v) \notin \epsilon\Theta(u,v)$ for any $\epsilon \in (0,1)$ and all $(u,v) \in \partial U$. Let $\epsilon \in (0,1)$ and $(u,v) \in \epsilon\Theta(u,v)$. Then there exist $\hat{f} \in S_{F,(u,v)}$ and $\hat{g} \in S_{G,(u,v)}$ such that, for $t \in [a,b]$, we have

$$u(t) = \epsilon\int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du + \frac{\epsilon}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Big]$$
$$+\frac{\epsilon}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$

$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big],$$

and

$$v(t) = \epsilon\int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du + \frac{\epsilon}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du$$
$$+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du$$
$$+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Big]$$
$$+\frac{\epsilon}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big].$$

Using the arguments employed in the second step, we find that

$$\|u\| \le \mathcal{D}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)] + \mathcal{D}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

and

$$\|v\| \le \mathcal{D}_3\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)] + \mathcal{D}_4\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)].$$

Then we have

$$\begin{aligned}\|(u,v)\|\|\|u\|+\|v\| &\le (\mathcal{D}_1+\mathcal{D}_3)\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)] + (\mathcal{D}_2+\mathcal{D}_4)\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)]\\ &\le \mathcal{E}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)] + \mathcal{E}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)],\end{aligned}$$

where $\mathcal{E}_i, i=1,2$, are given by (2.4). Consequently, we have

$$\frac{\|(u,v)\|}{\mathcal{E}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)] + \mathcal{E}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)]} \le 1.$$

According to (H_3), there exists N such that $\|(u,v)\| \neq N$. Let us set

$$U = \{(u,v) \in (\mathcal{F} \times \mathcal{F}) : \|(u,v)\| < N\}.$$

Observe that the operator $\Theta : \bar{U} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{F}) \times \mathcal{P}_{cp,cv}(\mathcal{F})$ is completely continuous and upper semicontinuous. From the choice of U, there is no $(u,v) \in \partial U$ such that $(u,v) \in \epsilon\Theta(u,v)$ for some $\epsilon \in (0,1)$. Therefore, by nonlinear alternative of Leray-Schauder type (Lemma 3.2), we deduce that Θ has a fixed point $(u,v) \in \bar{U}$ which is a solution of the problem (1.1). □

4. The Lipschitz case.

The forthcoming result is based on the fixed point theorem for contraction multivalued operators due to Covitz-Nadler [9], which is stated below.

Lemma 4.1. *(Covitz-Nadler) Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.*

Remark 4.2. *Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [21]).

Theorem 4.3. *Assume that the following conditions hold:*

(H_5) *$F, G : [a,b] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ are such that $F(\cdot,u,v), G(\cdot,u,v) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ are measurable for each $u,v \in \mathbb{R}$;*

(H_6) *For almost all $t \in [a,b]$ and $u,v,\bar{u},\bar{v} \in \mathbb{R}$ with $\mathcal{B}_1, \mathcal{B}_2 \in C([a,b],\mathbb{R}^+)$,*

$$H_d(F(t,u,v), F(t,\bar{u},\bar{v}) \leq \mathcal{B}_1(t)(|u-\bar{u}| + |v-\bar{v}|),\ H_d(G(t,u,v), G(t,\bar{u},\bar{v}) \leq \mathcal{B}_2(t)(|u-\bar{u}| + |v-\bar{v}|),$$

and $d(0, F(t,0,0)) \leq \mathcal{B}_1(t), d(0, G(t,0,0)) \leq \mathcal{B}_2(t)$.

Then the self-adjoint coupled multi-valued system (1.1) has at least one solution on $[a,b]$ if

$$\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| < 1,$$

where $\mathcal{E}_1, \mathcal{E}_2$ are given in (2.4).

Proof. Consider the operators $\Theta_1, \Theta_2 : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ defined by (2.2) and (2.3) respectively.

Notice that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty and consequently $\Theta \neq \emptyset$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Then, by the assumption (H_5), the multivalued maps $F(\cdot,(u,v))$ and $G(\cdot,(u,v))$ are measurable, and thus admit measurable selections.

Now we shall show that the operator $\Theta(u,v)$ satisfies the hypothesis of Lemma 4.1. Firstly, we verify that $\Theta(u,v) \in \mathcal{P}_{cl}(\mathcal{F}) \times \mathcal{P}_{cl}(\mathcal{F})$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Let $(h_k, \tilde{h}_k) \in \Theta(u_k, v_k)$ such that $(h_k, \tilde{h}_k)$ converges to $(h, \tilde{h})$ as $k \to \infty$ in $\mathcal{F} \times \mathcal{F}$. So $(h, \tilde{h}) \in \mathcal{F} \times \mathcal{F}$ and there exist $\hat{f}_k \in S_{F,(u_k,v_k)}$ and $\hat{g}_k \in S_{G,(u_k,v_k)}$ such that, for each $t \in [a,b]$, we have

$$\begin{aligned} h_k(u_k,v_k)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du \\ & +\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\, ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du \\ & +\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\, ds\Bigg] \\ & +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds \\ & +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds \end{aligned}$$

$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_k(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_k(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_k(z)dz\,ds\Big)\Big],$$

and

$$\tilde{h_k}(u_k,v_k)(t)=\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_k(z)dz\Big)du+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_k(z)dz\Big)du$$
$$+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_k(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_k(z)dz\Big)du$$
$$+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_k(z)dz\Big)du\,ds\Big]$$
$$+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_k(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_k(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_k(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_k(z)dz\,ds\Big)\Big].$$

Since F and G have compact values, we pass onto a subsequences (if necessary) to get that $\hat{f}_k$ and $\hat{g}_k$ converge to $\hat{f}$ and $\hat{g}$ in

$L^1([a,b],\mathbb{R})$ respectively. Then $\hat{f}\in S_{F,(u,v)}$ and $\hat{g}\in S_{G,(u,v)}$ and for each $t\in[a,b]$, we have

$$
\begin{aligned}
& h_k(u_k,v_k)(t)\to h(u,v)(t)\\
=& \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du+\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds-\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Bigg]\\
& +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
& -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
& +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
& +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)\\
& +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
& +E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
& -RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
& +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
& +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
& \tilde{h_k}(u_k,v_k)(t)\to \tilde{h}(u,v)(t)\\
=& \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du+\frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Bigg]\\
& +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
& -RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz
\end{aligned}
$$

$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big].$$

Therefore $(u,v)\in\Theta$ and hence $\Theta(u,v)$ is closed.

Next we show that Θ is a contraction on $\mathcal{P}_{cl}(\mathcal{F})\times\mathcal{P}_{cl}(\mathcal{F})$, that is, there exists a positive number $\gamma<1$ such that

$$H_d(\Theta(u,v),\Theta(\bar{u},\bar{v}))\le\gamma(\|u-\bar{u}\|+\|v-\bar{v}\|)\ \text{ for each }\ u,v,\bar{u},\bar{v}\in\mathcal{F}.$$

Let $(u,\bar{u}),(v,\bar{v})\in\mathcal{F}\times\mathcal{F}$, and $(h_1,\tilde{h_1})\in\Theta(u,v)$. Then there exist $\hat{f}_1(t)\in S_{F,(u,v)}$ and $\hat{g}_1(t)\in S_{G,(u,v)}$ such that, for each $t\in[a,b]$, we obtain

$$h_1(u,v)(t) = \int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\,ds\Big]$$
$$+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$

$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\,ds\Big)\Big],$$

and

$$\begin{aligned}\tilde{h}_1(u,v)(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\,ds\Big)\Big].\end{aligned}$$

By (H_6), we have that

$$H_d(F(t,u,v),F(t,\bar{u},\bar{v})) \le \mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|),$$

and

$$H_d(G(t,u,v),G(t,\bar{u},\bar{v})) \le \mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|).$$

So there exist $\hat{\vartheta}_f \in F(t,u(t),v(t))$ and $\hat{\vartheta}_g \in G(t,u(t),v(t))$ such that

$$|\hat{f}_1(t)-\hat{\vartheta}_f| \le \mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|),$$

$$|\hat{g}_1(t)-\hat{\vartheta}_g| \le \mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|).$$

Define $W_1, W_2 : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$W_1(t) = \{\hat{\vartheta}_f \in L^1([a,b),\mathbb{R}) : |\hat{f}_1(t)-\hat{\vartheta}_f| \le \mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|)\},$$

and

$$W_2(t) = \{\hat{\vartheta}_g \in L^1([a,b),\mathbb{R}) : |\hat{g}_1(t)-\hat{\vartheta}_g| \le \mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|)\}.$$

Since the multivalued operators $W_1(t) \cap F(t,u(t),v(t))$ and $W_2(t) \cap G(t,u(t),v(t))$ are measurable, there exist functions $\hat{f}_2(t), \hat{g}_2(t)$ which are measurable selections for W_1 and W_2. Thus $\hat{f}_2(t) \in F(t,u(t),v(t)), \hat{g}_2(t) \in G(t,u(t),v(t))$ and for each $t \in [a,b]$, we have

$$|\hat{f}_1(t) - \hat{f}_2(t)| \leq \mathcal{B}_1(t)(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|),$$

and

$$|\hat{g}_1(t) - \hat{g}_2(t)| \leq \mathcal{B}_2(t)(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|).$$

For each $t \in [a,b]$, let us define

$$\begin{aligned} h_2(u,v)(t) \,=\, & \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)dz\Big)du \\ & +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)dz\Big)du \\ & +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)dz\Big)du\,ds\Bigg] \\ & +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\ & +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\ & -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_2(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\ & +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\ & +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_2(z)dz\,ds\Big) \\ & +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\ & +E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\ & -RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_2(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\ & +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\ & +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_2(z)dz\,ds\Big)\Bigg], \end{aligned}$$

and

$$\begin{aligned} \tilde{h}_2(u,v)(t) \,=\, & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)dz\Big)du \\ & +\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)dz\Big)du \\ & +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)dz\Big)du\,ds\Bigg] \\ & +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\ & +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \end{aligned}$$

$$
\begin{aligned}
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_2(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_2(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_2(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_2(z)dz\,ds\Big)\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\big|h_1(u,v)(t)-h_2(u,v)(t)\big|\\
\le&\int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u\big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du\\
&+\frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big)du\,ds\\
&-\lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s\big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\,ds\Big)
\end{aligned}
$$

$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$

$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$

$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$

$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\,ds\Big)\Bigg]$$

$$\leq \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du\,ds$$

$$-\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du\,ds\Bigg]$$

$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$

$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)$$

$$+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)$$

$$\times\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\,ds\Big)$$

$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$

$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \mathcal{B}_2(\tau)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)$$

$$+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$-E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)$$

$$\times\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \mathcal{B}_1(\tau)(|u(\tau)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)|dz\,ds\Big)\Big]$$

$$\leq \Big\{\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\alpha_2(\beta_1+\beta_2)\Big)+\frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big]$$

$$+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}$$

$$+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big)$$

$$+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}$$

$$+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big]\Big\}$$

$$\times\|\mathcal{B}_1\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big)$$

$$+\Big\{\frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3}+\lambda_1\beta_2(\eta-a)(b-a)^2\Big]$$

$$+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}$$

$$+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big)$$

$$+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}$$

$$+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big]\Big\}$$

$$\times\|\mathcal{B}_2\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big)$$

$$\leq (\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|)(\|u-\bar{u}\|+\|v-\bar{v}\|),$$

which implies that

$$|h_1(u,v)(t)-h_2(u,v)(t)|\leq (\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|)(\|u-\bar{u}\|+\|v-\bar{v}\|).$$

In a similar manner, one can be establish that

$$|\tilde{h}_1(u,v)(t) - \tilde{h}_2(u,v)(t)| \leq (\mathcal{D}_3\|\mathcal{B}_1\| + \mathcal{D}_4\|\mathcal{B}_2\|)(\|u-\bar{u}\| + \|v-\bar{v}\|).$$

In consequence, we get

$$\begin{aligned}\|(h_1,h_2),(\tilde{h}_1,\tilde{h}_2)\| &\leq [(\mathcal{D}_1+\mathcal{D}_3)\|\mathcal{B}_1\| + (\mathcal{D}_2+\mathcal{D}_4)\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|)\\ &\leq [(\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|).\end{aligned}$$

Similarly, by interchanging the roles of (u,v) and $(\bar{u},\bar{v})$, we can obtain that

$$H_d(\Theta(u,v),\Theta(\bar{u},\bar{v})) \leq [(\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|).$$

Therefore, it follows by the assumption: $\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| < 1$ that Θ is a contraction, So, by Lemma 4.1, Θ has a fixed point (u,v), which is a solution of the problem (1.1). The proof is finished. □

5. Examples

Example 5.1. *Consider the following self-adjoint coupled system of second-order ordinary differential inclusions with boundary conditions*

$$\begin{cases} \Big(\Big(\dfrac{1}{t+13}\Big)u'(t)\Big)' \in \mu_1 F(t,u,v),\ t\in[0,3],\\ \Big(\dfrac{8}{4t^2+2t+12}\,v'(t)\Big)' \in \mu_2 G(t,u,v),\ t\in[0,3],\\ \dfrac{7}{3}u(0) + \dfrac{5}{3}u(3) = \dfrac{1}{7}\displaystyle\int_0^{\frac{1}{2}} v(s)ds,\ \ \dfrac{4}{3}u'(0) + u'(3) = \dfrac{2}{7}\displaystyle\int_0^{\frac{1}{2}} v'(s)ds,\\ \dfrac{1}{9}v(0) + \dfrac{2}{9}v(3) = \dfrac{3}{7}\displaystyle\int_{\frac{5}{2}}^{3} u(s)ds,\ \ \dfrac{3}{9}v'(0) + \dfrac{4}{9}v'(3) = \dfrac{4}{7}\displaystyle\int_{\frac{5}{2}}^{3} u'(s)ds. \end{cases} \tag{5.1}$$

Here $p(t) = 1/(t+13), q(t) = 8/(4t^2+2t+12), \mu_1 = 3/36, \mu_2 = 2/93, a = 0, b = 3, \eta = 1/2, \xi = 5/2, \lambda_1 = 1/7, \lambda_2 = 2/7, \lambda_3 = 3/7, \lambda_4 = 4/7, \alpha_1 = 7/3, \alpha_2 = 5/3, \alpha_3 = 4/3, \alpha_4 = 1, \beta_1 = 1/9, \beta_2 = 2/9, \beta_3 = 3/9, \beta_4 = 4/9$, and $F(t,u,v), G(t,u,v)$ will be fixed later.

Using the given data, we find that $|R| \approx 1.323129 \neq 0$, $|E| \approx 115.6354 \neq 0$, $\bar{p} \approx 0.0625$, $\bar{q} = 0.148148$, $\mathcal{D}_1 \approx 17.1389708$, $\mathcal{D}_2 \approx 0.06036034$, $\mathcal{D}_3 \approx 38.2023705$, $\mathcal{D}_4 \approx 4.565128967$, $\mathcal{E}_1 \approx 17.19933114$ and $\mathcal{E}_2 \approx 42.76749946$ ($\bar{p}, \bar{q}$ and $\mathcal{D}_i$ $(i = 1,2,3,4)$ are defined in (2.5), while $\mathcal{E}_1, \mathcal{E}_2$ are given in (2.4)).

(a) For illustration of Theorem 3.3, we choose

$$F(t,u,v) = \Big(\frac{t}{108t^2+32}\Big)\Big[\frac{|u(t)|}{\sqrt{|u(t)|^2+65}}, \frac{|v(t)|^2}{|v(t)|^2+1}\Big],$$

and

$$G(t,u,v) = \Big(\frac{t^2+1}{t^3+120}\Big)\Big[\frac{|u(t)|}{|u(t)|+1}, \frac{|v(t)|^3}{1+|v(t)|^3}\Big].$$

For $f\in F$, we have

$$|f| \leq \max\Big\{\Big(\frac{t}{108t^2+32}\Big[\frac{|u(t)|}{\sqrt{|u(t)|^2+65}}, \frac{|v(t)|^2}{|v(t)|^2+1}\Big]\Big\} \leq 2\Big\{\frac{t}{108t^2+32}\Big\}, u,v\in\mathbb{R}, t\in[0,3],$$

and for $g\in G$, we have

$$|g| \leq \max\Big\{\Big(\frac{t^2+1}{t^3+120}\Big)\Big[\frac{|u(t)|}{|u(t)|+1}, \frac{|v(t)|^3}{1+|v(t)|^3}\Big]\Big\} \leq 2\Big\{\frac{t^2+1}{t^3+120}\Big\}, u,v\in\mathbb{R}, t\in[0,3].$$

Thus

$$\|F(t,u,v)\|_{\mathcal{P}} := \sup\{|f| : f\in F(t,u,v)\} \leq 2\Big[\frac{t}{108t^2+32}\Big] = p_1(t)[\psi_1(\|u\|) + \phi_1(\|v\|)],$$

and

$$\|G(t,u,v)\|_{\mathcal{P}} := \sup\{|g| : g \in G(t,u,v)\} \le 2\left[\frac{t^2+1}{t^3+120}\right] = p_2(t)[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

with $p_1(t) = \dfrac{t}{108t^2+32}$, $p_2(t) = \dfrac{t^2+1}{t^3+120}$, $\psi_1(\|u\|) = \phi_1(\|v\|) = \psi_2(\|u\|) = \phi_2(\|v\|) = 1$. Furthermore, it is found that $N > N_1$, where $N_1 = 0.81272506$ (N is given in (H_3)). Clearly all the hypotheses of Theorem 3.3 are satisfied. Thus, there exists at least one solution for the problem (5.1) on $[0,3]$.

(b) For illustrating Theorem 4.3, we take the multivalued maps $F, G : [0,3]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ as

$$F(t,u,v) = \left[\Big(\frac{1}{4t+150}\Big)\Big(\frac{|u(t)|}{|u(t)|+1}, \sin v(t)\Big) + \frac{1}{175}, 0\right],$$

$$G(t,u,v) = \left[\Big(\frac{1}{3t^2+140}\Big)\Big(\tan^{-1}\, u(t), \frac{|v(t)|}{1+|v(t)|}\Big) + \frac{1}{170}, 0\right]. \tag{5.2}$$

Letting $\mathcal{B}_1(t) = \dfrac{1}{4t+150}$ and $\mathcal{B}_2(t) = \dfrac{1}{3t^2+140}$, we find that $H_d(F(t,u,v), F(t,\bar{u},\bar{v})) \le \mathcal{B}_1(t)(|u-\bar{u}|+|v-\bar{v}|)$ and $H_d(G(t,u,v), G(t,\bar{u},\bar{v})) \le \mathcal{B}_2(t)(|u-\bar{u}|+|v-\bar{v}|)$. Observe that $d(0, F(t,0,0)) = \dfrac{1}{175} \le \mathcal{B}_1(t)$ and $d(0, G(t,0,0)) = \dfrac{1}{170} \le \mathcal{B}_2(t)$ for almost all $t \in [0,3]$. Obviously $\|\mathcal{B}_1\| = 1/150$ and $\|\mathcal{B}_2\| = 1/140$ and

$$\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| \approx 0.4201443466 < 1.$$

Consequently, all the assumptions of Theorem 4.3 hold true. Therefore, by conclusion of Theorem 4.3, the problem (5.1) with F, G given by (5.2), has at least one solution on $[0,3]$.

6. Conclusions

We have developed the existence theory for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions supplemented with nonlocal integral multi-strip coupled boundary conditions on an arbitrary domain. Our study includes the cases of convex as well as non-convex multi-valued maps. Nonlinear alternative of Leray-Schauder type for multi-valued maps and Covitz and Nadler fixed point theorem for contractive multi-valued maps are applied to prove the main results. Numerical examples are constructed for the illustration of the obtained results. Our results are new in the given configuration and enrich the related literature. Moreover, several new results can be recorded as special cases of the present work by fixing the parameters appearing in the system. For example, we obtain the existence results for an anti-periodic multi-valued boundary value problem of self-adjoint coupled second-order ordinary differential inclusions by fixing $\alpha_i = 1, \beta_i = 1, \lambda_i = 0, i = 1,2,3,4$ in the results of this paper, which are indeed new.

7. Acknowledgement

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-MSc-53-130-1443). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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