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Stochastic delayed fractional-order differential equations driven by fractional Brownian motion

A.M. Sayed Ahmed *[1]

[1] *Department of Mathematics and Computer Science, Faculty of Sciences, Alexandria University, Alexandria, Egypt.*

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Abstract. In this paper, we presents results on existence and uniqueness of mild solutions to stochastic differential equations with time delay driven by fractional Brownian motion (fBM) with Hurst index $(1/2, 1)$ in a Hilbert space with non-Lipschitzian coefficients.

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Contents

1. Introduction and Background

Fractional differential equations have been widely applied in many fields of science and engineering, such as physics ([1]-[3]), chemical ([4]-[6]), etc. For example the nonlinear oscillation of earthquake can be modeled with fractional derivatives [7] and the fluid dynamic traffic model with fractional derivatives ([8]) can eliminate the deficiency arising from the assumption of continuum traffic flow. Actually, the concepts of fractional derivatives are not only generalization of the ordinary derivatives, but also it has been found that they can efficiently and properly describe the behavior of many real-life phenomena more accurately than integer order derivatives.

Stochastic differential equations (SDEs) are playing an increasingly important role in applications to finance, physics, and biology. A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process. Stochastic differential equations are considered by many authors (see for example, ([9])) where the stochastic disturbances are described by stochastic integrals with respect to

*Corresponding author. Email address: **ahmed.sibrahim@alexu.edu.eg** (A.M. Sayed Ahmed)

semimartingale (Brownian motion processes). However, the Brownian motion process is not suitable to represent a noise process if long-range dependence is modeled. It is then desirable to replace the Brownian motion process by fractional Brownian motion (fBM).

Fractional Brownian motion appears naturally in the modeling of many situations, for example , when describing the level of water in a river as a function of time, Financial turbulence. The existence of the fBM follows from the general existence theorem of centered Gaussian processes with given covariance functions ([10]). The fBM is divided into three very different families corresponding to $0 < H < 1/2$, $H = 1/2$ and $1/2 < H < 1$, respectively. The fBM (B^H) is not a semimartingale, as a result , the usual Itô calculus is not available for use. When $H > 1/2$, it happens that the regularity of the sample paths of B^H is enough and allows for using Young integral. In the case that $H < 1/2$ a powerful approach (Rough path theory) may be used.

In Ferrante and Rovira ([11]), the existence and uniqueness of solutions and the smoothness of the density for delayed SDEs driven by fBM is proved when $H > 1/2$, but under strong hypotheses, using only techniques of the classical stochastic calculus, and preventing, for instance, the presence of a hereditary drift in the equations. Neuenkirch et al. ([12]), using rough path theory, the authors prove existence and uniqueness of solutions to fractional equations with delays when $H > 1/3$. Recently, T. Caraballo et al. ([13]) prove the existence of solutions to stochastic delay evolution equations with a fBM.

Inspired by the above discussions, in this paper we study the following fractional stochastic differential equations (FSDEs) described in the form:

$$\begin{aligned} {}^{C}D_t^{\alpha}u(t) &= [Au(t) + f(t, u(\tau(t)))] + \sigma(t)\frac{dB_Q^H}{dt}, \quad 0 \le t \le T \\ u(t) &= \phi(t), \qquad -r \le t \le 0 \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup, $\{S(t)\}_{t\ge 0}$, of bounded linear operators in a separable Hilbert space $\mathfrak{X}$; B_Q^H is a fBM on a Hilbert space $\mathcal{Y}$, f and σ are given functions, $\tau : [0, \infty) \to [0, \infty)$ is a suitable delay function and $\phi : [-r, 0] \times \Omega \to \mathfrak{X}$ is the initial value.

The outline of this paper is structured as follows: section 2 contains some notations and preliminary facts. In section 3, the existence and uniqueness of solutions for equation (1.1) are established. The last section contains an example to illustrate our main results.

In the next part we give a brief review and preliminaries needed to establish our results.

Definition 1.1. *The Reimann-Liouville fractional derivative of f is defined as*

$$ {}^{R}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds $$

where $t > 0$, $n - 1 < \alpha < n$, $\Gamma(\cdot)$ stands for the gamma function and $n = [\alpha] + 1$ with $[\alpha]$ denotes the integer part of α (see e.g., [14]).

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^{α} proposed by M. Caputo in his work on the theory of viscoelasticity.

Definition 1.2. *The Caputo-type derivative of order α for a function f can be written as*

$$ {}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds $$

where $t > 0$, $n - 1 < \alpha < n$. (see e.g., [14]).

Remark 1.1. *1. The relationship between the Riemann-Liouville derivative and the Caputo-type derivative can be written as*

$$^{C}D_t^{\alpha} f(t) = {}^{R}D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)$$

2. The Caputo-type derivative of a constant is equal to zero.

$$\mathcal{I}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \qquad t > 0. \tag{1.2}$$

Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space and let $\{\beta^H(t), t \in [0,T]\}$ the one-dimensional fractional Brownian motion with Hurst index $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H})$$

It is known that β^H has the following Wiener integral representation (see, for example, [10]):

$$\beta^H(t) = \int_0^t \mathbb{K}_H(t,s) dB(s)$$

where $B = \{B(t) : t \in [0,T]\}$ is a standard Brownian motion process and $\mathbb{K}_H(t,s)$ is an explicit square integrable kernel given by

$$\mathbb{K}_H(t,s) = \mathbb{C}_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s$$

where

$$\mathbb{C}_H = \sqrt{\frac{H(2H-1)}{\int_0^t (1-x)^{1-2H} x^{H-\frac{3}{2}} dx}} = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$$

and $\beta(\cdot,\cdot)$ denotes the Beta function. Let $\mathcal{H}$ be the closure of the set of indicator functions $\{\mathbb{I}_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle \mathbb{I}_{[0,t]}, \mathbb{I}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$$

We recall that for $\varphi, \psi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by ([15]):

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \varphi(s)\psi(t) \mid t-s \mid^{2H-2} dsdt$$

Let the operator $\mathbb{K}_H^* : \mathcal{H} \to \mathcal{L}^2([0,T])$ defined by ([15]):

$$(\mathbb{K}_H^* \varphi)(s) = \int_s^T \varphi(\tau) \frac{\partial \mathbb{K}_H}{\partial \tau}(\tau, s) d\tau$$

and for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T \mathbb{K}_H^*(\varphi)(t) dB(t)$$

It is known that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $\mathcal{H}^*$ generated by the measurable functions ψ such that

$$\| \psi \|_{\mathcal{H}^*}^2 := H(2H-1) \int_0^T \int_0^T \mid \psi(\tau) \| \psi(s) \| \tau - s \mid^{2H-2} d\tau ds$$

It is clear that, the space $(\mathcal{H}^*; \| \psi \|^2_{\mathcal{H}^*})$ is a Banach space and we have, ([10]):

$$\mathcal{L}^2([0,T]) \subseteq \mathcal{L}^{\frac{1}{H}}([0,T]) \subseteq \mathcal{H}^* \subseteq \mathcal{H}$$

and for any $\varphi \in \mathcal{L}^2([0,T])$, we have

$$\|\psi\|^2_{\mathcal{H}^*} \leq 2HT^{2H-1} \int_0^T | \psi(s) |^2 \, ds$$

Let $\mathfrak{L}(\mathcal{Y}, \mathfrak{X})$ be the space of bounded linear operator from $\mathcal{Y}$ to $\mathfrak{X}$ and let $Q \in \mathfrak{L}(\mathcal{Y}, \mathcal{Y})$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $TrQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, $\lambda_n \geq 0$ are nonnegative real numbers and e_n is a complete orthonormal basis in $\mathcal{Y}$. Let $\mathcal{B}^H_Q = \{\mathcal{B}^H_Q(t)\}$ be $\mathcal{Y}$-valued fBM on $(\Omega, \Im, \mathbb{P})$ with covariance Q defined as:

$$\mathcal{B}^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t) e_n \sqrt{\lambda_n}$$

It is clear that the process $\mathcal{B}^H_Q$ is Gaussian, it starts from zero, has zero mean and covariance

$$\mathbb{E}[\langle \mathcal{B}^H_Q(t), x\rangle \langle \mathcal{B}^H_Q(s), y\rangle] = R(t,s)\langle Q(x), y\rangle, \quad x, y \in \mathcal{Y}, \quad t, s \in [0,T]$$

In order to define Wiener integrals with respect to the Q-fBM, we introduce the space $\mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ of all Q-Hilbert-Schmidt operators $\Psi : \mathcal{Y} \to \mathfrak{X}$. We recall that $\Psi \in \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ is called a a Q-Hilbert-Schmidt operator if

$$\| \Psi \|^2_{\mathbb{L}^2} := \sum_{n=1}^{\infty} \left\| \Psi e_n \sqrt{\lambda_n} \right\|^2 < \infty$$

We note that the space $\mathbb{L}^2$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathbb{L}^2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$$

is a separable Hilbert space ([13]). Now, the Wiener integral of $\varphi \in \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ with respect to $\mathcal{B}^H_Q$ is defined by:

$$\int_0^t \varphi(s) dB^H_Q(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) \sqrt{\lambda_n} e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \mathbb{K}^*_H(\varphi e_n)(s) \sqrt{\lambda_n} dB_n(s)$$

Lemma 1.1. *If $\Phi : [0,T] \to \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ satisfies $\int_0^T \| \Phi(s) \|^2_{\mathbb{L}^2} \, ds < \infty$. Then the above sum in the previous equation is well-defined as a $\mathfrak{X}$-valued random variable and we have:*

$$\mathbb{E} \left\| \int_0^t \Phi(s) d\mathcal{B}^H_Q \right\|^2 \leq 2HT^{2H-1} \int_0^t \|\Phi(s)\|^2_{\mathbb{L}^2} \, ds.$$

We recall that for any strongly continuous semigroup $\{S(t); t \geq 0\}$ on $\mathfrak{X}$, we define the generator

$$Au = \lim_{t \to 0^+} \frac{S(t)u - u}{t}.$$

Throughout this paper, let A is the infinitesimal generator of a strongly continuous semigroup $\{S(t); t \geq 0\}$ of operators on a Hilbert space $\mathfrak{X}$. Clearly,

$$M = \sup_{t \in [0,T]} \| S(t) \| < \infty.$$

We suppose that $\| S(t) \| \leq C_1$

Lemma 1.2. *([16],[18]) Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function and let also g, h and λ be non-negative functions on $\mathbb{R}_+$ such that:*

$$g(t) \leq h(t) + \int_0^t \lambda(s)\xi(g(s))ds, \quad t \geq 0$$

, then

$$g(t) \leq \rho^{-1}\left\{\rho(h^*(t)) + \int_0^t \lambda(s)ds\right\},$$

where $\rho(x) = \int_{t_0}^t \frac{dx}{\xi(x)}$ is well-defined for $t_0 > 0$ and $h^*(t) = \sup_{s\leq t} h(s)$. In particular, we have the Gronwall-Bellman Lemma: If

$$g(t) \leq h(t) + \int_0^t \lambda(s)g(s)ds, \quad t \geq 0$$

, then

$$g(t) \leq h^*(t)e^{\int_0^t \lambda(s)ds}.$$

Definition 1.3. *A $\mathfrak{X}$-valued process $\{u(t), t \in [-r, T]\}$ is called a mild solution of equation (1.1) if:*

1. *$u(t) \in \mathbf{C}([-r, T], \mathcal{L}^2(\Omega, \mathfrak{X}))$,*
2. *$u(t) = \phi(t), \qquad -r \leq t \leq 0$,*
3. *For any $t \in [0, T]$, we have*

$$u(t) = J(t)\phi(0) + \int_0^t J^*(t-s)f(s, u(\tau(s)))ds + \int_0^t J^*(t-s)\sigma(s)d\mathcal{B}_Q^H(s), \quad a.s.$$

where

$$J(t) = \int_0^\infty M_\alpha(\theta)S(t^\alpha\theta)d\theta,$$
$$J^*(t) = \alpha\int_0^\infty \theta t^{\alpha-1}M_\alpha(\theta)S(t^\alpha\theta)d\theta$$

and $M_\alpha(\theta) \geq 0$ is a probability function on $(0, \infty)$, that is

$$M_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\omega_\alpha(\theta^{-\frac{1}{\alpha}}),$$
$$\omega_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin n\pi\alpha.$$

and $\int_0^\infty M_\alpha(\theta)d\theta = 1$

Lemma 1.3. *([19]) The operators J and J^* have the following properties:*

1. *For any fixed $t \geq 0$, $J(t)$ and $J^*(t)$ are linear and bounded, i.e., for any $x \in \mathfrak{X}$*

$$\| J(t)x \| \leq C_2 \| x \|, \quad \| J^*(t)x \| \leq \frac{C_2 T^\alpha}{\Gamma(\alpha+1)} \| x \|$$

2. *$\{J(t), t \geq 0\}$ and $\{J^*(t), t \geq 0\}$ are strongly continuous.*
3. *For every $t > 0$, $J(t)$ and $J^*(t)$ are compact operators if $S(t)$ is compact.*

2. Main Results

To prove the existence and the uniqueness of mild solutions of equation (1.1), the following weaker conditions (instead of the Lipschitz and linear growth conditions, see, e.g., ([16],[17])) are listed:

($\mathcal{H}1$) $f:[0,T]\times\mathfrak{X}\to\mathfrak{X}$ and $\sigma:[0,T]\to\mathbb{L}^2(\mathcal{Y},\mathfrak{X})$ are satisfying the following conditions: there exists a function $\zeta:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that:

1. For all t, $\zeta(t,\cdot)$ is continuous non-decreasing , concave, and for each fixed $x\in\mathbb{R}_+$, $\int_0^T\zeta(s,x)ds<\infty$.
2. For any $t\in[0,T]$ and $x\in\mathfrak{X}$
$$\|f(t,x)\|^2\le\zeta(t,\|x\|^2),\quad \int_0^T\|f(t,x)\|^2_{\mathbb{L}^2}<\infty.$$
3. For any constant $q>0$, $x_0\ge 0$, the integral equation
$$x(t)=x_0+q\int_0^t\zeta(s,x(s))ds$$
has a global solution on $[0,T]$.

($\mathcal{H}2$) There exists a function $\eta:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that:

1. For all t, $\eta(t,\cdot)$ is continuous non-decreasing , concave, with $\eta(t,0)=0$, and for each fixed $x\in\mathbb{R}_+$, $\int_0^T\eta(s,x)ds<\infty$.
2. For any $t\in[0,T]$ and $x,y\in\mathfrak{X}$
$$\|f(t,x)-f(t,y)\|^2\le\eta(t,\|x-y\|^2).$$
3. For any constant $C_3>0$, if a non-negative function $h(t),t\in[0,T]$ satisfies $h(0)=0$ and
$$h(t)\le C_3\int_0^t\eta(s,h(s))ds$$
, then $h(t)=0,\forall t\in[0,T]$.

($\mathcal{H}3$) τ is a continuous function satisfying the condition:
$$-r\le\tau(t)\le t,\qquad \forall t\ge 0$$

($\mathcal{H}4$) we assume that $\phi\in\mathbf{C}([-r,T],\mathcal{L}^2(\Omega,\mathfrak{X}))$

Lemma 2.1. *Let $b\in\mathcal{L}^2([0,T],\mathfrak{X})$, $\tilde\sigma\in\mathcal{L}^2([0,T],\mathbb{L}^2)$ and consider the following equation:*

$$\begin{aligned}{}^CD_t^\alpha u(t)&=[Au(t)+b(t)]+\tilde\sigma(t)\frac{dB_Q^H}{dt},\quad 0\le t\le T\\ u(t)&=\phi(t),\qquad -r\le t\le 0\end{aligned}\tag{2.1}$$

, then equation (2.1) has a unique mild solution on $[-r,T]$

Proof. Let $\mathcal{C}_T := \mathbf{C}([-r,T],\mathcal{L}^2(\Omega,\mathfrak{X}))$ be a Banach space of all continuous functions from $[-r,T]$ into $\mathcal{L}^2(\Omega,\mathfrak{X})$, endowed with the norm

$$\|u\|_{\mathcal{C}_T}^2 = \sup_{-r\leq t\leq T} \|u(t,\omega)\|^2, \quad \omega\in\Omega. \tag{2.2}$$

Let us consider,

$$\mathcal{G}_T := \{u\in\mathcal{C}_T : u(s)=\phi(s), s\in[-r,0]\}.$$

It is clear that, $\mathcal{G}_T$ is a closed subset of $\mathcal{C}_T$ provided with the norm (2.2).
Let $\mathcal{F}$ be the function defined on $\mathcal{G}_T$ by:

$$\begin{aligned}
\mathcal{F}_x(t) &= \phi(t), \qquad t\in[-r,0],\\
\mathcal{F}_x(t) &= J(t)\phi(0) + \int_0^t J^*(t-s)b(s)ds + \int_0^t J^*(t-s)\tilde{\sigma}(s)d\mathcal{B}_Q^H(s), \quad t\in[0,T]\\
&= \sum_{k=1}^{3} I_k.
\end{aligned}$$

In the next step, we are going to prove that each function $t\mapsto I_k, k=1,2,3$ is contionuous on $[0,T]$ in the $\mathcal{L}^2(\Omega,\mathfrak{X})$ sense.

1. The contionuity of I_1 followes directly from the continuity of $t\mapsto J(t)z$ (see, Lemma (1.3)), and by using some simple computations we can show the continuity of I_2.

2. For the term I_3, by using (Lemma 1.1, Lemma 1.3), we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^{t+z} J^*(t+z-s)\tilde{\sigma}(s)d\mathcal{B}_Q^H(s) - \int_0^t J^*(t-s)\tilde{\sigma}(s)d\mathcal{B}_Q^H(s)\right\|\\
&\quad\leq \left\|\int_0^t (J^*(t+z-s)-J^*(t-s))\tilde{\sigma}(s)d\mathcal{B}_Q^H(s)\right\|\\
&\qquad+\left\|\int_t^{t+z} J^*(t+z-s)\tilde{\sigma}(s)d\mathcal{B}_Q^H(s)\right\|\\
&\leq I_{31}(z)+I_{32}(z).
\end{aligned}$$

It is clear that, $I_{31}, I_{32}\to 0$ as $z\to 0$, and then

$$\lim_{z\to 0}\mathbb{E}\|\mathcal{F}_x(t+z)-\mathcal{F}_x(t)\|^2 = 0.$$

Hence, we conclude that the function $\mathcal{F}_x(t)$ is continuous on $[0,T]$ in the $\mathcal{L}^2(\Omega,\mathfrak{X})$ sense. By using classical computations we can show that

$$\sup_{-r\leq t\leq T}\mathbb{E}\|\mathcal{F}_x(t)\|^2 < \infty.$$

Hence, we conclude that $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}$ is a contraction mapping in $\mathcal{G}_{T_1}$ with some $T_1\leq T$ and therefore has a unique fixed point, which is a mild solution of equation (2.1) on $[0,T_1]$. This procedure cen be repeated in order to extend the solution to the entire interval $[-r,T]$ in finitly many steps.

■

By using a Picard type iteration with the help of Lemma (2.1), we can construct a successive approximation sequence as: Let u_0 be the solution of equation (2.1) with $b \equiv 0$ and $\tilde{\sigma} \equiv 0$. For $n \geq 0$, let u_{n+1} ne the solution of equation (2.1) with $b(t) \equiv f(t, u(\tau(t)))$ and $\tilde{\sigma}(t) \equiv \sigma(t)$. Therefore,

$$\begin{aligned} u_{n+1}(t) &= \phi(t), \qquad t \in [-r, 0], \\ u_{n+1}(t) &= J(t)\phi(0) + \int_0^t J^*(t-s) f(t, u_n(\tau(t))) ds \\ &\quad + \int_0^t J^*(t-s)\sigma(s) d\mathcal{B}_Q^H(s), \quad t \in [0, T] \end{aligned} \tag{2.3}$$

Lemma 2.2. *Let* $(\mathcal{H}1 - \mathcal{H}4)$ *hold. The sequence* $\{u_n, n \geq 0\}$ *is well-defined and there exist positive constants* C_4, C_5, C_6 *such that* $\forall n, m \in \mathbb{N}$ *and* $t \in [0, T]$, *we have:*

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_{m+1}(s) - u_{n+1}(s)\|^2 \leq C_4 \int_0^t \eta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_m(\theta) - u_n(\theta)\|^2) ds \tag{2.4}$$

and

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_{n+1}(s)\|^2 \leq C_5 + C_6 \int_0^t \zeta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_n(\theta)\|^2) ds \tag{2.5}$$

Proof. For inequality (2.4), we have

$$\|u_{m+1}(t) - u_{n+1}(t)\|^2 = \left\| \int_0^t J^*(t-s)[f(t, u_m(\tau(t))) - f(t, u_n(\tau(t)))] ds \right\|^2.$$

By using condition $(\mathcal{H}2)$, we get

$$\sup_{-r \leq s \leq t} \mathbb{E} \left\| \int_0^t J^*(t-s)[f(t, u_m(\tau(t))) - f(t, u_n(\tau(t)))] ds \right\|^2 \leq C_4 \int_0^t \eta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_m(\theta) - u_n(\theta)\|^2) ds.$$

and hence the result. For inequality (2.5), we have

$$\|u_{n+1}(t)\|^2 = \left\| J(t)\phi(0) + \int_0^t J^*(t-s)[f(t, u_n(\tau(t)))] ds + \int_0^t J^*(t-s)\sigma(s) d\mathcal{B}_Q^H(s) \right\|^2.$$

By using the identity $\| x + y \|^2 \leq 2 \| x \|^2 + 2 \| y \|^2$, Lemma (1.3), condition $\mathcal{H}2$ and Lemma (1.1), we get

$$\sup_{-r \leq s \leq t} \|u_{n+1}(s)\|^2 \leq C_5 + C_6 \int_0^t \zeta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_n(\theta)\|^2.$$

■

Lemma 2.3. *Suppose that* $(\mathcal{H}1 - \mathcal{H}4)$ *are satisfied, then there exists an* $x(t)$ *such that*

$$x(t) = x_0 + q \int_0^t \zeta(s, x(s)) ds \tag{2.6}$$

for $x_0 \geq 0$, $q > 0$ *and the sequence* $\{u_n, n \geq 0\}$ *satisfies, for all* $n \in \mathbb{N}, t \in [0, T]$

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_n(s)\|^2 \leq x(t). \tag{2.7}$$

Proof. Let $x : [0, T] \to \mathbb{R}$ be a global solution of the integral equation (2.6) with an initial condition $x_0 = \max(C_6, \sup_{-r \leq t \leq T} \mathbb{E} \| u_0(t) \|^2)$, then by using mathematical induction we can prove that inequality (2.7). ■

Theorem 2.1. *Let $(\mathcal{H}1-\mathcal{H}4)$ be satisfied. Then for all $T>0$, the equation (1.1) has a unique mild solution on* $[-r,T]$.

Proof. Existence: For $t\in[0,T]$, by using Lemma (2.2), Lemma (2.3) and Fatou's theorem, we get:

$$\limsup_{m,n\to\infty}\{\sup_{-r\le s\le t}\mathbb{E}\,\|u_{m+1}(s)-u_{n+1}(s)\|^2\}\le C_4\int_0^t\eta(s,\limsup_{m,n\to\infty}\sup_{-r\le\theta\le s}\mathbb{E}\,\|u_m(\theta)-u_n(\theta)\|^2)ds.$$

By using condition $(\mathcal{H}2)$, we have

$$\lim_{m,n\to\infty}\sup_{-r\le s\le T}\mathbb{E}\,\|u_m(s)-u_n(s)\|^2)=0.$$

Hence, the sequence $\{u_n\}_{n\ge0}$ is a cauchy sequence in $\mathcal{C}_T$ and from the completeness of $\mathcal{C}_T$ we guarantees the existence of a process $u\in\mathcal{C}_T$ such that

$$\lim_{n\to\infty}\sup_{-r\le s\le T}\mathbb{E}\,\|u_n(s)-u(s)\|^2)=0,$$

and if $n\to\infty$ in equation (2.3), then we can see that u is a mild solution to equation (1.1) on $[-r,T]$.
Uniqueness: Let u,v be two mild solutions of equation (1.1), then

$$\sup_{-r\le s\le t}\mathbb{E}\,\|u(s)-v(s)\|^2\le C_4\int_0^t\eta(s,\sup_{-r\le\theta\le s}\mathbb{E}\,\|u(\theta)-v(\theta)\|^2)ds$$

and by using condition $(\mathcal{H}2)$, we get $\sup_{-r\le s\le t}\mathbb{E}\parallel u(s)-v(s)\parallel^2=0$, which implies that $u\equiv v$. ■

3. Applications.

In this section, we give an example to illustrate our main results.

Example 3.1.

$$\begin{aligned}
&{}^C D_t^{1/2}[u(t,\zeta)]=\frac{\partial^2}{\partial\zeta^2}u(t,\zeta)+\frac{e^{-2t}u(\frac{1}{3}(1+\cos t))}{70(1+u^2(\frac{1}{3}(1+\cos t))}+e^{-\pi^2 t}\frac{dB_Q^H}{dt},\quad t\in(0,T],\zeta\in[0,\pi]\\
&u(t,0)=u(t,\pi)=0,t\in(0,T],\\
&\qquad u(t,\zeta)=\phi(t,\zeta),\qquad -r\le t\le 0.
\end{aligned}\tag{3.1}$$

where $A:D(A)\subset\mathfrak{X}\to\mathfrak{X}$, which is defined by $A\varpi=\varpi''$ with $D(A)=\{u\in\mathfrak{X}:u''\in\mathfrak{X},u(0)=u(\pi)=0\}$, u,u' are absolutely continuous and then A can be written as $Au=\sum_{n=1}^{\infty}n^2\langle u,u_n\rangle u_n$ where $u_n(s)=\sqrt{\frac{2}{\pi}}\sin(nu)$ is the orthonormal set of eigenvectors of A. Also A is the the infinitesimal generator of an analytic semigroup, $\{S(t)\}_{t\ge0}$ in $\mathfrak{X}$ and there exists M, such that $\|S(t)\|\le M$. From (3.1), we know that the delay term $\frac{1}{3}(1+\cos t)$ and

$$f(t,u)=\frac{e^{-2t}u}{70(1+u^2)},$$
$$\sigma(t)=e^{-\pi^2 t}$$

and with the above choices (3.1) can be formulated in the abstract form of (1.1) and it is easy to verify the conditions of Theorem (2.1) all hold, and then (3.1) must have a mild solution on $[0,1]$.

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