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Uncertainty principles for the continuous wavelet transform associated with a Bessel type operator on the half line

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Abstract. This paper presents uncertainty principles pertaining to generalized wavelet transforms associated with a second-order differential operator on the half line, extending the concept of the Bessel operator. Specifically, we derive a Heisenberg-Pauli-Weyl type uncertainty principle, as well as other uncertainty relations involving sets of finite measure

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1. Introduction

Al Subaie and Mourou ([4]) have introduced and studied the following second order differential operator $\Delta_{\alpha,n}$, on the half line $(0,+\infty)$,

$$\Delta_{\alpha,n}(u) = u'' + \frac{2\alpha+1}{r}u' - \frac{4n(\alpha+n)}{r^2}u,$$

where $\alpha > -\frac{1}{2}$ and $n \in \mathbb{N}$. Its particularity resides in the fact that it generalizes the usual Bessel differential operator, indeed for $n = 0$, we recover the Bessel operator $\ell_\alpha = u'' + \frac{2\alpha+1}{r}u'$.

This paper focuses on exploring uncertainty principles concerning the generalized Fourier transform [4] and continuous wavelet transforms [3] associated with $\Delta_{\alpha,n}$. Essentially, a function and its Fourier transform cannot be sharply focused at the same time in harmonic analysis, according to the uncertainty principle. Various mathematical formulations express this principle, involving measurement of sets or norms. For further elaboration, interested readers can refer to [17, 24] and [5, 8, 12, 13, 16, 23]. Recently, similar uncertainty relations have been established for different integral transforms, such as continuous wavelet transforms and

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Gabor transforms, across various contexts. Relevant literature includes [7, 10, 22, 29, 33, 36] and related references.

In this context, we establish, among other, a sharp Heisenberg-Pauli-Weyl type uncertainty principle [35] for the generalized Fourier transform $\mathscr{F}_\Delta$ associated to $\Delta_{\alpha,n}$, which is defined on $\mathbb{R}_+ = [0, +\infty)$ by

$$\mathscr{F}_\Delta(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)\frac{x^{2\alpha+1}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)}dx; \quad \forall\lambda \in \mathbb{R}_+,$$

where $\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x)$ and j_α is the modified Bessel function (see [27, 34]).

We present Heisenberg-Pauli-Weyl type inequalities applicable to the generalized continuous wavelet transform associated with $\Delta_{\alpha,n}$. These inequalities encompass both the time and frequency variables, as well as their combination. Additionally, we explore other uncertainty relations pertinent to this transform, including Donoho and Stark type principles. Our investigation delves into the concentration of this transform on time-frequency sets, revealing that the generalized wavelet transforms of non-zero functions cannot have arbitrarily large support. Notably, extensive research has been conducted on this generalized Fourier transform, particularly within the realm of uncertainty principles [1, 2, 14, 15].

Numerous studies, including those on time-frequency representations such as Gabor and wavelet transforms, have been thoroughly explored in diverse contexts using various methodologies [6, 10, 18, 20, 22]. For further elucidation, refer to [21].

This document is structured as follows:

The first section revisits some harmonic analysis findings pertinent to the generalized Fourier transform, $\mathscr{F}_\Delta$. The second section focuses on the study of generalized continuous wavelet transforms associated with $\Delta_{\alpha,n}$. In the third section, we present results concerning finite sets of measurements, alongside discussions on Donoho-Stark and Benedicks-type uncertainty principles. Lastly, the fourth section addresses Heisenberg-type uncertainty principles for the generalized continuous wavelet transform.

2. Preliminaries

Within this section, we revisit essential concepts in harmonic analysis pertaining to the Bessel operator ℓ_α, as documented in references [9, 11, 26, 32]. These concepts serve as foundational knowledge for our examination of the Bessel-type operator $\Delta_{\alpha,n}$ (see [4]). For α greater than $-\frac{1}{2}$, the Bessel operator ℓ_α is defined over the interval $(0, +\infty)$ by

$$\ell_\alpha(u) = u'' + \frac{2\alpha+1}{r}u'.$$

Next, considering all values of λ in the complex number set, the following system

$$\ell_\alpha(u) = -\lambda^2 u, \; u(0) = 1, \; u'(0) = 0,$$

admits a unique solution given by the modified Bessel function $x \mapsto j_\alpha(\lambda x)$, where

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha+1)}{x^\alpha}J_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}(\frac{x}{2})^{2n}, \qquad x \in \mathbb{R};$$

and J_α is the Bessel function of the first kind and index α (see [27, 34]).

The Mehler integral representation of the modified Bessel function j_α is expressed as follows:

$$\forall x \in \mathbb{R}; \quad j_\alpha(x) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(xt)dt, & \text{if } \alpha > -1/2; \\ \cos(x), & \text{if } \alpha = -1/2. \end{cases}$$

Specifically, for each natural number n and real number x,

$$|j_\alpha^{(n)}(x)| \leq 1. \tag{2.1}$$

On the positive real numbers $\mathbb{R}_+$, define the measure μ_α as follows:

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{c_\alpha}dx; \text{ where } c_\alpha = 2^\alpha \Gamma(\alpha+1), \tag{2.2}$$

and represent the Lebesgue space on $\mathbb{R}_+$ with a focus on the measure μ_α by $L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p \in [1, +\infty]$, and the L^p-norm by $\|.\|_{p,\mu_\alpha}$.

The Bessel translation operators τ_x^α, where $x \geqslant 0$, operate on $L^1_{\mu_\alpha}(\mathbb{R}_+)$, with their definition as follows:

$$\tau_x^\alpha(f)(y) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\displaystyle\int_0^\pi f(\sqrt{x^2+y^2+2xycos\theta})(sin\theta)^{2\alpha}d\theta, & \text{if } \alpha > -1/2; \\ \dfrac{f(x+y)+f(|x-y|)}{2}, & \text{if } \alpha = -1/2. \end{cases} \tag{2.3}$$

Here, for every $x \in \mathbb{R}_+$, we have

$$\int_0^{+\infty} \tau_x^\alpha(f)(y)d\mu_\alpha(y) = \int_0^{+\infty} f(y)d\mu_\alpha(y).$$

For every $f \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p \in [1, +\infty]$ and for every $x \in \mathbb{R}_+$, the function $\tau_x^\alpha(f)$ belongs to the space $L^p_{\mu_\alpha}(\mathbb{R}_+)$ and

$$\| \tau_x^\alpha(f) \|_{p,\mu_\alpha} \leq \| f \|_{p,\mu_\alpha} . \tag{2.4}$$

In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the convolution operation between two functions f and g is defined by

$$f *_\alpha g(x) = \int_0^{+\infty} f(y)\tau_x^\alpha(g)(y)d\mu_\alpha(y), \quad \forall x \in \mathbb{R}_+.$$

In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the convolution product "$*_\alpha$" is both commutative and associative.

For the convolution product " $*_\alpha$ ", the Young's inequality states that if p, q, and $r \in [1, +\infty]$ are such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then for all f in $L^p_{\mu_\alpha}(\mathbb{R}_+)$ and g in $L^q_{\mu_\alpha}(\mathbb{R}_+)$, the function $f *_\alpha g$ belongs to $L^r_{\mu_\alpha}(\mathbb{R}_+)$ and

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq \| f \|_{p,\mu_\alpha} \| g \|_{q,\mu_\alpha} . \tag{2.5}$$

Furthermore, for any function f and g in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ and we have for each $x \in \mathbb{R}_+$,

$$\tau_x^\alpha(f *_\alpha g) = \tau_x^\alpha(f) *_\alpha g = f *_\alpha \tau_x^\alpha(g).$$

On $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the Hankel transform $\mathscr{H}_\alpha$ is defined, via

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r)j_\alpha(r\lambda)d\mu_\alpha(r), \quad \forall\ \lambda \in \mathbb{R}_+. \tag{2.6}$$

The following properties hold

• (Inversion formula for $\mathscr{H}_\alpha$) In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, for any function f, we have for almost all $x \in \mathbb{R}_+$

$$f(x) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)j_\alpha(x\lambda)d\mu_\alpha(\lambda).$$

• (Plancherel's theorem for $\mathscr{H}_\alpha$) An isometric isomorphism from $L^2_{\mu_\alpha}(\mathbb{R}_+)$ onto itself may be obtained by extending the Hankel transform $\mathscr{H}_\alpha$. Specifically, the Parseval's formula for each f and g in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ is as follows.

$$\int_0^{+\infty} f(x)\overline{g(x)}d\mu_\alpha(x) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda).$$

• For every $f \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p = 1$ or 2, and $x \in \mathbb{R}_+$, we have

$$\mathscr{H}_\alpha(\tau^\alpha_x(f))(\lambda) = j_\alpha(x\lambda)\mathscr{H}_\alpha(f)(\lambda), \;\; \forall\lambda \in \mathbb{R}_+. \tag{2.7}$$

• For every $f \in L^1_{\mu_\alpha}(\mathbb{R}_+)$ and $g \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p = 1, 2$ we have

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

•Suppose $f, g \in L^2_{\mu_\alpha}(\mathbb{R}_+)$. In $L^2_{\mu_\alpha}(\mathbb{R}_+)$, the function $f *_\alpha g$ is included if and only if, $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ belongs to $L^2_{\mu_\alpha}(\mathbb{R}_+)$ and in this case, we have

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

Let us consider the second-order singular differential operator on the half line (see [4])

$$\Delta_{\alpha,n}(u) = u^{''} + \frac{2\alpha+1}{r}u' - \frac{4n(\alpha+n)}{r^2}u$$

where $n \in \mathbb{N}$. We obtain the Bessel operator ℓ_α for $n = 0$.

For all $\lambda \in \mathbb{C}$, the function

$$\varphi_\lambda(x) = x^{2n}j_{\alpha+2n}(\lambda x). \tag{2.8}$$

is solution of $\Delta_{\alpha,n}(u) = -\lambda^2 u$.

The following characteristics apply to the function φ_λ

- For all $\lambda, x \in \mathbb{R}$,

$$\varphi_\lambda(x) = \frac{x^{2n}2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}}\cos(\lambda xt)dt.$$

  In particular,

$$|\varphi_\lambda(x)| \leqslant x^{2n}. \tag{2.9}$$

- For a measurable function on $\mathbb{R}$, we define the map M by $Mf(x) = x^{2n}f(x)$. Then, for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, the function φ_λ satisfy the following product formula

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{(xy)^{2n}\Gamma(\alpha+2n+1)}{\Gamma(\alpha+2n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi (M^{-1}\varphi_\lambda)(\sqrt{\lambda^2x^2+\lambda^2y^2+2\lambda^2xycos(\theta)})sin(\theta)^{\alpha+2n}d\theta.$$

In the sequel, we need the following notations.

- $L^p_{(\mu)}(\mathbb{R}_+)$, $p \in [1, +\infty]$, is the space of measurable functions f on $\mathbb{R}_+$ such that $\|M^{-1}f\|_{p,\mu_{\alpha+2n}} < \infty$. The space $L^p_{(\mu)}(\mathbb{R}_+)$ is equipped with the norm $\|.\|_{p,(\mu)}$ given by

$$\|f\|_{p,(\mu)} = \|M^{-1}f\|_{p,\mu_{\alpha+2n}}.$$

From the last product formula, we define the generalized translation operator, $\tau_x^{\Delta},\ x \in \mathbb{R}_+$ by

$$\tau_x^{\Delta}(f)(y) = \frac{(xy)^{2n}\Gamma(\alpha+2n+1)}{\Gamma(\alpha+2n+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\pi} (M^{-1}f)(\sqrt{x^2+y^2+2xycos(\theta)})sin(\theta)^{\alpha+2n}d\theta, \tag{2.10}$$

Whenever the integral of the right-hand side is well defined.

- We have the following relation between the generalized and Hankel translation operators

$$\tau_x^{\Delta}(f)(y) = (xy)^{2n}\tau_x^{\alpha+2n}(M^{-1}f)(y),$$

  where $\tau_x^{\alpha+2n}$ is given by the relation (2.3).

- For every $f \in L^p_{(\mu)}(\mathbb{R}_+)$, $p \in [1,+\infty]$ and for every $x \in \mathbb{R}_+$, the function $\tau_x^{\Delta}(f)$ belongs to the space $L^p_{(\mu)}(\mathbb{R}_+)$ and

$$\|\tau_x^{\Delta}(f)\|_{p,(\mu)} \leqslant x^{2n}\|f\|_{p,(\mu)}. \tag{2.11}$$

Given two functions $f, g \in L^1_{(\mu)}(\mathbb{R}_+)$, the generalized convolution product, "$\#$", is defined as

$$f\#g(x) = \int_0^{+\infty} f(y)\tau_x^{\Delta}(g)(y)\frac{y^{2\alpha+1}}{c_{\alpha+2n}}dy, \quad x \geqslant 0, \tag{2.12}$$

where the constant $c_{\alpha+2n}$ is given by the relation (2.2).

We have the following connection between "$\#$" and "$*_{\alpha+2n}$",

$$f\#g(x) = M\left(M^{-1}(f) *_{\alpha+2n} M^{-1}(g)\right)(x). \tag{2.13}$$

In $L^1_{(\mu)}(\mathbb{R}_+)$, the convolution product "$\#$" is both commutative and associative.

Young's inequality for the convolution product "$\#$" states that, for all f in $L^p_{(\mu)}(\mathbb{R}_+)$ and g in $L^q_{(\mu)}(\mathbb{R}_+)$, the function $f\#g$ belongs to $L^r_{(\mu)}(\mathbb{R}_+)$ and for all p, q and $r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ and

$$\|f\#g\|_{r,(\mu)} \leq \| f \|_{p,(\mu)} \| g \|_{q,(\mu)} . \tag{2.14}$$

On $L^1_{(\mu)}(\mathbb{R}_+)$, the generalized Fourier transform $\mathscr{F}_{\Delta}$ related to $\Delta_{\alpha,n}$ is defined by

$$\mathscr{F}_{\Delta}(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_{\lambda}(x)\frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx; \quad \forall\lambda \in \mathbb{R}_+, \tag{2.15}$$

where φ_{λ} is given by the relation (2.8).

We have the following properties

- For $f \in L^1_{(\mu)}(\mathbb{R}_+)$,

$$\mathscr{F}_{\Delta}(f)(\lambda) = \mathscr{H}_{\alpha+2n}(M^{-1}f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

- For each $f \in L^1_{(\mu)}(\mathbb{R}_+)$, $\mathscr{F}_{\Delta}(f)$ is a function that is a part of $\mathscr{C}_{*,0}(\mathbb{R})$ the space of continuous even functions f on $\mathbb{R}$ such that $\lim\limits_{|x|\to\infty} |f(x)| = 0$ in addition

$$\|\mathscr{F}_{\Delta}(f)\|_{\infty,\mu_{\alpha+2n}} \leqslant \|f\|_{1,(\mu)}$$

- Since $\mathscr{F}_\Delta(f) \in L^1_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ for every $x \in \mathbb{R}_+$, let $f \in L^1_{(\mu)}(\mathbb{R}_+)$

$$f(x) = \int_0^{+\infty} \mathscr{F}_\Delta(f)(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda).$$

  Applied to $L^1_{(\mu)}(\mathbb{R}_+)$, this demonstrates that $\mathscr{F}_\Delta$ is injective.

- (Plancherel's theorem for $\mathscr{F}_\Delta$) An isometric isomorphism from $L^2_{(\mu)}(\mathbb{R}_+)$ onto $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ may be obtained by extending the generalized Fourier transform $\mathscr{F}_\Delta$. Moreover, the following Parseval's formula holds for every f and g in $L^2_{(\mu)}(\mathbb{R}_+)$.

$$\langle f|g\rangle_{(\mu)} = \langle \mathscr{F}_\Delta(f)|\mathscr{F}_\Delta(g)\rangle_{\mu_{\alpha+2n}}. \tag{2.16}$$

  where the inner product defined on $L^2_{(\mu)}(\mathbb{R}_+)$ is $\langle\ .|.\ \rangle_{(\mu)}$, via

$$\langle f|g\rangle_{(\mu)} = \langle M^{-1}(f)|M^{-1}(g)\rangle_{\mu_{\alpha+2n}} \tag{2.17}$$

  and the inner product of the Hilbert space $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ is shown by the notation $\langle\ .|.\ \rangle_{\mu_{\alpha+2n}}$.

- When $x \in \mathbb{R}_+$, $p = 1$ or 2, and $f \in L^p_{(\mu)}(\mathbb{R}_+)$, we obtain

$$\mathscr{F}_\Delta(\tau_x^\Delta(f))(\lambda) = \varphi_\lambda(x)\mathscr{F}_\Delta(f)(\lambda), \quad \forall\lambda \in \mathbb{R}_+.$$

- For every $f \in L^1_{(\mu)}(\mathbb{R}_+)$ and $g \in L^p_{(\mu)}(\mathbb{R}_+)$, $p = 1, 2$ we have

$$\mathscr{F}_\Delta(f\#g) = \mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g).$$

- Suppose that $f, g \in L^2_{(\mu)}(\mathbb{R}_+)$. If and only if $\mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g)$ belongs to $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$, then the function $f\#g$ belongs to $L^2_{(\mu)}(\mathbb{R}_+)$. In this instance, we have

$$\mathscr{F}_\Delta(f\#g) = \mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g). \tag{2.18}$$

3. Generalized Continuous Wavelet Transforms Associated to $\mathscr{F}_\Delta$.

The theory of generalized continuous wavelet transforms, as studied by R.F. Al Subaie and M.A. Mourou [4], is briefly summarized in this section.

Let $a \in \mathbb{R}_+^* = (0, +\infty)$. The dilation operator $D_{\alpha,a}$ of a measurable function ψ, is defined by

$$D_{\alpha,a}(\psi)(s) = a^{\alpha+1}\psi(as), \quad \forall s \geqslant 0.$$

We have,

- For every $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$,

$$\|D_{\alpha,a}(\psi)\|_{2,(\mu)} = \|\psi\|_{2,(\mu)}. \tag{3.1}$$

- We obtain for any ψ and $\phi \in L^2_{(\mu)}(\mathbb{R}_+)$

$$\langle D_{\alpha,a}(\psi)|\phi\rangle_{(\mu)} = \langle \psi|D_{\alpha,\frac{1}{a}}(\phi)\rangle_{(\mu)},$$

- For every $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$, we have

$$\mathscr{F}_\Delta(D_{\alpha,a}(\psi)) = D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_\Delta(\psi). \tag{3.2}$$

We indicate by

- $\vartheta_{\alpha,n}$ the measure defined on $\mathbb{R}_+^* \times \mathbb{R}_+$, by

$$d\vartheta_{\alpha,n}(a,x) = d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(x),$$

- The Lebesgue space on $\mathbb{R}_+^* \times \mathbb{R}_+$ with regard to the measure $\vartheta_{\alpha,n}$ with the L^p-norm represented by $\| \,.\, \|_{p,\vartheta_{\alpha,n}}$ is $L^p_{\vartheta_{\alpha,n}}(\mathbb{R}_+^* \times \mathbb{R}_+)$, $p \in [1,+\infty]$.
- $\langle \,.| \,.\, \rangle_{\vartheta_{\alpha,n}}$ the inner product of the Hilbert space $L^2_{\vartheta_{\alpha,n}}(\mathbb{R}_+^* \times \mathbb{R}_+)$.
- For a measurable function f on $\mathbb{R}_+^* \times \mathbb{R}_+$, the mapping M_2 is defined by

$$M_2(f)(a,x) = x^{2n}f(a,x).$$

- $L^p_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$, $p \in [1,+\infty]$ the space of measurable functions f on $\mathbb{R}_+^* \times \mathbb{R}_+$ such that $\| M_2^{-1}(f) \|_{p,\vartheta_{\alpha,n}} < +\infty$. The space $L^p_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ is equipped with the norm $\|.\|_{p,(\vartheta)}$ given by

$$\|f\|_{p,(\vartheta)} = \| M_2^{-1}(f) \|_{p,\vartheta_{\alpha,n}}.$$

- $\langle \,.\, | \,.\, \rangle_{(\vartheta)}$ the inner product of the Hilbert space $L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ defined by

$$\langle \, f| \, g \, \rangle_{(\vartheta)} = \langle \, M_2^{-1}(f)| \, M_2^{-1}(g) \, \rangle_{\vartheta_{\alpha,n}}.$$

A generalized admissible wavelet is defined as $\psi \in L^2_{(\mu)}(\mathbb{R}_+)\backslash\{0\}$ if

$$0 < C_\psi^\Delta = \frac{1}{c_{\alpha+2n}} \int_0^\infty |\mathscr{F}_\Delta(\psi)(a)|^2 \frac{da}{a} < \infty. \tag{3.3}$$

The generalized continuous wavelet transform $\mathcal{W}_\psi^\Delta$, for such ψ, is defined on $L^2_{(\mu)}(\mathbb{R}_+)$ by

$$\mathcal{W}_\psi^\Delta(f)(a,x) = \int_0^\infty f(s)\overline{\psi_{a,x}^\Delta(s)}\frac{s^{2\alpha+1}}{c_{\alpha+2n}}ds, \quad (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+ \tag{3.4}$$

where

$$\psi_{a,x}^\Delta(s) = \tau_x^\Delta D_{\alpha,a}(\psi)(s). \tag{3.5}$$

Another way to express the transform $\mathcal{W}_\psi^\Delta$ is

$$\begin{aligned} \mathcal{W}_\psi^\Delta(f)(a,x) &= f\#D_{\alpha,a}(\overline{\psi})(x) \\ &= \langle \, f \, |\psi_{a,x}^\Delta \, \rangle_{(\mu)}. \end{aligned} \tag{3.6}$$

Then, in virtue of relations (3.6), (2.14) and (3.1), we deduce that the function $\mathcal{W}_\psi^\Delta(f)$ belongs to the space $L^\infty_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ and

$$\left\|\mathcal{W}_\psi^\Delta(f)\right\|_{\infty,(\vartheta)} \leqslant \|f\|_{2,(\mu)}\|\psi\|_{2,(\mu)}. \tag{3.7}$$

Theorem 3.1. *Let $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$ be a generalized admissible wavelet.*

(i) (Plancherel's formula for $\mathcal{W}^{\Delta}_{\psi}$) For every function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, the function $\mathcal{W}^{\Delta}_{\psi}(f)$ belongs to the space $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$ and we have

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)} = \sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)}. \tag{3.8}$$

(ii) (Parseval's formula for $\mathcal{W}^{\Delta}_{\psi}$) For all functions $f, g \in L^2_{(\mu)}(\mathbb{R}_+)$ we have

$$\langle\, f \mid g\,\rangle_{(\mu)} = \frac{1}{C^{\Delta}_{\psi}}\langle\, \mathcal{W}^{\Delta}_{\psi}(f) \mid \mathcal{W}^{\Delta}_{\psi}(g)\,\rangle_{(\vartheta)}, \tag{3.9}$$

Proof. (i) Let $f \in L^2_{(\mu)}(\mathbb{R}_+)$, we have from relations (2.16), (2.18) and (3.6),

$$\begin{aligned}
\|\mathcal{W}^{\Delta}_{\psi}(f)\|^2_{2,(\vartheta)} &= \int_0^\infty\int_0^\infty |M_2^{-1}(\mathcal{W}^{\Delta}_{\psi}(f))(a,x)|^2 d\vartheta_{\alpha,n}(a,x)\\
&= \int_0^\infty\left[\int_0^\infty |f\# D_{\alpha,a}(\overline{\psi})(x)|^2 x^{-4n} d\mu_{\alpha+2n}(x)\right] d\mu_{\alpha+2n}(a)\\
&= \int_0^\infty\left[\int_0^\infty |\mathscr{F}_{\Delta}(f\# D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right] d\mu_{\alpha+2n}(a)\\
&= \int_0^\infty\left[\int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2|\mathscr{F}_{\Delta}(D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right] d\mu_{\alpha+2n}(a).
\end{aligned}$$

Now using relations (3.2) and (3.3) we get

$$\begin{aligned}
\|\mathcal{W}^{\Delta}_{\psi}(f)\|^2_{2,(\nu)} &= \int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2\left[\int_0^\infty |D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_{\Delta}(\overline{\psi})(\lambda)|^2 d\mu_{\alpha+2n}(a)\right] d\mu_{\alpha+2n}(\lambda)\\
&= \int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2\left[\frac{1}{c_{\alpha+2n}}\int_0^\infty |\mathscr{F}_{\Delta}(\overline{\psi})(a)|^2\frac{da}{a}\right] d\mu_{\alpha+2n}(\lambda)\\
&= C^{\Delta}_{\psi}\int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C^{\Delta}_{\psi}\|f\|^2_{2,(\mu)}.
\end{aligned}$$

(ii) The outcome is derived from the polarization identity and (i).

■

Theorem 3.2. *Let ψ be a generalized admissible wavelet. For every $f \in L^2_{(\mu)}(\mathbb{R}_+)$, the function $\mathcal{W}^{\Delta}_{\psi}(f) \in L^p_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$, $p \in [2,\infty]$ and we have*

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{p,(\vartheta)} \leqslant (C^{\Delta}_{\psi})^{\frac{1}{p}}\|\psi\|^{1-\frac{2}{p}}_{2,(\mu)}\|f\|_{2,(\mu)}. \tag{3.10}$$

Proof. According to the relation (3.8), the Plancherel's theorem for the generalized continuous wavelet transform for $p = 2$ produces

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)} = \sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)}.$$

For $p = \infty$, we have by the relation (3.7)

$$\left\|\mathcal{W}^{\Delta}_{\psi}(f)\right\|_{\infty,(\vartheta)} \leqslant \|f\|_{2,(\mu)}\|\psi\|_{2,(\mu)}.$$

The outcome of the Riez-Thorin Theorem is obtained. ■

Proposition 3.3. *For $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$ be a generalized admissible wavelet. Then, $\mathcal{W}^\Delta_\psi(L^2_{(\mu)}(\mathbb{R}_+))$ is a reproducing kernel Hilbert space in $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$, with kernel*

$$\mathscr{K}^\Delta_\psi((a,x),(a',x')) = \frac{1}{C^\Delta_\psi}\langle\, \psi^\Delta_{a,x} \mid \psi^\Delta_{a',x'}\,\rangle_{(\mu)}. \tag{3.11}$$

The kernel $\mathscr{K}^\Delta_\psi$ satisfies the following

$$\forall (a,x),(a',x') \in \mathbb{R}^*_+ \times \mathbb{R}_+, \quad \left|\mathscr{K}^\Delta_\psi((a,x),(a',x'))\right| \leqslant \frac{(xx')^{2n}}{C^\Delta_\psi}\|\psi\|^2_{2,(\mu)}.$$

Proof. From the relation (3.6), we have for all $(a,x),(a',x') \in \mathbb{R}^*_+ \times \mathbb{R}_+$,

$$\mathscr{K}^\Delta_\psi((a,x),(a',x')) = \frac{1}{C^\Delta_\psi}\mathcal{W}^\Delta_\psi(\psi^\Delta_{a,x})(a',x').$$

Thus, from Theorem 3.1, we deduce that for all $(a,x) \in \mathbb{R}^*_+ \times \mathbb{R}_+$ the function $\mathscr{K}^\Delta_\psi((a,x),(.,.))$ belongs to $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$.

Let $F \in \mathcal{W}^\Delta_\psi(L^2_{(\mu)}(\mathbb{R}_+))$; $F = \mathcal{W}^\Delta_\psi(f)$, $f \in L^2_{(\mu)}(\mathbb{R}_+)$, by relations (3.6) and (3.9), we have for all $(a,x) \in \mathbb{R}^*_+ \times \mathbb{R}_+$

$$\begin{aligned} F(a,x) = \mathcal{W}^\Delta_\psi(f)(a,x) &= \langle\, f \mid \psi^\Delta_{a,x}\,\rangle_{(\mu)} \\ &= \frac{1}{C^\Delta_\psi}\langle\, \mathcal{W}^\Delta_\psi(f) \mid \mathcal{W}^\Delta_\psi(\psi^\Delta_{a,x})\,\rangle_{(\vartheta)} \\ &= \langle\, \mathcal{W}^\Delta_\psi(f) \mid \mathscr{K}^\Delta_\psi((a,x),(.,.))\,\rangle_{(\vartheta)} \end{aligned}$$

This demonstrates that given the Hilbert space $\mathcal{W}^\Delta_\psi(L^2_{(\mu)}(\mathbb{R}_+))$, $\mathscr{K}^\Delta_\psi$ is a reproducing Kernel.
Then, we obtain from relations (3.5), (2.11), and (3.1),

$$\begin{aligned} \left|\mathscr{K}^\Delta_\psi((a,x),(a',x'))\right| &= \frac{1}{C^\Delta_\psi}|\langle\, \psi^\Delta_{a,x} \mid \psi^\Delta_{a',x'}\,\rangle_{(\mu)}| \\ &\leqslant \frac{1}{C^\Delta_\psi}\|\psi^\Delta_{a,x}\|_{2,(\mu)}\|\psi^\Delta_{a',x'}\|_{2,(\mu)} \\ &\leqslant \frac{(xx')^{2n}}{C^\Delta_\psi}\|\psi\|^2_{2,(\mu)}. \end{aligned}$$

This achieves the proof. ■

4. Approximate Concentration

In this part, we introduce a weak uncertainty principle [16], which is adapted for the generalized continuous wavelet transforms. It is a Donoho and Stark type uncertainty principle. Such results were first reported by Gröchenig in [22], first for the Gabor transform. We also examine how concentrated these generalized continuous wavelet transformations are on subsets of $\mathbb{R}^*_+ \times \mathbb{R}_+$ with finite measures. Finally, we present a Benedicks-type uncertainty principle, subject to some assumptions on the wavelet function. Comparable outcomes are reported in [7, 36].

Proposition 4.1. *Consider a generalized wavelet ψ with the property that $\|\psi\|_{2,(\mu)} = 1$. For any function f belonging to the space $L^2_{(\mu)}(\mathbb{R}_+)$ satisfying the condition $\|f\|_{2,(\mu)} = 1$, and for any subset Ω of $\mathbb{R}^*_+ \times \mathbb{R}_+$ and $\xi \geq 0$, the following holds:*

$$1 - \xi \leq \iint_\Omega |M_2^{-1}\mathcal{W}^\Delta_\psi(f)(a,x)|^2 d\vartheta_{\alpha,n}(a,x),$$

we obtain,

$$1-\xi \leq \vartheta_{\alpha,n}(\Omega).$$

Proof. Based on equation (3.7), we obtain the following relation:

$$1-\xi \leq \iint_{\Omega} |M_2^{-1}\mathcal{W}_{\psi}^{\Delta}(f)(a,x)|^2 d\vartheta_{\alpha,n}(a,x) \leq \|\mathcal{W}_{\psi}^{\Delta}(f)\|_{\infty,(\vartheta)}\vartheta_{\alpha,n}(\Omega) \leq \vartheta_{\alpha,n}(\Omega).$$

■

Theorem 4.2. *Suppose ψ represents a generalized wavelet such that its norm, denoted by $\|\psi\|_{2,(\mu)} = 1$, and let Ω be a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ satisfying*

$$C_{\psi}^{\Delta} > \vartheta_{\alpha,n}(\Omega),$$

Therefore, given a function f in $L^2_{(\mu)}(\mathbb{R}_+)$, we get

$$\|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)} \geq \sqrt{1-\frac{\vartheta_{\alpha,n}(\Omega)}{C_{\psi}^{\Delta}}}\sqrt{C_{\psi}^{\Delta}}\|f\|_{2,(\mu)}.$$

Proof. According to equation (3.7), it follows that for any function f belonging to the space $L^2_{(\mu)}(\mathbb{R}_+)$

$$\begin{aligned}
\|\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)}^2 &= \|\chi_{\Omega}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)}^2 + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)}^2 \\
&\leq \vartheta_{\alpha,n}(\Omega)\|\mathcal{W}_{\psi}^{\Delta}(f)\|_{\infty,(\vartheta)}^2 + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)}^2 \\
&\leq \vartheta_{\alpha,n}(\Omega)\|f\|_{2,(\mu)}^2\|\psi\|_{2,(\mu)}^2 + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)}^2.
\end{aligned}$$

We obtain the necessary result by using Plancherel's formula to $\mathcal{W}_{\psi}^{\Delta}$ as stated in relation (3.8) and the inequality $\vartheta_{\alpha,n}(\Omega) < C_{\psi}^{\Delta}$. ■

Remark 4.3. *It implies that the generalized wavelet transform $\mathcal{W}_{\psi}^{\Delta}(f)$ cannot be substantially focused on a set whose volume is smaller than the minimum C_{ψ}^{Δ} for any non-zero function f. In particular, we have*

$$\vartheta_{\alpha,n}(supp\mathcal{W}_{\psi}^{\Delta}(f)) < C_{\psi}^{\Delta} \;\Rightarrow\; f=0.$$

We take into account the following orthogonal projections:

1. P_{ψ}: This projection operates from $L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ to $\mathcal{W}_{\psi}^{\Delta}(L^2_{(\mu)}(\mathbb{R}_+))$. Its range is denoted by ImP_{ψ}.
2. P_{Ω}: Defined as the orthogonal projection onto $L^2_{(\vartheta)}(\mathbb{R}^*+\times\mathbb{R}+)$, given by

$$P_{\Omega}F = \chi_{\Omega}F, \qquad F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+),$$

where $F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$, and Ω is a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$. The range of P_{Ω} is denoted by ImP_{Ω}.

We define

$$\|P_{\Omega}P_{\psi}\| = \sup\{\|P_{\Omega}P_{\psi}(F)\|_{2,(\vartheta)}, F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+); \|F\|_{2,(\vartheta)} = 1\}.$$

Proposition 4.4. *Consider ψ be a generalized wavelet with a unit norm. A Hilbert Schmidt operator $P_{\Omega}P_{\psi}$ is defined for each subset $\Omega \subset \mathbb{R}_+^* \times \mathbb{R}_+$ of a finite measure $\vartheta_{\alpha,n}(\Omega)$ and we have*

$$\|P_{\Omega}P_{\psi}\|^2 \leq \frac{\vartheta_{\alpha,n}(\Omega)}{C_{\psi}^{\Delta}}. \tag{4.1}$$

Proof. For each function $F \in L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$, we obtain since P_ψ is a projection onto a reproducing kernel Hilbert space

$$P_\psi(F)(a,x) = \langle\, F \mid \mathscr{K}^{\Delta}_{\psi}((a,x),(.,.))\,\rangle_{(\vartheta)}$$

as defined by (3.11) for $\mathscr{K}^{\Delta}_{\psi}$. Thus,

$$P_\Omega P_\psi(F)(a,x) = \langle\, F \mid \chi_\Omega(a,x)\mathscr{K}^{\Delta}_{\psi}((a,x),(.,.))\,\rangle_{(\vartheta)}$$

Now, using the definition of the kernel provided by the relation (3.11), Fubini's theorem, relations (3.5), Plancherel's formula for the generalized wavelet transform (3.8), (2.11), and (3.1), we obtain

$$\begin{aligned}
\|P_\Omega P_\psi\|^2_{HS} &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xx')^{-4n}|\chi_\Omega(a,x)|^2|\mathscr{K}^{\Delta}_{\psi}((a,x),(a',x'))|^2 d\vartheta_{\alpha,n}(a',x')|d\vartheta_{\alpha,n}(a,x)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{-4n}\chi_\Omega(a,x)|\frac{1}{C^{\Delta}_{\psi}}M_2^{-1}\mathcal{W}^{\Delta}_{\psi}(\psi_{a,x})(a',x')|^2 d\vartheta_{\alpha,n}(a',x')d\vartheta_{\alpha,n}(a,x)\\
&= \frac{1}{C^{\Delta}_{\psi}}\int\int_\Omega x^{-4n}\frac{1}{C^{\Delta}_{\psi}}\|\mathcal{W}^{\Delta}_{\psi}(\psi_{a,x})\|^2_{2,(\vartheta)}d\vartheta_{\alpha,n}(a,x)\\
&= \frac{1}{C^{\Delta}_{\psi}}\int\int_\Omega x^{-4n}\|\psi_{a,x}\|^2_{2,(\mu)}d\vartheta_{\alpha,n}(a,x) = \frac{1}{C^{\Delta}_{\psi}}\int\int_\Omega x^{-4n}\|\tau^{\Delta}_x D_{\alpha,a}(\psi)\|^2_{2,(\mu)}d\vartheta_{\alpha,n}(a,x)\\
&\leq \frac{\|\psi\|^2_{2,(\mu)}}{C^{\Delta}_{\psi}}\vartheta_{\alpha,n}(\Omega) = \frac{\vartheta_{\alpha,n}(\Omega)}{C^{\Delta}_{\psi}}.
\end{aligned}$$

The integral operator $P_\Omega P_\psi$ has a Hilbert Schmidt kernel as a result. The fact that $\|P_\Omega P_\psi\| \leq \|P_\Omega P_\psi\|_{HS}$ implies the outcome. ■

According to Havin and Jöricke [25, 1.A, p.88], we have the following

Proposition 4.5. *Let Ω be a subset of $\mathbb{R}^*_+ \times \mathbb{R}_+$ and let ψ be a generalized wavelet. The following is our equivalency*

1. *In $L^2_{(\mu)}(\mathbb{R}_+)$, there is a constant c= c(Ω, ψ) > 0 such that for any function f*

$$\sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)} \leq c\|\chi_{\Omega^c}\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)}. \tag{4.2}$$

2. $\|P_\Omega P_\psi\| < 1$.

Remark 4.6. 1. *If the relation (4.2) is met, then (P_Ω, P_ψ) is considered a strong a-pair.*

2. *If $\|P_\Omega P_\psi\| < 1$, then*

$$\sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)} \leq \frac{1}{\sqrt{1-\|P_\Omega P_\psi\|^2}}\|\chi_{\Omega^c}\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)}. \tag{4.3}$$

3. *Relative to (4.1) and (4.3), Theorem 4.2 can be obtained.*

Theorem 4.7. ***(Benedicks-type uncertainty principle for $\mathcal{W}^{\Delta}_{\psi}$)*** *For each generalized wavelet ψ, allow $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(\psi)\neq 0\}) < \infty$. Let $\int_0^\infty \chi_\Omega(a,x)d\mu_{\alpha+2n}(x) < \infty$ be any subset Ω of $\mathbb{R}^*_+ \times \mathbb{R}_+$ such that for virtually every $a > 0$, we have*

$$\mathcal{W}^{\Delta}_{\psi}(L^2_{(\mu)}(\mathbb{R}_+)) \cap ImP_\Omega = \{0\}. \tag{4.4}$$

Proof. If F is a non-trivial function in $\mathcal{W}_\psi^\Delta(L^2_{(\mu)}(\mathbb{R}_+)) \cap ImP_\Omega$, then $F = \mathcal{W}_\psi(f)$ and $SuppF \subset \Omega$ exist for some function f in $L^2_{(\mu)}(\mathbb{R}_+)$. Suppose $a > 0$. Then, $\int_0^\infty \chi_\Omega(a,x)d\mu_{\alpha+2n}(x) < \infty$. Examine the function

$$F_a(x) = \mathcal{W}_\psi^\Delta(f)(a,x), \qquad x \geq 0.$$

After that,

$$suppF_a \subset \{x \geq 0; (a,x) \in \Omega\},$$

additionally

$$\mu_{\alpha+2n}(suppF_a) < \infty.$$

Currently, we have by utilizing the relation (2.18),

$$\mathscr{F}_\Delta(F_a)(\lambda) = \mathscr{F}_\Delta(f)(\lambda)\mathscr{F}_\Delta(D_a^\alpha\psi)(\lambda), \qquad a.e.$$

Consequently

$$\{\mathscr{F}_\Delta(F_a) \neq 0\} \subset \{\mathscr{F}_\Delta(\psi) \neq 0\}.$$

Furthermore, we derive the following from the hypothesis: $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(F_a) \neq 0\}) < \infty$. By applying the Fourier-Bessel transform Benedicks-type theorem [19], we may infer that, for any $a > 0$, F(a,.) = 0, implying that F=0. ■

The outcome that follows is a direct result of [[24], 2. A) p. 90].

Proposition 4.8. *If ψ is a generalized wavelet with $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(\psi) \neq 0\}) < \infty$, and Ω is a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ with $\vartheta_{\alpha,n}(\Omega) < \infty$, then $c(\Omega,\psi) > 0$, such that the inequality (4.2) holds.*

Here, we rephrase the proof given in [5].

Proof. Because P_Ω, P_ψ are projections, the equation $\|P_\Omega P_\psi(F)\|_{2,(\vartheta)} = \|F\|_{2,(\vartheta)}$, implies P_Ω(F)= P_ψ(F) =F. Now, the fact that

$$\vartheta_{\alpha,n}(\Omega) = \int_0^\infty \int_0^\infty \chi_\Omega(a,x)d\vartheta_{\alpha,n}(a,x) < \infty,$$

implies that for almost every $a > 0$,

$$\int_0^\infty \chi_\Omega(a,x)d\mu_\alpha(a,x) < \infty.$$

Then, from relation (4.4), we get $F = 0$ and therefore, for F $\neq$ 0 we have $\|P_\Omega P_\psi(F)\|_{2,(\vartheta)} < \|F\|_{2,(\vartheta)}$. Using the fact that $P_\Omega P_\psi$ is a Hilbert-Schmidt operator, we deduce that its largest eigenvalue λ satisfies $|\lambda| < 1$ and $\|P_\Omega P_\psi\| = |\lambda| < 1$.

The result follows from Proposition 4.5. ■

5. Heisenberg-Pauli-Weyl Type Inequalities for $\mathcal{W}_\psi^\Delta$.

The primary findings of this study, the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$ and the generalised wavelet transform $\mathcal{W}_\psi^\Delta$, are presented in this section. We consult Rassias [30] for his study on the classical Fourier transform. Rösler and Voit demonstrated the Heisenberg-Pauli-Weyl uncertainty principle for the Hankel transform in [31]. It asserts that for any function $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$,

$$\|rf\|_{2,\mu_\alpha}\|\lambda\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} \geqslant (\alpha+1)\|f\|^2_{2,\mu_\alpha},$$

with equality for any $d \in \mathbb{C}$ and $b > 0$, if and only if $f(r) = de^{-br^2/2}$.

The previous inequality was extended by Ma in his paper [28] to a more general setting, namely the Chébli-Triméche hypergroups. Specifically, he established that, for $s, t > 0$, there exists a constant $c = c(\alpha, s, t) > 0$ such that, for every function $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$, we have

$$\|r^s f\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geqslant c\|f\|_{2,\mu_\alpha}.$$

Subsequently, Soltani provided the constant c in the cases $s \geqslant 1$ and $t \geqslant 1$ explicitly in his article [33], which is $c = (\alpha+1)^{\frac{st}{s+t}}$. If and only if $s = t = 1$ and $f(r) = de^{-br^2/2}$ for some $d \in \mathbb{C}$ and $b > 0$, then we have equality.

Combining these outcomes, we obtain

Theorem 5.1. *Let $t, s > 0$. For any $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$, there is a constant $c = c(\alpha, s, t) > 0$ such that*

$$\|r^s f\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geqslant c\|f\|_{2,\mu_\alpha}, \tag{5.1}$$

Moreover, for $s, t \geq 1$ the constant $c = (\alpha+1)^{\frac{st}{s+t}}$ with equality if and only if $s = t = 1$ and $f(r) = de^{-br^2/2}$ for some $d \in \mathbb{C}$ and $b > 0$.

In the following theorem we give the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$.

Theorem 5.2. *Assume $s, t > 0$. There is a constant $c = c(\alpha, n, s, t) > 0$, for any function $f \in L^2_{(\mu)}(\mathbb{R}_+)$ such that*

$$\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \geq c\|f\|_{2,(\mu)}. \tag{5.2}$$

Moreover, for $s, t \geq 1$ the constant c is given by $(\alpha+2n+1)^{\frac{st}{s+t}}$ with equality if and only if $s = t = 1$ and $f(r) = dr^{2n}e^{\frac{-br^2}{2}}$ for some $d \in \mathbb{C}$ and $b > 0$.

Proof. Assume $f \in L^2_{(\mu)}(\mathbb{R}_+)$. Using the relation (5.1) to apply the Heisenberg-Pauli-Weyl inequality for Hankel transform with index $\alpha + 2n$, we obtain

$$\begin{aligned}
\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} &= \|M^{-1}(r^s f)\|_{2,\mu_{\alpha+2n}}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_{\alpha+2n}(M^{-1}f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \\
&= \|r^s M^{-1}(f)\|_{2,\mu_{\alpha+2n}}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_{\alpha+2n}(M^{-1}f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \\
&\geq c\|M^{-1}f\|_{2,\mu_{\alpha+2n}} \\
&\geq c\|f\|_{2,(\mu)}.
\end{aligned}$$

If and only if $s = t = 1$ and $f(r) = dr^{2n}e^{\frac{-br^2}{2}}$, then $c = (\alpha+2n+1)^{\frac{st}{s+t}}$ with equality for $s, t \geq 1$. ■

In the next theorems, we establish inequalities that we will use to prove Heisenberg-Pauli-Weyl type inequality for $\mathcal{W}^\Delta_\psi$.

Theorem 5.3. *Let ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$ and $s, t > 0$. Then, for any function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, there is a constant $c = c(\alpha, n, s, t) > 0$, such that*

$$\|x^s \mathcal{W}^\Delta_\psi(f)\|_{2,(\nu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \geq c(\sqrt{C_\psi})^{\frac{t}{s+t}} \|f\|_{2,(\mu)},$$

Furthermore, $c = (\alpha+2n+1)^{\frac{st}{s+t}}$ if $s, t \geq 1$.

Proof. Considering that both of the integrals on the left-hand side are finite is a non-trivial situation. The function $x \mapsto \mathcal{W}_\psi^\Delta(f)(a,x)$ may be obtained by using the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$. For every $a \in \mathbb{R}_+^*$,

$$\left(\int_0^\infty x^{2s}|\mathcal{W}_\psi^\Delta(f)(a,x)|^2 \frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx\right)^{\frac{t}{s+t}} \left(\int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{s}{s+t}}$$
$$\geq c^2 \int_0^\infty |\mathcal{W}_\psi^\Delta(f)(a,x)|^2 \frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx.$$

Therefore, by integrating over $d\mu_{\alpha+2n}(a)$ and using Plancherel's theorem and Hölder's inequality for $\mathcal{W}_\psi^\Delta$, we obtain

$$\|x^s \mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{2t}{s+t}} \left(\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)\right)^{\frac{s}{s+t}}$$
$$\geq c^2\|\mathcal{W}_\psi^\Delta(f)\|_{(\vartheta)}^2 = c^2 C_\psi^\Delta \|f\|_{2,(\mu)}^2.$$

But, relations (3.6) and (2.18) yield

$$\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)$$
$$= \int_0^\infty \int_0^\infty \lambda^{2t}\left|\mathscr{F}_\Delta\left(f\# D_{\alpha,a}(\overline{\psi})\right)(\lambda)\right|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)$$
$$= \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty |\mathscr{F}_\Delta(D_{\alpha,a}(\overline{\psi}))|^2(\lambda)d\mu_{\alpha+2n}(a)\right)d\mu_{\alpha+2n}(\lambda).$$

Then, from relations (3.2) and (3.3) it follows

$$\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda) = C_\psi^\Delta \|\lambda^t \mathscr{F}_\Delta(f)\|_{\alpha,\mu_{\alpha+2n}}^2.$$

Then,

$$\|x^s \mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} = \|x^s \mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}} (\sqrt{C_\psi^\Delta})^{\frac{s}{s+t}} \|\lambda \mathscr{F}_\Delta(f)\|_{\alpha,\mu_{\alpha+2n}}^{\frac{s}{s+t}}$$
$$\geq c\sqrt{C_\psi^\Delta}\|f\|_{2,(\mu)}.$$

it yields the outcome. ■

Theorem 5.4. *Let ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$ and $s,t > 0$. Then, there exists a constant $c = c(\alpha,n,s,t) > 0$, such that*

$$\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}} \|a^t \mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\left(|\mathscr{F}_\Delta(\psi)|^2\right)(2t)}\right)^{\frac{s}{s+t}} \|f\|_{2,(\mu)},$$

for every function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, where $\mathcal{M} : f \mapsto \mathcal{M}(f)(z) = \int_0^\infty f(x)\frac{dx}{x^{z+1}}$ denotes the classical Mellin transform and $c_{\alpha+2n}$ is the constant given in (2.2).

Moreover, if $s,t \geqslant 1$ then $c = (\alpha+2n+1)^{\frac{st}{s+t}}$ and we have equality if and only if $s = t = 1$ and $f(r) = dr^{2n}e^{-br^2/2}$, $d \in \mathbb{C}$, $b > 0$.

Proof. Let us assume the non-trivial case that both integrals on the left-hand side are finite.

Using Fubini's theorem, Plancherel's theorem for $\mathscr{F}_\Delta$ given by (2.16) and the relation (2.18), we get

$$\begin{aligned}
\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\nu)}^2 &= \int_0^\infty a^{2t}\left(\int_0^\infty |\mathcal{W}_\psi^\Delta f(a,x)|^2\frac{x^{2\alpha+1}}{c_{\alpha+2n}}\right)d\mu_{\alpha+2n}(a)\\
&= \int_0^\infty a^{2t}\left(\int_0^\infty |\mathscr{F}_\Delta(f\#D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)d\mu_{\alpha+2n}(a)\\
&= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty a^{2t}|\mathscr{F}_\Delta(D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 a^{2\alpha+4n+1}da\right)d\mu_{\alpha+2n}(\lambda)\\
&= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty a^{2t}|D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_\Delta(\overline{\psi})(\lambda)|^2 d\mu_{\alpha+2n}(a)\right)d\mu_{\alpha+2n}(\lambda)\\
&= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\frac{1}{c_{\alpha+2n}}\int_0^\infty a^{2t}|\mathscr{F}_\Delta(\overline{\psi})(\frac{\lambda}{a})|^2\frac{da}{a}\right)d\mu_{\alpha+2n}(\lambda),
\end{aligned}$$

by a change of variables $b=\frac{\lambda}{a}$, it follows

$$\begin{aligned}
\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^2 &= \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(f)(\lambda)|^2\left(\frac{1}{c_{\alpha+2n}}\int_0^\infty |\mathscr{F}_\Delta(\overline{\psi})(b)|^2\frac{db}{b^{2t+1}}\right)d\mu_{\alpha+2n}(\lambda)\\
&= \big(\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)\big)\|\lambda^t\mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^2
\end{aligned} \tag{5.3}$$

Now, applying Heisenberg-Pauli-Weyl inequality for $\mathscr{F}_\Delta$ given in the relation (5.2), we get

$$\begin{aligned}
\|r^s f\|_{2,(\mu)}^{\frac{t}{t+s}}\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{t+s}} &= \left(\frac{1}{c_{\alpha+2n}}\sqrt{\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{t+s}}\|r^s f\|_{2,(\mu)}^{\frac{t}{t+s}}\|\lambda^t\mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{t+s}}\\
&\geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{t+s}}\|f\|_{2,(\mu)}.
\end{aligned}$$

■

The next theorem proves the Heisenberg-Pauli-Weyl uncertainty principle for $\mathcal{W}_\psi^\Delta$ which involves the two variables of the time-frequency plan.

Theorem 5.5. *Let $s,t>0$ and ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$. Then, there exists a constant $c=c(\alpha,n,s,t)>0$, such that*

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}}\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\big(|\mathscr{F}_\Delta(\psi)|^2\big)(2t)}\right)^{\frac{s}{s+t}}\big(\sqrt{C_\psi^\Delta}\big)^{\frac{t}{s+t}}\|f\|_{2,(\mu)},$$

for every function $f\in L^2_{(\mu)}(\mathbb{R}_+)$. Moreover, if $s,t\geq 1$ then $c=(\alpha+2n+1)^{st/(s+t)}$.

Proof. From the equality (5.3),

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}}\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}} = \|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}}\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{s+t}})\|\lambda^t\mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}},$$

thus, using Theorem 5.3, we get

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}}\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\nu)}^{\frac{s}{s+t}} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\big(|\mathscr{F}_\Delta(\psi)|^2\big)(2t)}\right)^{\frac{s}{s+t}}\big(\sqrt{C_\psi^\Delta}\big)^{\frac{t}{s+t}}\|f\|_{2,(\mu)}.$$

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