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Analysis and optimal control for SEIR mathematical modeling of COVID-19

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Abstract. In this paper a mathematical model of SEIR type is formulated. represented by modeling the coronavirus epidemic. In this present study, we consider a mathematical model that incorporates the whole population and variability in transmission between reported and unreported populations. The global stability of the disease free equilibrium (DFE) point is established. The basic reproduction number R_0 is calculated. We introduce into our model two controls which are vaccination of susceptible humans denoted by u and treatment of infected humans designed by v. In addition, this model takes into consideration the control of contact (γ) between infectious individuals and susceptible persons. A numerical simulation of the model is made.

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Contents

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1. Introduction

The pandemic of COVID-19 is an infectious disease caused by the virus Sars-CoV-2 and characterized by severe acute respiratory syndrome. In late December 2019, the disease COVID-19 was first identified in China precisely in city of Wuhan ([37]). This virus has caused several deaths in the world and deserves the attention of researchers. In Burkina Faso, the first case is detected on March 9, 2020 ([28, 35]). Today, there is no effective treatment that has been accepted. In response to this epidemic, the state has taken a number of measures to reduce the spread of the virus. The best way to fight COVID-19 is to find ways to limit the spread of the virus in public spaces. The whole world is now concerned with the transmission of the disease by trying out vaccines, treatments and barrier measures in order to control the disease. In the literature, several mathematical models have been studied in order to show the dynamics of the infectious disease (see the references [8, 22, 38, 39]). Wu et al. ([38]) developed a susceptible exposed infectious recovered model (SEIR) to clarify the transmission dynamics and global spread of disease. Tang et al. ([32]) proposed a compartmental deterministic model that would combine the clinical development of the disease, the patient's state of health and intervention measures. Researchers found that the amount of control reproduction number may be as high as $R_0 = 6.47$, and that the methods of intervention, including contact followed by quarantine and isolation would effectively minimize COVID-19 cases ([9, 33]). Several modeling studies have already been performed for the COVID-19 outbreak (see [20, 27, 31–33]). Recent mathematical models with optimal control have been developed to study the COVID-19 pandemic. Hongzhi Lin and Yongping Zhang are studying a COVID-19 model to determine the optimal deployment of cordon sanitaires in terms of minimum queueing delay time with available health testing resources (see [14]). Shou Chens and Chen Xiao are studying a COVID-19 model to determine the associated credit risk contagion among financial institutions (see [2]). According to the models and the epidemiological characteristics of COVID-19 ([5]), we propose a SEIR type model to study the dynamics of this current pandemic (see [11, 20, 25, 29]). Our model is described by differential equations system and gives a comprehensive mechanism for the dynamics of COVID-19 transmission. In this model, we take into consideration the control of contact (γ) between infectious individuals and susceptible persons. We introduce into our model two controls which are vaccination of susceptible humans denoted by u and treatment of infected humans designed by v.

The organization of this paper is as follow: In Section 2, we formulate the mathematical model for COVID-19. In Section 3, we give Mathematical properties of the model (estimation of R_0, parameters with biological interpretation of model, positivity and boundedness of the solution). In Section 4, we establish the global stability of disease free equilibrium (DFE). In Section 5, we give a numerical simulation in order to illustrate the theoretical results. In Section 6, we give the optimal control problem and we derive the necessary condition for existence optimal control and we present the resulting numerical simulation. Finally, in Section 7, we give the conclusion.

2. Mathematical model

In this section, we formulate the mathematical model. Considering the characteristics of the COVID-19 pandemic, we have the following compartments:

- $S(t)$ Susceptible persons at time t.
- $E(t)$ Exposed and infectious persons at time t.
- $I(t)$ Infected and infectious persons at time t.
- $I_r(t)$ Symptomatic infected and infectious persons at time t (the number of persons infected who are reported and isolated at time t).
- $I_u(t)$ Asymptomatic infected and infectious persons at time t (the number of persons who are infected but do not have symptoms at time t).

- $R_r(t)$ Recovery of infected reported persons at time t.
- $R_u(t)$ Recovery of infected unreported persons at time t.

The class of infected individuals I is subdivided into two classes infected reported persons (I_r) and infected unreported persons (I_u) for the following reasons. Firstly COVID-19 patients do not test for COVID-19 because the test is expensive in this country. As a result, infected people do not show signs of the disease. These infectious persons move freely in the susceptible population and continue to infect them. They are the most vulnerable in the infection of COVID-19 and spread the disease more. Each individual in this class called infected unreported persons (I_u), heals alone and enters the class R_u. Secondly, those infected with COVID-19 who are tested positive are detected, isolated then treated. They are less infectious. Each individual of this class infected reported persons (I_r), heals by treatment and enters in the class R_r.
Therefore, we have the following transfer diagram:

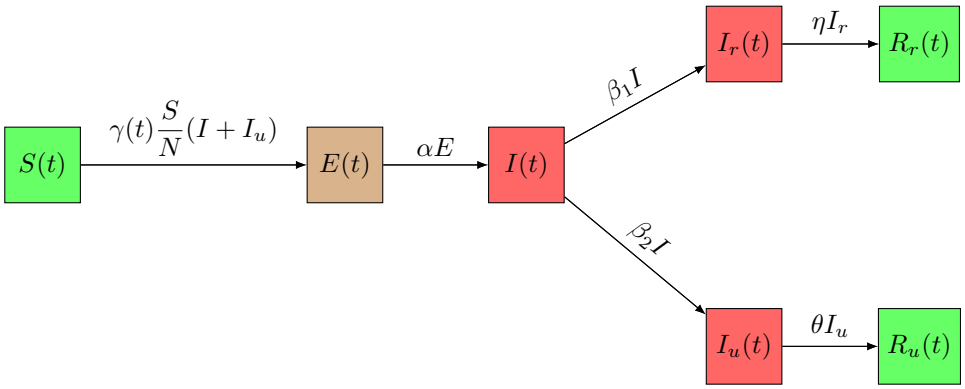

Figure 1: The transfer diagram.

According to the Figure 1 the corona virus mathematical model is

$$\begin{cases} \frac{dS}{dt} = \frac{-\gamma(t)S(I+I_u)}{N}, \\ \frac{dE}{dt} = \frac{\gamma(t)S(I+I_u)}{N} - \alpha E, \\ \frac{dI}{dt} = \alpha E - (\beta_1+\beta_2)I, \\ \frac{dI_r}{dt} = \beta_1 I - \eta I_r, \\ \frac{dI_u}{dt} = \beta_2 I - \theta I_u, \\ \frac{dR_r}{dt} = \eta I_r, \\ \frac{dR_u}{dt} = \theta I_u. \end{cases} \tag{2.1}$$

The initial conditions are:

$$S(0) = S_0 > 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad I_r(0) = I_{r0} \geq 0, \quad I_u(0) = I_{u0} \geq 0,$$

$$R_u(0) = R_{u0} \geq 0, \quad R_r(0) = R_{r0} \geq 0.$$

The total population at time t is given by:

$$N(t) = S(t) + E(t) + I(t) + R_r(t) + I_r(t) + I_u(t) + R_u(t).$$

and the total population $N_0 = S_0 + E_0 + I_0 + I_{r0} + I_{u0} + R_{r0} + R_{u0}$ at the initial time $t_0 = 0$ is constant. Parameters with biological interpretation of model (2.1)

- $\gamma(t)$: the contact rate of a person in state S at time t.
- α : the transition rate of a person in state E.
- β_1 : the transition rate between E and I_r.
- β_2 : the transition rate between E and I_u.
- θ : the transition rate of a person in state I_u to the state R_u.
- η : the transition rate of a person in state I_r to the state R_r.

3. Mathematical properties of the model

3.1. Estimation of R_0

The disease free equilibrium (DFE) of the model (2.1) is $X_0 = (S^0, E^0, I^0, I_u^0, I_r^0, R_r^0, R_u^0) = (N_0, 0, 0, 0, 0, 0, 0)$. We determine the basic reproduction number R_0 by applying Van Den Driesche and Watmougth method ([36]).

Proposition 3.1. *The basic reproduction number of model (2.1) is defined by*

$$R_0 = \frac{\gamma_0(\theta + \beta_2)}{\theta(\beta_1 + \beta_2)}. \tag{3.1}$$

Proof.

$$\mathcal{F} = \begin{pmatrix} \frac{\gamma S(I + I_u)}{N} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} -\alpha E \\ \alpha E - \beta I \\ \beta_1 I - \eta I_R \\ \beta_2 I - \theta I_u \end{pmatrix} \quad \text{where} \quad \beta = \beta_1 + \beta_2. \tag{3.2}$$

$\mathcal{F}$ is the new infection or contact function and $\mathcal{V}$ is the transition function.
$F = \left(\frac{\partial \mathcal{F}_j}{\partial x_i}\right)$ with $1 \leq i, j \leq 4$ and similarly $V = \left(\frac{\partial \mathcal{V}_j}{\partial x_i}\right)$ with $1 \leq i, j \leq 4$ and

$$X = \begin{pmatrix} E \\ I \\ I_r \\ I_u \end{pmatrix}.$$

which gives

$$V = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ \alpha & -\beta & 0 & 0 \\ 0 & -\beta_1 & -\eta & 0 \\ 0 & \beta_2 & 0 & -\theta \end{pmatrix} \quad \Leftrightarrow \quad V^{-1} = \begin{pmatrix} \frac{-1}{\alpha} & 0 & 0 & 0 \\ \frac{-1}{\beta} & \frac{-1}{\beta} & 0 & 0 \\ \frac{-\beta_1}{\beta\eta} & \frac{-\beta_1}{\beta\eta} & \frac{-1}{\eta} & 0 \\ \frac{-\beta_2}{\beta\theta} & \frac{-\beta_2}{\beta\theta} & 0 & \frac{-1}{\theta} \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & \frac{\gamma_0 S^0}{N_0} & 0 & \frac{\gamma_0 S^0}{N_0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

V^{-1} is determined by $VX = Y$, then we express the coordinates of vector X as a function of Y.

$$-FV^{-1} = \begin{pmatrix} \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} & \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} & 0 & \frac{\gamma_0 S^0}{N_0\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

this gives

$$\rho(-FV^{-1}) = \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} = R_0.$$

The basic reproduction number with γ_0 constant is:

$$R_0 = \frac{\gamma_0\,(\theta+\beta_2)}{\theta(\beta_1+\beta_2)} \tag{3.3}$$

Therefore

$$R_e(t) = \frac{\gamma(t)S(t)\,(\theta+\beta_2)}{N\theta(\beta_1+\beta_2)}$$

$R_e(t)$ is called the effective reproduction number at time t, it is defined as the number of cases that one infected person generates during his infectious period at time t in the presence of barrier measures controlled by $\gamma(t)$. After taking the measures, the number of contacts decreases and $\gamma(t)$ decreases as a function of time t. The disease slows when $R_e(t) < 1$. The basic reproduction number R_0 is defined as the number of cases that one infected person generates on average during his infectious period, in an uninfected population and without any special control measures. This number does not change during the spread of the disease. Furthermore, $\gamma(0) = \gamma_0$ and $R_e(0) = R_0$.

3.2. Positivity and boundedness of the solution for the Model

In this subsection, we show the positivity of the solution of model (2.1), let pose

$$\Omega = \left\{ (S(t), E(t), I(t), I_r(t), I_u(t), R_r(t), R_u(t)) \in \mathbb{R}_+^7; \begin{pmatrix} S(0) \\ E(0) \\ I(0) \\ I_r(0) \\ I_u(0) \\ R_r(0) \\ R_u(0) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

In general, the following lemma is used to show the positivity of the solutions of time-delay system where τ is the time-delay. In our model, the time-delay $\tau = 0$, therefore if the initial conditions are positive the lemma can be used. In the literature, the following lemma is used by T. Sarda et al. ([30]) and O. Harouna et al. (see [23]) to show the positivity of the solutions of ordinary differential equations.

Lemma 3.2. *([12]) Let $\Omega \subset \mathbb{R} \times \mathbb{C}^n$ an open and $f_i \in C(\Omega, \mathbb{R})$, $i = 1, ..., n$, if $f_i|_{x_i=0} \geq 0$ for $(x_1, ..., x_n) = X_t \in \mathbb{C}_{+0}^n$ then $\mathbb{C}_{+0}^n = \{\phi = (\phi_1, ..., \phi_n) : \quad \phi \in C([-\tau; 0], \mathbb{R}_+^n)\}$ is the invarious domain of the following equations:*

$$\frac{dx_i(t)}{dt} = f_i(t, X_t), \quad t \geq \tau, \quad i = 1, ..., n. \tag{3.4}$$

Where $\mathbb{R}_+^n = \{(X_1, ..., X_n) \in \mathbb{R}^n : \quad X_i \geq 0; \quad i = 1, ..., n\}$.

Proof. We consider the following equation

$$\frac{dx_i(t)}{dt} = f_i(t, X(t)) + \frac{1}{m}, \quad t \geq \tau, \quad i = 1, ..., n, \quad n, m \in \mathbb{N}^*. \tag{3.5}$$

Let $x_i(t)$ be the solution of (3.5) and $x_i(t) \geq 0$, $t \in [l - t, l]$, with $x_i(l) > 0$, $i = 1, ..., n$. If there is a $\tau > l$, $X_\tau \notin \mathbb{C}_{+0}^n$, then there must be i and t_0 such that $x_i(t_0) = 0$, $X_{it_0} \geq 0$, $t \in [l, t_0]$. This implies $\frac{dx_i(t_0)}{dt} \leq 0$. It contradicts because $\frac{dx_i(t_0)}{dt} = f_i(t_0, X_{t_0}) + \frac{1}{m} > 0$. So we can say that $\mathbb{C}_{+0}^n$ is the invarious domain of (3.5). Letting $m \longrightarrow +\infty$ we get that $\mathbb{C}_{+0}^n$ is the invarious domain of (3.4)

Proposition 3.3. *The set Ω is positively invariant, moreover the system (2.1) has a unique solution in Ω.*

Proof. We use the same technique as Harouna et al. ([23]) and Sardar et al. ([30]) to show the positivity of the solutions of system (2.1). The system (2.1) can be rewrited as follow

$$\frac{dX_i(t)}{dt} = f_i(t, X(t)), \quad X(0) = X_0 \geq 0, \quad i = 1, ..., 7,$$

where $X(t) = (S, E, I, I_r, I_u, R_r, R_u)$.

We can note that $\frac{dS}{dt}|_{(S=0)} = 0 \geq 0$, $\frac{dE}{dt}|_{(E=0)} = \frac{\gamma(t)S(I + I_u)}{N} \geq 0$,

$\frac{dI}{dt}|_{(I=0)} = \alpha E \geq 0$, $\frac{dI_u}{dt}|_{(I_u=0)} = \beta_2 I \geq 0$,

$\frac{dI_r}{dt}|_{(I_r=0)} = \beta_1 I \geq 0$, $\frac{dR_r}{dt}|_{(R_r=0)} = \eta I_r \geq 0$,

$\frac{dR_u}{dt}|_{(R_u=0)} = \theta I_u \geq 0$.
Then it follows from the Lemma 3.2 that Ω is an invariant set for the system (2.1).
For the second part of the proof, we use the same techniques as [34] to show the uniqueness of the system solutions (2.1). Let's now consider the following function

$$\dot{Y}(t) = g(t, Y(t)), \text{ where } \quad Y \in \Omega \tag{3.6}$$

and

$$g : \mathbb{R}^+ \times \mathbb{R}^7 \longrightarrow \mathbb{R}^7, \tag{3.7}$$

such as

$$g(t, Y(t)) = \begin{pmatrix} -\frac{\gamma(t)S(t)(I(t)+I_u(t))}{N} \\ \frac{\gamma(t)S(t)(I(t)+I_u(t))}{N} - \alpha E(t) \\ \alpha E(t) - (\beta_1 + \beta_2)I(t) \\ \beta_1 I(t) - \eta I_r(t) \\ \beta_2 I(t) - \theta I_r(t) \\ \eta I_r(t) \\ \theta I_u(t) \end{pmatrix}.$$

The function $g(.,.)$ is continuous and $t \mapsto g(t,.)$ is lipschitzian. By application of theorem.2.2.1 and theorem.2.2.3 of Hale and Verduyn Lunel ([7]), the system (2.1) has a unique solution in Ω.

Proposition 3.4. *The solution of system* (2.1) *is bounded in*
$\Omega_1 = \{(S, E, I, I_r, I_u, R_r, R_u) \in \Omega : \quad S + E + I + I_r + I_u + R_r + R_u \leq N_0\}$.

Proof. $N(t) = S(t) + E(t) + I(t) + I_u(t) + I_r(t) + R_r(t) + R_u(t)$, by using the system (2.1) we get

$\frac{dN}{dt} = 0 \Longleftrightarrow N$ is constant i.e $\forall t \geq 0, \quad N(t) = N(0) = N_0$. Therefore, for any $t \geq 0$ we obtain

$$0 \leq S(t) \leq N_0; \quad 0 \leq E(t) \leq N_0; \quad 0 \leq I(t) \leq N_0; \quad 0 \leq I_r(t) \leq N_0;$$

$$0 \leq R_u(t) \leq N_0; \quad 0 \leq R_r(t) \leq N_0.$$

Hence the system (2.1) is bounded in Ω_1.

4. Global stability of disease-free equilibrium (DFE)

In this section, we prove the global stability of the disease free equilibrium (DFE) point.

Theorem 4.1. *The DFE of the model* (2.1) *is globally asymptotically stable in* Ω *whenever* $R_0 \leq 1$.

Proof. We use the Lyapunov function technique. Let consider the follow candidate Lyapunov function:

$$V = \theta(E + I) + \gamma_0 I_u.$$

By definition, V is positive because the parameters of model (2.1) are positive. V is zero at DFE (X_0). We take the function V derivated with respect to t.

$$\begin{aligned}
\dot{V} &= \theta(\dot{E}+\dot{I})+\gamma_0\dot{I}_u \\
&= \theta\left[\frac{\gamma(t)S}{N}(I+I_u)-\alpha E+\alpha E-\beta I\right]+\gamma_0(\beta_2 I-\theta I_u) \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)-\theta\beta I+\gamma_0\beta_2 I-\gamma_0\theta I_u \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)-\theta\beta I+\gamma_0\beta_2 I-\theta\gamma_0 I_u+\beta_2\gamma(t)I_u-\beta_2\gamma(t)I_u \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)+\gamma_0\beta_2(I+I_u)-\beta\theta I-\theta\gamma_0 I_u-\gamma_0\beta_2 I_u \\
&\leq (\theta\gamma_0+\beta_2\gamma_0)(I+I_u)-\beta\theta(I+I_u)+[\beta\theta-\theta\gamma_0-\beta_2\gamma_0]I_u \\
&\leq [\theta\gamma_0+\beta_2\gamma_0](I+I_u)-\beta\theta(I+I_u)+\beta\theta\left(1-\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}\right)I_u \\
&\leq (\theta\gamma_0+\beta_2\gamma_0-\beta\theta)(I+I_u)+\beta\theta\left(1-\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}\right)I_u \\
&\leq \beta\theta\left(\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}-1\right)(I+I_u)+\beta\theta(1-R_0)I_u \\
&\leq \beta\theta(R_0-1)(I+I_u)+\beta\theta[1-R_0]I_u \\
&\leq \beta\theta[(R_0-1)I+(R_0-1)I_u-(R_0-1)I_u] \\
&\leq \beta\theta(R_0-1)I.
\end{aligned}$$

Since all the parameters of the model (2.1) are non negative, it follows that $\dot{V}\leq 0$ for $R_0\leq 1$. Hence V is Lyapunov function on Ω. Therefore, by using the Lasalle invariance principle ([12]), we have : $(E(t),I(t),I_u(t))\longrightarrow(0,0,0)$ as $t\longrightarrow+\infty$.
Since $\lim\limits_{t\to+\infty} supE(t)=0$, $\lim\limits_{t\to+\infty} supI(t)=0$, $\lim\limits_{t\longrightarrow+\infty} supI_u(t)=0$. It follows that for sufficiently small $\epsilon\geq 0$, there exist constant $t_1\geq 0$, $t_2\geq 0$ and $t_3\geq 0$ such that
$\lim\limits_{t\to+\infty} supE(t)\leq\epsilon$, for all $t\geq t_1$
$\lim\limits_{t\to+\infty} supI(t)\leq\epsilon$, for all $t\geq t_2$ and $\lim\limits_{t\to+\infty} supI_u(t)\leq\epsilon$, for all $t\geq t_3$
Hence, it follows from the fifth equations of the model (2.1)
$\frac{dI_r}{dt}\leq\beta\epsilon-\eta I_r$. Therefore using comparison theorem

$$I_r^{\infty}=\lim_{t\to+\infty} supI_r(t)\leq\frac{\beta\epsilon}{\eta}\longrightarrow 0 \text{ as } \epsilon\longrightarrow 0. \tag{4.1}$$

Similarity (by using $\lim\limits_{t\to+\infty} infI_r(t)=0$)

$$I_{r\infty}=\lim_{t\to+\infty} infI_r(t)=0. \tag{4.2}$$

It follows from the two relations (4.1) and (4.2) above
$\lim\limits_{t\to+\infty} I_r(t)=0$.
It can also be shown that
$\lim\limits_{t\to+\infty} R_u(t)=0$, $\lim\limits_{t\to+\infty} R_r(t)=0$, $\lim\limits_{t\to+\infty} S(t)=N_0$.
Therefore by combining all equations above, it follows that each solution of the model equation (2.1), with initial conditions in Ω, approaches X_0 as $t\to+\infty$ for $R_0\leq 1$.

5. Numerical simulation

In this section, we propose the numerical simulation of mathematical model (2.1). The following curves are obtained by using scilab. Estimated values of the model (2.1) parameters and unknown initial conditions $(S_0, E_0, I_0, I_{r0}, I_{u0}, R_{r0}, R_{u0}) = (8000, 198, 2, 2, 0, 0, 0)$ are provided by [5]. The parameters values are given by the table 1.

Symbol	Values of model (2.1)	Source	Values of model (6.1)	source
α	0.1818	[5]	0.1818	[5]
γ_0	0.19	[5]	0.19	[5]
θ	0.0714	fixed	1/14	[5]
η	0.823	[5]	$1/14$	[5]
β_1	0.418	[5]	0.28	fixed
β_2	0.415	[5]	0.31	fixed
μ	0.127	[5]	0.42	fixed
A_1			8	fixed
A_2			10	[1]

Table 1: The values of the parameters for the simulation of model (2.1) and (6.1)

After 14 days, strong government measures in the country, such as isolation, quarantine, and the wearing of face mask, allowed the reduction of the transmission of new cases. For that we use an exponential decrease for the transmission rate $\gamma(t)$ given by ([5, 19])

$$\gamma(t) = \begin{cases} \gamma_0, & 0 < t < 14, \\ \\ \gamma_0 \exp(-\mu(t-14)), & t \geq 14. \end{cases}$$

For the simulation of model (2.1), we use the ode method in scilab given by following algorithm. The system (2.1) can be rewrited $\dot{x} = f(t, x)$ where $f(t, x) = f_i(t, x)$, i=1,...,7 and $x = (S, E, I, I_r, I_u, R_r, R_u)$.

Algorithm
function Xdot=f(t,X)
X_1dot=$f_1(t, X)$
X_2dot=$f_2(t, X)$
...
X_7dot=$f_7(t, X)$
endfunction
X=ode(X_0, t_0, f)
X_0 is the initial conditions at $t_0 = 0$, $t = 0 : 0.1 : 900$

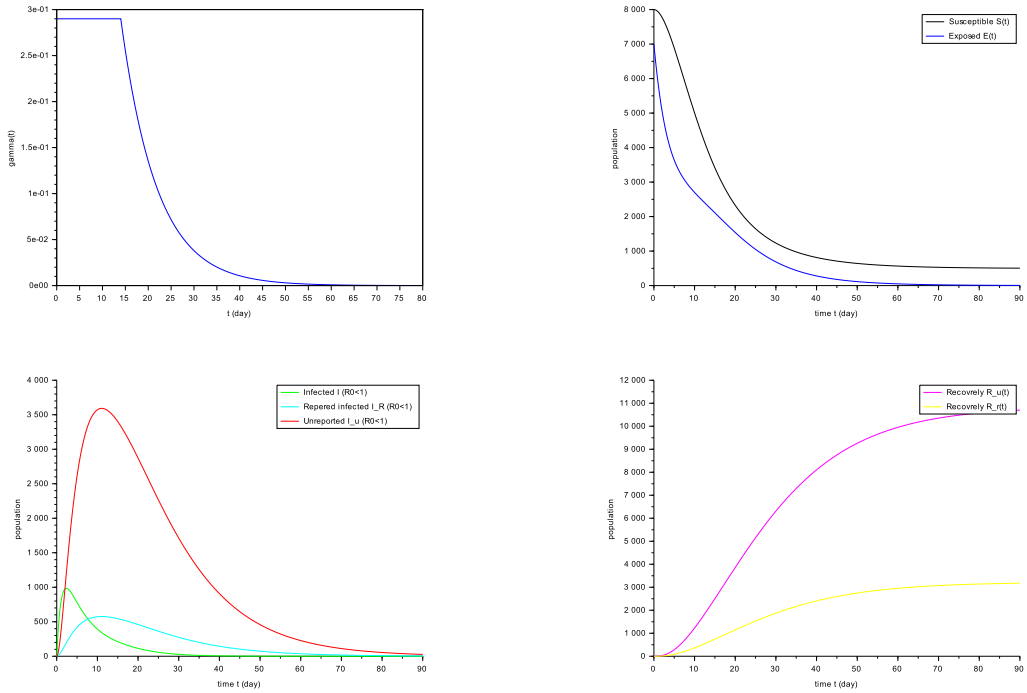

Figure 2: The variation of contact $\gamma(t)$ and population dynamic

The $\gamma(t)$ curve in the Figure 2 represents the variation of the contact : from 0 to 14 days, the infected remained in constant contact with the susceptible individuals. After the 14 days, the measures taken by the government permitted to reduce the contact between the infected and susceptible persons . In this case the contact function decreases and is canceled after 65 days when all measures taken by the government are respected.

The curves describing the dynamics of the susceptible (S) and the exposed (E) in Figure 2 decrease and stabilizes after 50 days. This decrease is due to the respect of the barrier measures taken by the government.

The curves describing the dynamics of infected individuals in Figure 2 show two phases. The increase of the curves in the first phase is due to the fact that there were no measures before the 14 days. After the 14 days, the measures that are taken allowed the reduction of the infected. If all the measures are respected then the disease disappears after 40 days.

6. The optimal control problem

The best way to control the COVID-19 epidemic is to respect the barrier measures which are represented here by $\gamma(t)$. The implementation of these measures is very complicated in practice because there are unreported infectious diseases. For this we need another alternative to control the disease. Furthermore, we first prove the existence of the two optimal controls u^*, v^* and we give their characterization.

6.1. Presentation of the problem

In this section we use the optimal control theory to analyze the behavior of the model (6.1). Our goal is to maximize the number of persons who have survived the disease (recovered) and to minimize the infected individuals during the course of an epidemic and the cost of this strategy. In the model (2.1), we introduce two controls $u; v$ which are defined as follow.

- The function $u(t) \in [0,1]$ is the control corresponding to the vaccination ([26]). The rate at which individuals gain immunity through vaccination is denoted by $u(t) \in [0,1]$ with $t \in [0, t_f]$. Because asymptomatic infected may not be aware of their infection, we assume that susceptible and asymptomatic infected are indistinguishable with respect to vaccination. Vaccinating asymptomatic infected individuals has no effect, but still implies a cost. The ideal is to vaccinate the entire population in this case $u = 1$. In reality this is not possible, so we try to vaccinate as many people. To find the maximum number of people we take $u = u_{max}$. u_{max} represents the proportion of susceptible persons receiving serum of vaccine.
- The second control $v(t) \in [0,1]$ represents the treatment of patients over the interval $[0; t_f]$. The control v that we consider here can therefore represent the treatment of symptomatic or the isolation of patients in hospitals to avoid possible new contamination.

By inserting the controls u and v in the model (2.1), we obtain the following controlled equations:

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{-(1-u)\gamma(t)S(I+I_u)}{N}, \\ \dfrac{dE}{dt} = \dfrac{(1-u)\gamma(t)S(I+I_u)}{N} - \alpha E, \\ \dfrac{dI}{dt} = \alpha E - (\beta_1+\beta_2)I, \\ \dfrac{dI_r}{dt} = \beta_1 I - (\eta+v)I_r, \\ \dfrac{dI_u}{dt} = \beta_2 I - \theta I_u, \\ \dfrac{dR_r}{dt} = (\eta+v)I_r, \\ \dfrac{dR_u}{dt} = \theta I_u. \end{cases} \tag{6.1}$$

$$S(t_0) = S_0 > 0, \quad E(t_0) = E_0 > 0, \quad I(t_0) = I_0 > 0, \quad I_r(t_0) = I_{r0} > 0, \quad I_u(t_0) = I_{u0} > 0,$$
$$R_u(t_0) = R_{u0} > 0, \quad R(t_0) = R_{r0} > 0.$$

Mathematically, for a fixed terminal time t_f, we minimize the functional objective J on $[0, t_f]$.

$$J(u,v) = \int_0^{t_f} \left(I_r(t) - R_r(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) \right) dt. \tag{6.2}$$

$A_1 > 0$ is the weight which allows to regulate the control u and $A_2 > 0$ the weight which allows to regulate the control v.

6.2. Study of optimal control problem

In this section, we define the Hamiltonian associated with the control problem. Then, we characterize the solutions of control problem (6.1) after proving their existence. Our work is to determine the optimal controls (u^*, v^*) such as

$$J(u^*, v^*) = min\{J(u,v) : (u,v) \in U \times V\} \tag{6.3}$$

U and V are the set of admissible controls defined by:

$$U = \left\{u(t) \in \mathbb{R}/ \quad 0 \leq u(t) \leq u_{max} < 1, \quad t \in [0, t_f], \quad u \in L^2([0; t_f], \mathbb{R})\right\}$$

and

$$V = \left\{v(t) \in \mathbb{R}/ \quad 0 \leq v(t) \leq 1; v \in L^2([0; t_f], \mathbb{R})\right\}.$$

Definition 6.1. *(Hamiltonian of the minimization problem)*
The Pontryagin's maximum principle [21] converted (6.1) , (6.2) *and* (6.3) *into problem of minimizing an Hamiltonian, H, defined by:*

$$H = I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{7} \lambda_i f_i.$$

Where f_i are the right side of the differential equations state variable and λ_i, $i = 1, ..., 7$ are the adjoints variables associated with their respective states.

Theorem 6.2. *Consider the optimal control problem* (6.1) *subject to* (6.2). *Then there exists an optimal pair of controls (u^*, v^*) and a corresponding optimal states $(S^*, E^*, I^*, I_u^*, I_r^*, R_r^* R_u^*)$ that minimizes the objective function $J(u, v)$ over set of admissible controls $U \times V$.*

Proof. The existence of optimal control can be proved by using the results from ([13] see Theorem 2.1) and Fleming's results (Theorem III.4.1, [4]), we must verify the following conditions:

- the set of admissible controls is nonempty,
- the admissible sets U, V are convex and closed,
- the vector field of the state system is bounded by a linear function of control,
- the objective function is convex,
- there exists constants $c_1, c_2 > 0$ such as the integrand of the objective function be bounded by $c_1(|u|^2 + |v|^2)^{\frac{p}{2}} - c_2$.

(1) We verify these conditions, thanks to a result of Lukes et al. [24] which assures the existence of solutions for the state system (6.1).

(2) The set U and V are convex and bounded by definition.

(3) The right-hand side of the state system (6.1) is bounded by a linear function in the state and control variables.

(4) The integrand of the objective functional is

$$f^0(x, u, v) = I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t).$$

The hessian matrix of $f^0(X, u, v)$ is given by :

$$M_{f^0} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

$spec(M_{f^0}) = \{A_1, A_2\} \subset \mathbb{R}_+^*$.
So f^0 is strictly convex over $U \times V$.

(5) We have,

$$\begin{aligned} f^0(x,u,v) &= I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) \\ &\geq \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) - R_r \\ &\geq \frac{1}{2}\min\{A_1, A_2\}\left(|u|^2(t) + |v|^2(t)\right)^k - R_r \\ &\geq c_1\left(|u|^2(t) + |v|^2(t)\right)^k - c_2 \end{aligned}$$

where $c_1 = \frac{1}{2}\min\{A_1, A_2\} > 0$, $c_1 \leq R_r \leq c_2$ and $k \geq 1$. Therefore the last assertion is verified.

6.3. Characterization of optimal control

In this section, we characterize the solutions of system (6.1).

Theorem 6.3. *Given an optimal $w^* = (u^*, v^*) \in U \times V$ and corresponding states $X^* = (S^*, E^*, I^*, I_u^*, I_r^*, R_r^*\ R_u^*)$ of system (6.1), there exist adjoint functions satisfying the following system.*

$$\begin{cases} \frac{d\lambda_1(t)}{dt} = \frac{(\lambda_1(t) - \lambda_2(t))\gamma(t)}{N}(1-u(t))(I + I_u), \\ \frac{d\lambda_2(t)}{dt} = (\lambda_2 - \lambda_3)\alpha, \\ \frac{d\lambda_3(t)}{dt} = \frac{(\lambda_1(t) - \lambda_2(t))\gamma(t)S(t)}{N}(1-u(t))(\lambda_3 - \lambda_4)\beta_4 + (\lambda_3 - \lambda_5)\beta_2, \\ \frac{d\lambda_4(t)}{dt} = -1 + \lambda_4(\beta_1 + \beta_2) - \lambda_6(\eta + v), \\ \frac{d\lambda_5(t)}{dt} = \frac{(\lambda_1(t) - \lambda_2(t))\gamma(t)S(t)}{N}(1-u(t)) + \theta(\lambda_5 - \lambda_7), \\ \frac{d\lambda_6(t)}{dt} = 1, \\ \frac{d\lambda_7(t)}{dt} = 0 \end{cases} \tag{6.4}$$

with the transversality conditions
$\lambda_1(t) = 0, \lambda_2(t) = 0, \lambda_3(t) = 0, \lambda_4(t) = 0, \lambda_5(t) = 0, \lambda_6(t) = 0, \lambda_7(t) = 0.$
Let's up $N^* = S^* + E^* + I^* + I_u^* + I_r^* + R_u^* + R_r^*$.
Furthermore, the optimal controls are characterized by:

$$\begin{aligned} u^* &= \max\left\{0, \min\left\{u_{max}, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\}\right\}, \\ v^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2}I_r^*\right\}\right\}. \end{aligned} \tag{6.5}$$

Proof. The differential equations for the adjoints are standard results from Pontryagin's Maximum Principle. Let $w^* = (u^*, v^*)$ corresponding solution $X^* = (S^*, E^*, I^*, I_R^*, I_u^*, R_r^*\ R_u^*)$ that minimizes $J(u, v)$ over $U \times V$. By applying the Pontryagin's maximum principle (see [21]) there exists adjoint functions,

$p(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t), \lambda_7(t))$ $(t \in [0, t_f])$ verifying the following conditions:

$$\frac{dp(t)}{dt} = -\frac{\partial H}{\partial X} \tag{6.6}$$

$$\frac{dX(t)}{dt} = \frac{\partial H}{\partial p} \tag{6.7}$$

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0. \tag{6.8}$$

$$\frac{dp(t)}{dt} = -\frac{\partial H}{\partial X} \Longleftrightarrow \begin{cases} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S} \\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial E} \\ \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I} \\ \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial I_r} \\ \frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial I_u} \\ \frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial R_r} \\ \frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial R_u}, \end{cases} \tag{6.9}$$

$$\lambda_i(t_f) = 0 \quad (i = 1, ..., 7).$$

Therefore, the system (6.9) yields (6.4).
By applying the optimality conditions to the (6.8), we obtain:

$$\frac{\partial H}{\partial u} |_{u^*} = 0, \tag{6.10}$$

$$\frac{\partial H}{\partial v} |_{v^*} = 0. \tag{6.11}$$

$\Rightarrow$ On the set $\{0 \leq u^*(t) \leq u_{max}\}$,$\{0 \leq v^*(t) \leq 1\}$, we have:
the conditions (6.10) and (6.11) give:

$$\begin{cases} \lambda_1(t)\gamma(t)\frac{S^*}{N}(I^* + I_u^*) - \lambda_2\gamma(t)\frac{S^*}{N}(I^* + I_u^*) + A_1u^* = 0 \\ -\lambda_4 I_r^* + \lambda_6 I_r^* + A_2v^* = 0. \end{cases} \tag{6.12}$$

As $-A_1 < 0$ and $-A_2 < 0$, so (6.12) becomes:

$$\begin{cases} 0 \geq \frac{(-\lambda_2(t) + \lambda_1(t))\,\gamma(t)S^*(I^* + I_u^*)}{-N^*A_1}, \\ 0 \geq \frac{-\lambda_4 I_r^* + \lambda_6 I_r^*}{-A_2}. \end{cases} \tag{6.13}$$

We obtain

$$\begin{aligned} u^* &= \max\left\{0, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\}, \\ v^* &= \max\left\{0, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2}I_r^*\right\}. \end{aligned} \tag{6.14}$$

$\Rightarrow \{u^*(t) = u_{max}\}$ and $\{v^*(t) = 1\}$.
The equation (6.12) gives

$$\begin{cases} -NA_1u_{max} \geq (-\lambda_2(t) + \lambda_1(t))\,\gamma(t)S^*(I^* + I_u^*) \\ -A_2 \geq -\lambda_4 I_r^* + \lambda_6 I_r^*. \end{cases}$$

This gives,

$$\begin{cases} u_{max} \leq \dfrac{(-\lambda_2(t) + \lambda_1(t))\,\gamma(t)S^*(I^* + I_u^*)}{-N^*A_1} \\ 1 \leq \dfrac{-\lambda_4 I_r^* + \lambda_6 I_r^*}{-A_2} \end{cases}$$

and thus

$$\begin{aligned} u^* &= \min\left\{u_{max}, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\} \\ v^* &= \min\left\{1, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2}I_r^*\right\}. \end{aligned} \tag{6.15}$$

The systems (6.15) and (6.14) give the result :

$$\begin{aligned} u^* &= \max\left\{0, \min\left\{u_{max}, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\}\right\} \\ u^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2}I_r^*\right\}\right\}. \end{aligned}$$

6.4. Numerical simulation of the controlled model

Several modeling studies have already been performed for the simulation of optimal contol model like Liu et al. ([16–18]). Here, we present the numerical results of the system (6.1) by using python and the same method of [1]. The boundary conditions of optimality system at times $t_0 = 0$ and t_f are separated. We put $N_0 = 200000$ representing the number of the total population of a city in our country. We use the Euler method of step h=0.1 to solve the optimality system (6.1). We discretize the model in interval $[t_0, t_f]$ at time $t_i = t_0 + ih$ (i= 0,1,...,n), where $h = 0.1$ is the time step such that $t_n = t_f = 90$ days, $t_0 = 0$. The value n=900 is the number of points of the discretization. Our algorithm is inspired by [1, 3, 6, 10, 15] to approximate the solutions. A combination of forward and backward difference, we obtain the following approximation:

$$\begin{cases} \frac{S_{i+1}-S_i}{h} = -(1-u_i)\gamma_i \frac{S_{i+1}}{N_0}(I_i + I_u^i) \\ \frac{E_{i+1}-E_i}{h} = (1-u_i)\gamma_i \frac{S_{i+1}}{N_0}(I_i + I_u^i) - \alpha E_{i+1} \\ \frac{I_{i+1}-I_i}{h} = \alpha E_{i+1} - (\beta_1+\beta_2) I_{i+1} \\ \frac{I_r^{i+1}-I_r^i}{h} = \beta_1 I_{i+1} - (\eta + v_i) I_r^{i+1} \\ \frac{I_u^{i+1}-I_u^i}{h} = \beta_2 I_{i+1} - \theta I_u^{i+1} \\ \frac{R_r^{i+1}-R_r^i}{h} = (\eta + v_i) I_r^{i+1}. \\ \frac{R_u^{i+1}-R_u^i}{h} = \theta I_u^{i+1}. \end{cases}$$

By using a similar technique in [1], we approximate the time derivative of the adjoint variables by their first order backward difference and we use the appropriate scheme as follows:

$$\begin{cases} \frac{\lambda_1^{n-i}-\lambda_1^{n-i-1}}{h} = \frac{(\lambda_1^{n-i-1}-\lambda_2^{n-i})\gamma_i}{N_0}(1-u_i)(I_{i+1}+I_u^{i+1}) \\ \frac{\lambda_2^{n-i-1}-\lambda_2^{n-i}}{h} = \alpha(\lambda_2^{n-i-1}-\lambda_3^{n-i}) \\ \frac{\lambda_3^{n-i}-\lambda_3^{n-i-1}}{h} = \frac{(\lambda_1^{n-i-1}-\lambda_2^{n-i-1})\gamma_i}{N_0}(1-u_i)S_{i+1} + \beta_1(\lambda_3^{n-i-1}-\lambda_4^{n-i}) \\ \qquad + \beta_2(\lambda_3^{n-i-1}-\lambda_3^{n-i}) - \beta_1\lambda_4^{n-i} \\ \frac{\lambda_4^{n-i}-\lambda_4^{n-i-1}}{h} = \lambda_4^{n-i-1}(\eta+v_i) - \lambda_6^{n-i}(\eta+v_i) - 1 \\ \frac{\lambda_5^{n-i}-\lambda_5^{n-i-1}}{h} = \frac{(\lambda_1^{n-i-1}-\lambda_2^{n-i-1})\gamma_i S_{i+1}}{N_0}(1-u_i) + \theta(\lambda_5^{n-i-1}-\lambda_7^{n-i}) \\ \frac{\lambda_6^{n-i}-\lambda_6^{n-i-1}}{h} = 1 \\ \frac{\lambda_7^{n-i}-\lambda_7^{n-i-1}}{h} = 0. \end{cases}.$$

The algorithm describing the approximation method to give the optimal control is the following.
Algorithm2.
Step1.

$$S(0)=S_0, \quad E(0)=E_0, \quad I(0)=I_0, \quad I_r(0)=I_{r0}, \quad I_u(0)=I_{u0},$$
$$R_u(0)=R_{u0}, \quad R_r(0)=R_{r0}, \quad \lambda_i(t_f)=0, \quad (i=1,...,7), \quad u(0)=v(0)=0.$$

Step2.
For $i=1,...,n+1$ do,

$$S_{i+1} = \frac{N_0 S_i}{N_0 + \gamma_i h(1-u_i)(I_i+I_u^i)}, \qquad E_{i+1} = \frac{N_0 E_i + h(1-u_i)\gamma_i S_{i+1}(I_i+I_u^i)}{N_0(1+h\alpha)}$$

$$I_{i+1} = \frac{I_i + h\alpha E_{i+1}}{1+h\beta_1+h\beta_2}, \qquad I_r^{i+1} = \frac{I_r^i + h\beta_1 I_{i+1}}{1+h(\eta+v_i)},$$

$$I_u^{i+1} = \frac{I_u^i + \beta_2 h I_{i+1}}{1+h\theta}, \qquad R_r^{i+1} = R_r^i + h(\eta + v_i) I_r^{i+1}.$$

$$\lambda_1^{n-i-1} = \frac{\lambda_1^{n-i} N_0 + h\lambda_2^{n-i}(1-u_i)(I_{i+1}+I_u^{i+1})}{N_0 + h(1-u_i)(I_{i+1}+I_u^{i+1})\gamma_i}$$

$$\lambda_2^{n-i-1} = \frac{\lambda_1^{n-i} + h\alpha\lambda_3^{n-i}}{1+h\alpha}$$

$$\lambda_3^{n-i-1} = \frac{\lambda_3^{n-i} + h(\lambda_2^{n-i-1} - \lambda_1^{n-i-1})\gamma_i(1-u_i)S_{i+1} + N_0 h\beta_2\lambda_5^{n-i} + N_0 h\beta_1\lambda_4^{n-i}}{N_0(1+h\beta_1+h\beta_2)}$$

$$\lambda_4^{n-i-1} = \frac{\lambda_4^{n-i} + h(\eta+v_i)\lambda_6^{n-i} + h}{1+h\beta_1+hv_i}$$

$$\lambda_5^{n-i-1} = \frac{N_0\lambda_5^{n-i} + h(\lambda_2^{n-i} - \lambda_1^{n-i-1})\gamma_i(1-u_i)S_{i+1} + hN_0\theta\lambda_7^{n-i}}{N_0 + N_0 h\theta}$$

$$\lambda_6^{n-i-1} = h + \lambda_6^{n-i}$$

$$\lambda_7^{n-i-1} = \lambda_7^{n-i}$$

$$M_{i+1} = \left(\frac{(\lambda_1^{n-i-1} - \lambda_2^{n-i-1})}{A_1}\right)\gamma_i \frac{S_{i+1}^*}{N_0}(I_{i+1}^* + I_u^{*(i+1)})$$

$$Z_{i+1} = \frac{\lambda_4^{n-i-1} - \lambda_6^{n-i-1}}{A_2} I_r^{*(i+1)}$$

$$u_{i+1} = \max(0, \min(u_{max}, M_{i+1}))$$

$$v_{i+1} = \max(0, \min(1, Z_{i+1})).$$

Step3.
For i =0,...,n, do
$S^*(t_i) = S_i$, $E^*(t_i) = E_i$, $I^*(t_i) = I_i$, $I_r^*(t_i) = I_r^i$, $I_u^*(t_i) = I_u^i$, $R_r^*(t_i) = R_r^i$,
$u^*(t_i) = u_i$, $v^*(t_i) = v_i$. The curves in this simulation are obtained by python. Certain values of the simulation are taken in [1, 5] and $(S_0, E_0, I_0, I_{r0}, I_{u0}, R_{r0}, R_{u0}) = (N_0, 198, 2, 2, 0, 0, 0)$.

The curves of infected reported persons in the Figure 3 are obtained by simulating the symptomatic infectious population. If left unchecked, the disease infection stabilizes within 120 days. But after application of control u (vaccination) and taking the control of the barrier measures $\gamma(t)$, the reported infected immediately decrease and stabilize in I_0. This is explained by the treatment of patients who are immediately isolated. The curves describing the dynamics of recovered persons in Figure 3 show the evolution of individuals cured of the disease by applying the reported individuals (I_r) the treatment (control v). The curves of unreported infectious persons in the Figure 3 show the evolution of individuals unreported by applying in the susceptible individuals (S) the vaccination (control u). After vaccination of susceptible, there is no effect of contact with unreported infected.

The curves of susceptible persons in the Figure 3 represent the dynamics of the susceptible population (S) for different aspects of control. After operation of the measures, the vaccination u and treatment v, the populations susceptible stabilizes.

The curves of exposed persons in the Figure 3 represent the dynamic of exposed population (E) for different

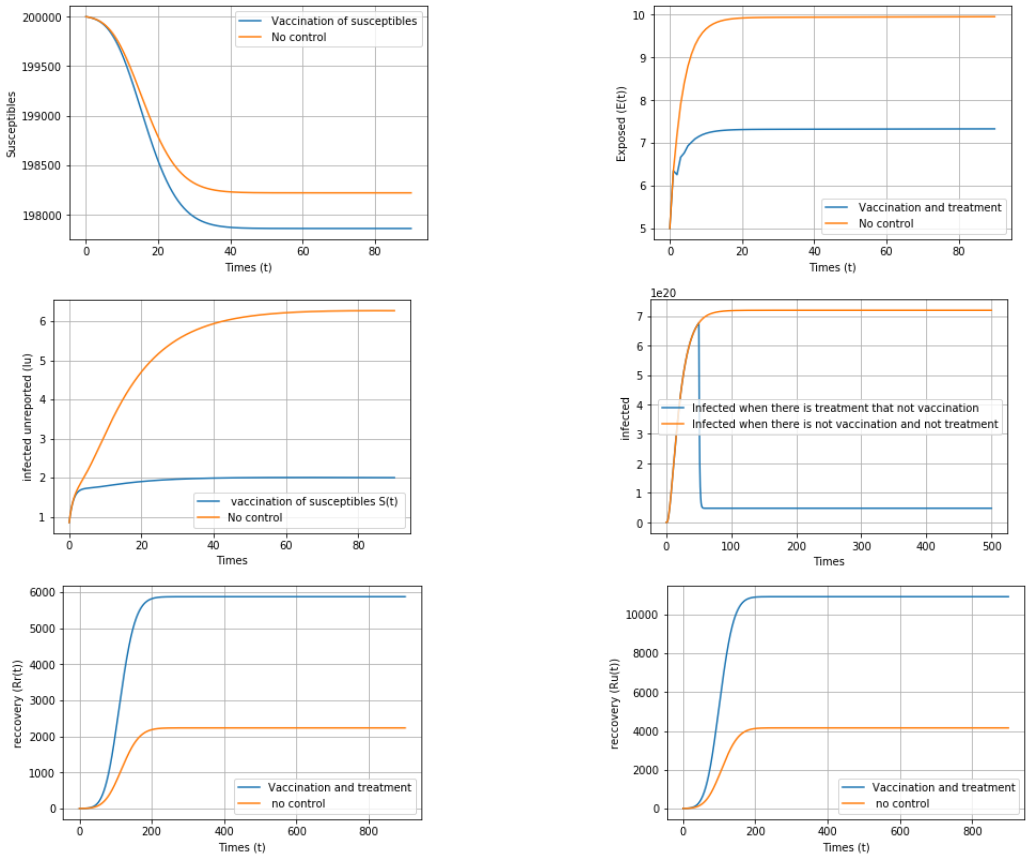

Figure 3: Population dynamic with and without control

aspects of control. The orange curve is the evolution of exposed population in application of u (vaccination) and v (treatment) controls. The blue curve is uncontrolled ($u = 0$ and $v = 0$).

The curves of unreported infectious persons in the Figure 3 represent the dynamics of the unreported infected and infectious population for different aspects of control u (vaccination) and v (treatment). The blue curve represents the evolution of the infected unreported population (I_u) with u and v control ($u \neq 0$ and $v \neq 0$). The orange curve represents the evolution of infected people who have not been brought back without control ($u = 0$ and $v = 0$).

The curves of recovered persons in the Figure 3 represent the dynamics of the reported cured (R_r) population for different aspects of controls. The blue curve is the evolution of cured reported in application of controls ($u \neq 0$ and $v \neq 0$). The orange curve is without control ($u = 0$ and $v = 0$).

The curves of unreported persons in the Figure 3 represent the dynamics of the unreported cured (R_u) population for different aspects of control u (vaccination) and v (treatment).

7. Conclusion

We have developed a model of the COVID-19 epidemic in China (see [20, 27, 31–33]). In this present study, we consider a mathematical model of COVID-19 transmission that incorporates the exposed populations. In our model, we also consider transmission variability between symptomatic and asymptomatic population with former being a fast spreader of the disease. The basic reproduction number is calculated by applying the Van den Driesch method [36]. We also construct the Lyapunov function to show the global stability of disease free equilibrium.

Next, we consider model (2.1) with the controls u (vaccination) and v (treatment of infected). In this model, the existence and uniqueness of the solution associated to the optimal controls are proven. The Hamiltonian function is constructed converting (6.1) into problem of minimizing an Hamiltonian. The $\gamma(t)$ function makes it possible to control the contact between infected individuals and those susceptible at time t. It takes into account all the measures taken by the government of a country. Finally a numerical simulation allows us to interpret the results on the curves. The study shows that the most infectious individuals are the unreported infected.

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