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Approximate and exact solution of Korteweg de Vries problem using Aboodh Adomian polynomial method

OLUDAPO OMOTOLA OLUBANWO[1], JULIUS TEMITATO ADEPOJU*[1], ABOIODUN SUFIAT AJANI[1], SENAYON SUNDAY IDOWU[1]

[1] *Department of Mathematical Sciences, Olabisi Onabanjo University, Ago-Iwoye, Nigeria.*

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Abstract. This study introduce Aboodh Adomian polynomial Method (AAPM) to solve nonlinear third order KdV problems providing it approximate and exact solution. To get the approximate analytical answers to the issues, the Aboodh transform approach was used. Given that the Aboodh transform cannot handle the nonlinear elements in the equation, the Adomian polynomial was thought to be a crucial tool for linearizing the associated nonlinearities. All of the issues examined demonstrated the strength and effectiveness of the Adomian polynomial and Aboodh transform in solving various nonlinear equations when compared to other well-known methods. To show how this strategy may be applied and is beneficial, three cases were examined.
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Contents

1. Introduction

In engineering, physics, and other fields of study, nonlinear models play a crucial role in explaining new phenomena. However, in certain situations, it could be difficult to provide an exact analytical solution for nonlinear problems [3]. To address nonlinear issues, a variety of numerical techniques were employed, and as these techniques improved, so did the analytical techniques. Particular focus has recently been paid to the merging of numerical and analytical methodologies. A technique for solving nonlinear differential equations in series is the homotopy approach, which was developed by He [9, 10]. Easy and straightforward execution are

*Corresponding author. Email address: **olubanwo.oludapo@oouagoiwoye.edu.ng** (Oludapo Omotola Olubanwo), **adepojujulius58@gmail.com** (Julius Temitayo Adepoju), **ajani.abiodun@oouagoiwoye.edu.ng** (Abiodun Sufiat Ajani), **idowusunday288@gmail.com** (Senayon Sunday, Idowu)

the technique's advantages. In engineering, physics, and other disciplines, nonlinear models play a crucial role in explaining new phenomena. However, it might occasionally be challenging to provide a precise analytical solution for nonlinear situations [3]. Numerous numerical techniques were employed to address nonlinear issues, and advancements in these techniques led to advancements in analytical techniques. Combining analytical and numerical approaches has drawn a lot of interest lately. Among these techniques is the homtopy approach, which He developed in order to solve nonlinear differential equations in series [9, 10]. The method's ease of use and effortlessness are advantages.

In the field of nonlinear physics, the KdV problem is a crucial PDE that appears while studying solitons and waves. The KdV equation, which bears the names of the Dutch scientists D.J. Korteweg and G. de Vries who first proposed it in 1895, provides a mathematical explanation for a variety of scientific phenomena, including shallow water waves and electrical pulses in nerve fibers. [6]. The term "soliton," referring to a solution of a non-linear PDE, was initially utilized by Zabusky and Kruskal [13].

A KdV problem in third order is expressed as [4]:

$$\phi_\tau + a\phi\phi_z + b\phi\phi_{zzz} = 0 \tag{1.1}$$

with

$$\phi\,(z,0) = \phi(z,\tau) \tag{1.2}$$

a and b	are arbitrary constants
ϕ_z	Partial derivatives w.r.t z
ϕ_τ	Partial derivatives w.r.t. τ

Various methods, including the combination of LTHPM [8], have been employed to obtain approximate analytical solutions and numerical results for KdV equations and other nonlinear PDE. Achieving exact solutions involved utilizing graphical representation for the KdV equation [4], as well as employing the homotopy perturbation method with Elzaki transform [5, 8] combined with Aboodh transform for approximate solutions of certain third-order KDV equations with initial conditions. Additionally, numerical techniques for partial differential equations [1], methods like the Adomian Polynomial and Elzaki Transform [11], NTHPM [2], and HPM using Mahgoub Transform [6] have been investigated as computational approaches for KdV problem on an unrestricted domain. The combination of Elzaki Transformation with Adomian Polynomial is also considered [11].

In this study, we want to improve the efficiency of the Aboodh transform method by its integration with the Adomian polynomial approach. The combined approach known as "Aboodh Transform and Adomian Polynomial (AAPM) for Solving Third Order Korteweg-De Vries (KDV) Equation" is used in this situation. Usually, this approach takes several stages to get an accurate result, presenting the outcome as an approximation analytic solution in a series structure.

2. Main Results

2.1. Aboodh Transform Method (ATM)

Differential equations are solved with the application of the Aboodh transform and some of its basic characteristics. For exponentially ordered functions, the Aboodh Transform is defined.

Definition 2.1. The Aboodh transform, defined on set A of functions, is the new term for the integral transform.

$$A = \{\phi(\tau)|\ \exists\ M, k_1, k_2 > 0, |\phi(\tau)| < Me^{-v\tau}\} \tag{2.1}$$

where $k_1, k_2 \subset M$. The symbol for the Aboodh transform is $\mathcal{A}[\phi(\tau)]$. The integral equation of the form is represented by .

$$\mathcal{A}[\phi(\tau)] = q(v) = \frac{1}{v}\int_0^\infty \phi(\tau)e^{-v\tau}d\tau, \quad \tau \geq 0,\ k_1 \leq v \leq k_2 \tag{2.2}$$

Definition 2.2. Aboodh Transform for function $\phi(\tau)$ of exponential order over the set of function is defined as

$$\mathcal{A}\left\{f : |\phi(\tau) < Me_j^k|\tau|,\ \text{if}\ \ \tau \in (-1)^j \times [0, \infty], \quad j = 1, 2, ...(M, k_1, k_2 > 0)\right\} \tag{2.3}$$

where $\phi(\tau)$ is denoted by

$$\mathcal{A}[\phi(\tau)] = \mathcal{H}(v)$$

and defined as

$$\mathcal{A}[\phi(\tau)] = \frac{1}{v}\int_0^\infty \phi(\tau)e^{-v\tau}d\tau = \mathcal{H}(v), \qquad t < 0,\ k_1 \leq v \leq k_2$$

2.2. Aboodh of basic functions

Using definition 2.2, one can show as follows.

1. Let $\phi(z, \tau) = e^{az+b\tau}$, then its Aboodh transform w.r.t τ is given by

$$\begin{aligned}
\mathcal{A}\{e^{az+b\tau}\} &= \frac{1}{v}\int_0^\infty e^{az+b\tau}e^{-v\tau}d\tau \\
&= \frac{e^{az}}{v}\int_0^\infty e^{b\tau-v\tau}d\tau \\
&= \frac{e^{az}}{v}.\frac{-1}{b-v} = \frac{e^az}{v(v-b)}, \quad v > b
\end{aligned}$$

2. Let $\phi(z, \tau) = z^m\tau^n$, then its Aboodh transform w.r.t τ is given by

$$\begin{aligned}
\mathcal{A}\{z^m\tau^n\} &= \int_0^\infty z^m\tau^n e^{-v\tau}d\tau \\
&= \frac{z^m}{v}\int_0^\infty \tau^n e^{-v\tau}d\tau = \frac{z^m}{v}\int_0^\infty \tau^n e^{-v\tau}d\tau \\
&= \frac{z^m}{v}.\frac{\Gamma(n+1)}{v^(n+1)} \\
&= \frac{n!z^m}{v^{n+2}}, \quad v > 0
\end{aligned}$$

3. Let $\phi(z, \tau) = \sin(az + b\tau)$, then its Aboodh transform w.r.t τ is given by

$$\mathcal{A}\{\sin(az + b\tau) = \frac{1}{v}\int_0^\infty \sin(az + b\tau)e^{-v\tau}\}$$

Applying repeated integration by parts leads to

$$\begin{aligned}
&= \mathcal{A}\{\sin(az + b\tau)\} = \frac{1}{v}\left(-\frac{e^{-v\tau}(v\sin(b\tau + az) + b\cos(b\tau + az))}{v^2 + b^2}\right)\Bigg|_{t=0}^{\infty} \\
&= \frac{v\sin(az) + b\cos(az)}{v(v^2 + b^2)}, \qquad v > 0
\end{aligned}$$

4. Let $\phi(z,\tau)=\sinh(az+b\tau)$ then its Aboodh transform w.r.t τ is given by

$$\mathcal{A}\{\sinh(az+b\tau)\}=\frac{1}{v}\sinh(az+b\tau)e^{-v\tau}d\tau.$$

By using properties of hyperbolic functions and integration by parts, we have

$$\begin{aligned}\mathcal{A}\{\sinh(az+b\tau)\}&=\frac{1}{v}\int_0^{\infty}[\cosh(az)\sinh(b\tau)+\sinh(az)\cosh(b\tau)]e^{-v\tau}d\tau\\&=\frac{1}{v}\left(-\frac{((v+b)e^{2(b\tau+az)}-v+b)e^{(v+b)\tau-az}}{2(v-b)(v+b)}\right)\Bigg|_{t=0}^{\infty}\\&=\frac{v\sinh(az)+b\cos(az)}{v(v^2-b^2)},\quad |v|>|b|\end{aligned}$$

If $\mathcal{A}[\phi(\tau)]=q(v)=\frac{1}{v}\int_0^{\infty}\phi(\tau)e^{-v\tau}d\tau,\quad \tau\geq 0,\ k_1\leq v\leq k_2$. Then, the Aboodh and Inverse Aboodh Transform of some Elementary functions are given below.

S/N	$\mathcal{A}^{-1}\{k(v)\}=\phi(\tau)$	$\mathcal{A}\phi(\tau)=k(v)$
1.	1	$\frac{1}{v^2}$
2.	τ^n	$\frac{n!}{v^{n+2}}$
3.	$e^{a\tau}$	$\frac{1}{v^2+a^2}$
4.	$\sin(a\tau)$	$\frac{a}{v(v^2+a^2)}$
5.	$\cos(a\tau)$	$\frac{1}{(v^2+a^2)}$
6.	$\sinh(a\tau)$	$\frac{a}{v(v^2-a^2)}$
7.	$\cosh(a\tau)$	$\frac{1}{v^2-a^2}$

Table 1: Aboodh $\{\mathcal{A}\}$ and inverse Aboodh Transform $\{\mathcal{A}^{-1}\}$ of some functions

also the Aboodh and Inverse Aboodh Transform of some derivatives is given below

S/N	$\mathcal{A}\{u(z,\tau)\}=k(v)$	$\mathcal{A}^{-1}\left[\frac{\partial^n u(z,\tau)}{\partial\tau^n}\right]$
1.	$\mathcal{A}[u(z,\tau)]$	$K(z,v)$
2	$\mathcal{A}\left[\frac{\partial u(z,\tau)}{\partial t}\right]$	$vK(z,v)-\frac{u(z,0)}{v}$
3.	$\mathcal{A}\left[\frac{\partial^2 u(z,\tau)}{d\tau}\right]$	$v^2\left[K(z,v)-u(z,0)-\frac{1}{v}u_t(z,0)\right]$
4.	$\mathcal{A}\left[\frac{\partial^n u(z,\tau)}{\partial\tau^n}\right]$	$v^nK(z,v)-\sum_{k=0}^{n-1}\frac{f^{(k)}(z,0)}{v^{2-n+k}}$

2.3. Properties of Aboodh Transform

The following properties of Aboodh transform are derived from the definition and which will be applied in the following chapter to solve the Schrödinger equation, both linear and nonlinear.

Lemma 2.3 (Linearity Property of Aboodh)**.** Let $\phi(z,\tau)$ and $\varphi(z,\tau)$ be any two functions whose Aboodh transform w.r.t exist. Then for arbitrary constants a and b, we have

$$\mathcal{A}\{a\phi(z,\tau)+b\varphi(z,\tau)\}=a\mathcal{A}\{\phi(z,\tau)\}+b\mathcal{A}\{\varphi(z,\tau)\}\tag{2.4}$$

Proof. By definition of Aboodh transform w.r.t τ, we obtain

$$\begin{aligned}\mathcal{A}\{a\phi(z,\tau)+b\varphi(z,\tau)\} &= \frac{1}{v}\int_0^\infty (a\phi(z,\tau)+b\varphi(z,\tau))e^{-v\tau}d\tau \\ &= a\left(\frac{1}{v}\int_0^\infty (\phi(z,\tau))e^{-v\tau}d\tau\right)+b\left(\frac{1}{v}\int_0^\infty \varphi(z,\tau)e^{-v\tau}d\tau\right) \\ &= a\mathcal{A}\{\phi(z,\tau)\}+b\mathcal{A}\{\varphi(z,\tau)\}\end{aligned}$$

■

2.4. Aboodh Adomian Polynomial Method (AAPM)

This study's main idea is to demonstrate the Adomian polynomial method with the Aboodh Transform by applying it to a broad category of nonlinear partial differential equations.

$$\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}+\mathbb{R}\phi(z,\tau)+\mathbb{N}\phi(z,\tau)=f(z,\tau) \tag{2.5}$$

where $q=1,2,3,\cdots$
with

$$\frac{\partial^{q-1}\phi(z,\tau)}{\partial \tau^{q-1}}(z,0)=g_{q-1}(x) \tag{2.6}$$

Taking Aboodh transform of (2.5)

$\left.\frac{\partial^q\phi(z,\tau)}{\partial\tau^q}\right\|_{\tau=0}$	q^{th} order partial derivative of $\phi(z,\tau)$
$\mathbb{R}$	linear term
$\mathbb{N}$	nonlinear terms
$\phi(z,\tau)$	represents the source term.

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}+R\phi(z,\tau)+N\phi(z,\tau)=f(z,\tau)\right] \tag{2.7}$$

Applying Aboodh linearity property to (2.7)

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}\right]+\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]=\mathcal{A}[f(z,\tau)] \tag{2.8}$$

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}\right]=\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}-\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k} \tag{2.9}$$

Substituting equation (2.9) into (2.8)

$$\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}-\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k}+\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]=\mathcal{A}[f(z,\tau)] \tag{2.10}$$

(2.11)

This result into

$$\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}=\mathcal{A}[f(z,\tau)]+\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k}-\{\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]\} \tag{2.12}$$

By simplification, equation (2.12) becomes

$$\mathcal{A}\left[\phi\left(z,\tau\right)\right]=v^{q}\left[\mathcal{A}\left[f\left(z,\tau\right)\right]\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}-v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\} \tag{2.13}$$

Taking Aboodh inverse transform of(2.13), gives

$$\mathcal{A}^{-1}\left[\phi\left(z,\tau\right)\right]=\mathcal{A}^{-1}\left[v^{q}\mathcal{A}\left[f\left(z,\tau\right)\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}\right]-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.14}$$

$$\phi\left(z,\tau\right)=\mathcal{A}^{-1}\left[v^{q}\mathcal{A}\left[f\left(z,\tau\right)\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}\right]-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.15}$$

Equation (2.15) is expressed below as

$$\phi\left(z,\tau\right)=F(z,\tau)-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.16}$$

$F(z,\tau)$ is obtained from the initial conditions given. The result obtained in (2.16) is

$$\phi\left(z,\tau\right)=\sum_{r=0}^{\infty}\phi_{r}(z,\tau) \tag{2.17}$$

The non-linearity in the equation can be simplified using Adomian polynomial as

$$N\phi\left(z,\tau\right)=\sum_{r=0}^{\infty}A_{r} \tag{2.18}$$

Where A_r represents the Adomian polynomials. It is obtained using the expression in (2.19)

$$A_{r}=\frac{1}{r!}\frac{d^{r}}{d\lambda^{r}}f\left[\sum_{i=0}^{\infty}\lambda^{i}\phi_{i}\right]_{\lambda=0}\quad r=0,\ 1,\ \cdots \tag{2.19}$$

Substituting equation (2.18) and (2.19) into (2.17) leads to

$$\sum_{r=0}^{\infty}\phi_{r}\left(z,\tau\right)=F\left(z,\tau\right)-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\sum_{r=0}^{\infty}\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]+\mathcal{A}\left[\sum_{r=0}^{\infty}A_{r}\right]\right\}\right] \tag{2.20}$$

From equation $\phi_0\left(z,\tau\right)=F\left(z,\tau\right)$. Thus, the recursive expression is hereby obtained as

$$\phi_{r+1}=-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\sum_{r=0}^{\infty}\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]+\mathcal{A}\left[\sum_{r=0}^{\infty}A_{r}\right]\right\}\right],\quad r\geq 0, \tag{2.21}$$

With truncated series, one can approximate the analytical answer $\phi(z,\tau)$.

$$\phi\left(z,\tau\right)=\lim_{r\to\infty}\sum_{r=0}^{N}\ \phi_{r}\left(z,\tau\right) \tag{2.22}$$

3. Application

In this section, we wll work through the examples that follow to demonstrate the Adomian polynomial method to illustrate how well the third-order nonlinear KDV problems may be solved by using the Aboodh Adomian Polynomial Method..

Example 3.1. *Examine the nonlinear KDV problem [6]*

.

$$\phi_\tau - 6\phi\phi_z + \phi_{zzz}(z,\tau) = 0, \tag{3.1}$$

with

$$\phi(z,0) = 6x \tag{3.2}$$

Applying Aboodh to both sides of the equation (3.1)

$$\mathcal{A}[\phi_\tau] = -\mathcal{A}[\phi_{zzz}(z,\tau) - 6\phi\phi_z] \tag{3.3}$$

Making use of the Aboodh differential properties, equation (3.3) becomes

$$v\mathcal{A}[\phi(z,\tau)] - \frac{1}{v}\phi(z,0) = -\mathcal{A}[\phi_{zzz}(z,\tau) - 6\phi\phi_z] \tag{3.4}$$

Applying the initial condition Equation (3.2) on Equation (3.4), we obtain

$$\mathcal{A}[\phi(z,\tau)] = \frac{6z}{v^2} - \left[\frac{1}{v}\mathcal{A}[\phi_{zzz} - 6\phi\phi_z]\right] \tag{3.5}$$

Equation (3.5), when transformed using the inverse Aboodh Transform, yields

$$\phi(z,\tau) = 6z - \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{zzz} - 6\phi\phi_z]\right] \tag{3.6}$$

$$\phi_0 = 6x \tag{3.7}$$

The following is the recursive relation:

$$\phi_{r+1}(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_r - \phi_{rzzz}]\right] \tag{3.8}$$

Where A_r is the Adomian polynomial.
Let the representation of the nonlinear term be:

$$A_r = \frac{1}{r!}\frac{d^r}{d\lambda^r} f\left[\sum_{i=0}^{\infty} \lambda^i \phi_i\right]_{\lambda=0} \tag{3.9}$$

By using equation (3.9), we obtain

$$A_0 = \phi_0[\phi_{0z}]$$
$$A_1 = \phi_1[\phi_{0z}] + \phi_0\phi_{1z}$$
$$A_2 = \phi_2[\phi_{0z}] + \phi_1[\phi_{1z}] + \phi_0[\phi_{2z}]$$

From Equation (3.8)
When $r = 0$, $\phi_1 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_0 - \phi_{0zzz}]\right]$

$$\phi_1 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi_0\phi_{0z} - \phi_{0zzz}]\right]$$
$$\phi_1 = 216z\tau$$

When $r = 1$, $\phi_2 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_1 - \phi_{1zzz}]\right]$

$$\phi_2 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}\left[6\phi_1\phi_{0z} + 6w_0\phi_{1z} - \phi_{1zzz}\right]\right]$$
$$\phi_2 = \frac{1552}{2}z\tau^2$$
$$\phi_2 = 7776z\tau^2$$
$$r = 2,\ \phi_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}\left[6A_2 - \phi_{2zzz}\right]\right]$$
$$\phi_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi_2\phi_{0z} + 6w_1\phi_{1z} + 6\phi_0\phi_{2z} - \phi_{2zzz}]\right]$$
$$\phi_3 = 419904z\tau^3$$

The approximate series solution is:

$$\phi(z,\tau) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\phi(z,\tau) = 6z + 216z\tau + 7776z\tau^2 + 419904z\tau^3 + \cdots$$

$$\phi(z,\tau) = 6z(1 + 36\tau + (36\tau)^2 + (36\tau)^3 + \cdots) \tag{3.10}$$

Equation (3.10) may be expressed in closed form using Taylor's series.:

$$\phi(z,\tau) = \frac{6z}{1 - 36\tau}\ ,\ |36\tau| < 1 \tag{3.11}$$

The solution obtained in equation (3.10) is in good agreement with the result obtained by Mahgoub Transform method [6].

Example 3.2. *Examine the nonlinear KDV problem [2]*

$$\phi_\tau + \phi\phi_z + \phi_{zzz}(z,\tau) = 0 \tag{3.12}$$

With

$$\phi(z,0) = 1 - z \tag{3.13}$$

Applying Aboodh to both sides of the equation (3.12).

$$\mathcal{A}[\phi_\tau] = -\mathcal{A}[\phi_{zzz}(z,\tau) + \phi\phi_z] \tag{3.14}$$

Making use of the Aboodh differential properties, equation (3.14) becomes

$$v\mathcal{A}[\phi(z,\tau)] - \frac{1}{v}\phi(z,0) = -\mathcal{A}[\phi_{zzz}(z,\tau) + \phi\phi_z] \tag{3.15}$$

Applying the initial condition equation (3.13) on equation (3.15) we obtain,

$$\mathcal{A}\left[\phi(z,\tau)\right] = \frac{z-1}{v^2} - \left[\frac{1}{v}\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z]\right] \tag{3.16}$$

Now, we use the Aboodh inverse Transform of (3.16).Thus,

$$\phi(z,\tau) = (z-1) - \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z]\right] \tag{3.17}$$

$$\phi_0 = 1 - z \tag{3.18}$$

The recursive expression is can now be written as

$$\phi_{r+1} = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{rzzz}\left(z,\tau\right) + A_r]\right] \tag{3.19}$$

where the Adomian polynomial denoted by A_r is used to decompose the nonlinear terms.

$$A_r = \frac{1}{r!}\frac{d^r}{d\lambda^r} f\left[\sum_{i=0}^{\infty}\lambda^i\phi_i\right]_{\lambda=0} \tag{3.20}$$

The nonlinear term is represented by

$$f\left(u\right) = \phi\phi_z \tag{3.21}$$

By using equation (3.20) , we obtain

$$\begin{aligned} A_0 &= \phi_0[\phi_{0z}] \\ A_1 &= \phi_1\left[\phi_{0z}\right] + \phi_0\phi_{1z} \\ A_2 &= \phi_2\left[\phi_{0z}\right] + \phi_1\left[\phi_{1z}\right] + \phi_0\left[\phi_{2z}\right] \end{aligned}$$

From equation (3.19)

When $r = 0$, we obtain

$$\phi_1 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{0zzz}(z,\tau) + A_0]\right]$$
$$\phi_1 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)(-1)]\right]$$
$$\phi_1 = (1-z)\tau$$
$$r = 1,\;\; w_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{0zzz}(z,\tau) + A_1]\right]$$
$$\phi_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{1zzz}(z,\tau) + \phi_1\phi_{0z} + \phi_0\phi_{1z}]\right]$$
$$\phi_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)\tau(-1) + (1-z)(-\tau)]\right]$$
$$\phi_2 = (1-z)\tau^2$$
$$r = 2,\;\; w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{2zzz}(z,\tau) + A_2]\right]$$
$$w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{2zzz}(z,\tau) + \phi_0\phi_{2z} + \phi_1\phi_{1z} + \phi_2\phi_{0z}]\right]$$
$$w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)\left(-\tau^2\right) + (1-z)\left(\tau^2\right) + (1-z)\left(\tau^2\right)(-1)]\right]$$
$$\phi_3 = (1-z)\tau^3$$
$$\phi(z,\tau) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$

$$\phi(z,\tau) = (1-z) + (1-z)\tau + (1-z)\tau^2 + (1-z)\tau^3 + \cdots \tag{3.22}$$

$$\phi(z,\tau) = (1-z) + (1 + \tau + \tau^2 + \tau^3 + \cdots) \tag{3.23}$$

Applying Taylor's series, equation (3.23) is expressed in exact form as:

$$\phi(z,\tau) = \frac{1-z}{1-\tau},\;\; |\tau| < 1 \tag{3.24}$$

The solution obtained in equation (3.24) is the same as with the result obtained by Natural Transform and Homotopy Methods [2].

Example 3.3. *Examine the nonlinear KDV problem [11]*

$$\phi_\tau - 6\phi\phi_z + \phi_{zzz} = 0 \tag{3.25}$$

With

$$\phi(z,0) = \frac{2}{(z-3)^2} \tag{3.26}$$

Taking the Aboodh of (3.25)

$$\mathcal{A}[\phi_\tau] = \mathcal{A}[6\phi\phi_z - \phi_{zzz}] \tag{3.27}$$

Using the Aboodh differential properties, we obtain

$$v\mathcal{A}[\phi_\tau] - \frac{1}{v}\phi(z,0) = \mathcal{A}[6\phi\phi_z - \phi_{zzz}] \tag{3.28}$$

putting equation (3.25) into equation (3.28), we get

$$\mathcal{A}\left[\phi(z,\tau)\right] = \frac{2}{v(z-3)^2} + \left[\frac{1}{v}\mathcal{A}[6\phi\phi_z - w_{zzz}\left(z,\tau\right)]\right] \tag{3.29}$$

Now, we take the Aboodh inverse of (3.29). We have

$$\phi\left(z,\tau\right) = \frac{2}{v\left(z-3\right)^2} + \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi\phi_z - w_{zzz}\left(z,\tau\right)]\right] \tag{3.30}$$

$$\phi_0 = \frac{2}{\left(z-3\right)^2} \tag{3.31}$$

The recursive relation is given as

$$\phi_{r+1} = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_r - \phi_{rzzz}\left(z,\tau\right)]\right] \tag{3.32}$$

Where A_r represents the Adomian polynomials. It is obtained using the expression in

$$A_r = \frac{1}{r!}\frac{d^r}{d\lambda^r} f\left[\sum_{i=0}^{\infty} \lambda^i \phi_i\right]_{\lambda=0} \tag{3.33}$$

The nonlinear term is represented by

$$f\left(u\right) = \phi\phi_z \tag{3.34}$$

By using Equation, we obtain

$$\begin{aligned} A_0 &= \phi_0\phi_{0z} \\ A_1 &= \phi_1\phi_{0z} + \phi_0\phi_{1z} \\ A_2 &= \phi_2\phi_{0z} + \phi_1\phi_{1z} + \phi_0\phi_{2z} \end{aligned}$$

From equation (3.33)
When $r = 0$, we obtain

$$\begin{aligned} \phi_1 &= \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_0 - \phi_{0zzz}\left(z,\tau\right)]\right] \\ \phi_1 &= \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6w_0\phi_{0z} - \phi_{0zzz}]\right] \\ \phi_1 &= \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\left(\frac{2}{\left(z-3\right)^2}\left(\frac{-4}{\left(z-3\right)}\right)^3\right) + \frac{48}{\left(z-3\right)^5}]\right] \end{aligned}$$

$$\begin{aligned} \phi_1 &= 0 \\ r = 1,\ \ w_2 &= \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_1 - \phi_{1zzz}\left(z,\tau\right)]\right] \\ \phi_2 &= -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_1\phi_{0z} + \phi_0\phi_{1z} - \phi_{1zzz}]\right] \\ \phi_2 &= -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0]\right] \end{aligned}$$

$$\phi_2 = 0$$
$$r = 2,\ \ w_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_2 - \phi_{2zzz}(z,\tau)]\right]$$
$$\phi_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_2\phi_{0z} + \phi_1\phi_{1z} + \phi_0\phi_{2z} - \phi_{2zzz}]\right]$$
$$\phi_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0]\right]$$
$$\phi_3 = 0$$

The approximate series solution is expressed below as

$$\phi(z,\tau) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\phi(z,\tau) = \frac{2}{(z-3)^2} + 0 + 0 + 0 + \cdots$$

The approximate solution $\phi(z,\tau)$ is given by

$$\phi(z,\tau) = \frac{2}{(z-3)^2}. \tag{3.35}$$

The solution obtained in equation (3.35) is in good agreement with result obtained by Adomian Polynomial and Elzaki Transform Method [11]

4. Conclusion

This study presents the solution of third-order nonlinear KdV problem using AAPM. The examples under consideration demonstrated how successful this technique is at solving third-order KDV equations and how well it works as a system to produce outcomes that are realistic and closely aligned with precise solutions after a minimal number of repetitions. The answers found using this approach concur with additional answers found in the cited literature.

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