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# Letter on the Chinese-T-game

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**Abstract.** This letter presents three new minimization problems related to a new notion called the *Chinese-T-game* in a simple connected graph. The minimization problems stem from the *Chinese-T-walks* generated by the Chinese-T-game rules. It is a letter because some results rely on *axiomatic reasoning* which the author find sufficient. However, some readers may find the reasoning not sufficiently rigorous. It is foreseen that the *Chinese-T-game* will find application in at least, graph data science, robotics, AI, facial recognition, consumer preference analysis and alike. The ideas presented can easily be generalized to non-simple connected graphs.

**AMS Subject Classifications**: 05C05, 05C12, 05C30, 05C38, 05C45, 05C57.

**Keywords**: Chinese-T-game, Chinese-T-walk, Theresa pair, T-pair.

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## Contents



## 1. Introduction

Only finite, undirected and connected simple graphs of order $n \geq 2$ are considered. The size of a graph $G$ i.e. the number of edges of $G$ is denoted by $\varepsilon(G)$. Useful definitions will be recalled from [1].

If the edge $v_iv_j$ exists in a graph $G$ then moving from vertex $v_i$ to $v_j$ is called *traversing the edge* $v_iv_j$ or it is said, *to traverse the edge* $v_iv_j$. Recall that a *walk* in $G$ is a non-null sequence of neighboring vertices say, $W = v_0v_1v_2\cdots v_k$ which represent the sequential traversing of the edges $v_0v_1$, $v_1v_2$, $v_2v_3$, $\ldots$, $v_{k-1}v_k$. Vertex $v_0$ is called the *origin* of $W$ and vertex $v_k$ is called the *terminus* of $W$. Furthermore, there is no requirement that the vertices or edges are distinct. Assuming that in walk $W = v_0v_1v_2\cdots v_k$ the vertices are distinct then the value $k$ is called the *length* of the walk and is denoted by $\ell(W)$. In general however, if $t$ is the number of distinct edges of $G$ which are traversed in a walk $W$ then, $\ell(W) \geq t$. A *section* of a walk $W = v_0v_1v_2\cdots v_k$ is a subsequential part of a walk say, $v_iv_{i+1}v_{i+2}\cdots v_s$. Such section is called a $(v_i, v_s)$-*section* of $W$. The walk $W$ can also be written as,

$$W = (v_0, v_i)(v_i, v_s)(v_s, v_k).$$

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The sectional notation of a walk can be viewed as, ”section-wise (or sectional) walking the walk”. Generalization of the aforesaid notation is straightforward.

*Chinese Postman Problem:* A postman has to deliver letters to a given neighborhood. He needs to walk through all the streets in the neighborhood and back to the post-office. How can he plan his route so that he walks the shortest distance? See [3–5] and the references thereto. Note that in graph theoretic terms a simple connected graph has unweighted edges each of unit length. Hence, the Chinese Postman Problem translates to finding a minimum closed walk which traverse each edge at least once. It also implies that the common origin and terminus is predetermined or fixed. This problem attracted wide attention in the fields of mathematical programming, optimization theory, operations research and other blends of scientific disciplines. From the earlier work it is worthy to mention [2]. Many of the results from the mentioned disciplines can substitute the approach used in this letter. However, the graph theoretical approach is considered suitable for the new primary objectives stated later.

**Definition 1.1.** *A Chinese-T-walk in a graph $G$ is defined to be:*
*(i) For any $v_i \in V(G)$ as the origin, select a $S_1 = (v_i, v_s)$*-section *such that, $\ell(S_1) \leq deg(v_i)$.*
*(ii) Repeat step (i) in respect of vertex $v_s$ as the next origin to select a section $S_2$ and so-forth.*
*(iii) The Chinese-T-walk terminates at any finite step provided that each edge of $G$ has been traversed at least once.*
*(iv) If the Chinese-T-walk terminates after section $S_q$ the walk is given by $W = S_1S_2S_3\cdots S_q$.*

Since termination of a Chinese-T-walk is arbitrary after all edges have been traversed at least once it is axiomatically true that any graph $G$ has infinitely many Chinese-T-walks. Associated with a Chinese-T-walk $W$ over $q$ sections is the ordered pair called the *Theresa pair*[‡](for brevity, $T$-pair),

$$(q, \ell),\, \ell = \ell(W) = \sum_{i=1}^{q} \ell(S_i).$$

Clearly, for a graph $G$ we have $1 \leq q < \infty$ and $\varepsilon(G) \leq \ell < \infty$. To clarify the bounds let us consider the path $G = P_2$ on the vertices $v_1, v_2$. The Chinese-T-walk $W_1 = v_1v_2$ has the $T$-pair $(q, \ell) = (1, 1)$. On the other hand the Chinese-T-walk

$$W_2 = \underbrace{(v_1, v_2)(v_2, v_1)(v_1, v_2)\cdots(v_2, v_1)(v_1, v_2)}_{(v_1v_2)\ repeated\ t\ times} \text{ with } t < \infty$$

has the $T$-pair $(q, \ell) = (2t - 1, 2t - 1)$. Hence, both $q, \ell < \infty$. A $T$-pair is minimal in respect of $q$ or $\ell$ if and only if the Chinese-T-walk terminates immediately on the least step-count (section-count) required to traverse the last untraversed edge of $G$, once.

The motivation for this study is firstly, that it is a derivative of the Chinese Postman Problem and secondly, that three types of minimization problems come to the fore. For a graph $G$:
Type 1: Find a minimum Chinese-T($q^*$)-walk $W_1$ such that $q^* = min\{q : \forall$ minimal $(q, \ell)$ of $G\}$.
Type 2. Find a minimum Chinese-T($\ell^*$)-walk $W_2$ such that $\ell^* = min\{\ell : \forall$ minimal $(q, \ell)$ of $G\}$.
Type 3. Find a minimum Chinese-T($r^*$)-walk $W_3$ such that $r^* = min\{r = q + \ell : \forall$ minimal $(q, \ell)$ of $G\}$.
Note that Type 2 is equivalent to solving a derivative of the classical Chinese Postman Problem. For some graphs it is possible to find a minimum Chinese-T-walk which yields the $T$-pair, $(q^*, \ell^*)$. Such walk is called an *optimal* Chinese-T($r^*$)-walk. We can in terms of minimization improve on the bound i.e. $1 \leq q \leq \varepsilon(G)$. To illustrate the distinction between the minimization types, consider the path $P_3 = v_1v_2v_3$. Without loss of generality the following three minimal Chinese-T-walks can be found.

[‡]See dedication for an explanation.

$W_1 = S_1S_2$ with $S_1 = v_1v_2$, $S_2 = v_2v_3$ and $(q, \ell) = (2, 2)$.
$W_2 = S_1S_2S_3$ with $S_1 = v_2v_3$, $S_2 = v_3v_2$, $S_3 = v_2v_1$ and $(q, \ell) = (3, 3)$.
$W_3 = S_1S_2$ with $S_1 = v_2v_3v_2$, $S_2 = v_2v_1$ and $(q, \ell) = (2, 3)$.

In respect of Type 1 both $W_1, W_3$ yield minimum Chinese-T($q^*$)-walks. In respect of Type 2 the walk $W_1$ yields a minimum Chinese-T($\ell^*$)-walk. In respect of Type 3 the walk $W_1$ yields a optimal Chinese-T($r^*$)-walk. The walk $W_3$ remains a minimal Chinese-T-walk in that, following the unfortunate section $S_1$, though maximum itself, resulted in minimality only. An attempt to find minimization of Type 1, Type 2 or Type 3 respectively, the associated $T$-pairs of minimal Chinese-T-walks $W_1$, $W_2$, $W_3$ may be used by writing $(q^*, \ell_1) = (\leq q_2, \ell_2)$, $(q_1, \ell^*) = (q_2, \leq \ell_2)$ and $(q^*, \ell^*) = (\leq q_1, \leq \ell_1)$.

## 2. Chinese-T-game for certain graphs

Recall from [1] that if in a walk $W$ the edges are distinct (or put differently, an edge is traversed once) then $W$ is called a *trail*. If a trail in $G$ traverse all edges of $G$ the trail is called an *Euler trail*. Furthermore, if an Euler trail in $G$ is a closed trail it is called an *Euler tour* of $G$. A graph which has an Euler tour is said to be *Eulerian*. Note that an Eulerian graph inherently has an Euler trail. However, the converse is not necessarily true. It is well known that a graph $G$ is Eulerian if and only if $G$ has no vertex of odd degree. As a corollary it follows that a graph $G$ has an Euler trail if and only if $G$ has at most two vertices of odd degree.

**Theorem 2.1.** *A graph $G$ is Eulerian or has an Euler trail if and only if there exists a minimal Chinese-T-walk $W$ such that a minimal $T$-pair is of the form $(q, \varepsilon(G))$.*

**Proof**. If a graph $G$ is Eulerian or has an Euler trail $W$ and $q$ is not prescribed then the stepwise traversing procedure defined in Definition 1.1 if applied to minimal $W$ can only yield a minimal $T$-pair of the form $(q, \varepsilon(G))$. Conversely, if a closed Chinese-T-walk $W'$ exists with the minimal $T$-pair $(q, \varepsilon(G))$ then $W'$ complies with the definition of an Eulerian graph else if an open Chinese-T-walk exists with the minimal $T$-pair $(q, \varepsilon(G))$ then it complies with the existence of an Euler trail. ■

A direct consequence of Theorem 2.1 is stated as a corollary.

**Corollary 2.2.** *If a graph $G$ does not contain an Euler trail then any minimum Chinese-T-walk $W$ which yields $q^*$, $\ell^*$ or $r^*$ has $\ell^* > \varepsilon(G)$.*

Recall that a labeled path $P_n, n \geq 2$ on the consecutively labeled vertices say, $v_1, v_2, v_3, \ldots, v_n$ has the edge set $E(P_n) = \{v_iv_{i+1} : i = 1, 2, 3, \ldots, n-1\}$. The cycle $C_n$, $n \geq 3$ is obtained by closing the path $P_n$ with the edge $v_1v_n$. It is obvious that a path $P_n$ has an Euler trail and a cycle $C_n$ has an Euler tour.

**Proposition 2.3.** *(i) For a path $P_n$, $n \geq 2$ the optimal Chinese-T($r^*$)-walk yields the $T$-pair $(\frac{n}{2}, n-1)$ if $n$ is even and $(\frac{n+1}{2}, n-1)$ if $n$ is odd.*
*(ii) For a cycle $C_n$, $n \geq 3$ the optimal Chinese-T($r^*$)-walk yields the $T$-pair $(\frac{n}{2}, n)$ if $n$ is even and $(\frac{n+1}{2}, n)$ if $n$ is odd.*

**Proof**. Part 1, $P_n, n \geq 2$ is even: For $P_2$ the result $W = S_1 = (v_1v_2)$ is obvious . Assume it holds for $P_k$, $k > 2$ and even. Let the optimal Chinese-T($r^*$)-walk be, $W = (v_1v_2)(v_2v_3v_4)(v_4v_5v_6)\cdots(v_{k-2}v_{k-1}v_k)$. Hence, the $T$-pair for $P_k$ is $(\frac{k}{2}, k-1)$. Consider the path $P_{k+2}$. Obviously, at least one more section $(v_kv_{k+1}v_{k+2})$ will be required. Since $deg_{P_{k+2}}(v_k) = 2$ only one addition step is required which is optimal. It follows that the $T$-pair for the path $P_{k+2}$ is $(\frac{k}{2}+1, (k-1)+2) = (\frac{k+2}{2}, (k+2)-1)$. By induction the results holds for all even $n \geq 2$.
Part 1, $P_n$, $n$ is odd: The proof follows in similar fashion as in Part 1, $n$ is even.
Part 2, $C_n$, $n \geq 3$ and odd or even: The proofs of the two cases follow in similar fashion as in Part 1. ■

**Theorem 2.4.** *Consider a graph $G$.*
*(i) If $G$ is Eulerian then an optimal Chinese-T($r^*$)-walk $W$ in $G$ has the $T$-pair $(\leq \frac{\varepsilon(G)}{2}, \varepsilon)$ if $\varepsilon(G)$ is even and $(\leq \frac{\varepsilon(G)+1}{2}, \varepsilon(G))$ if $\varepsilon(G)$ is odd.*
*(ii) If a non-Eulerian graph $G$ has an Euler trail then a minimum Chinese-T($r^*$)-walk $W$ in $G$ has the $T$-pair $(\leq \frac{\varepsilon(G)}{2}, \varepsilon(G))$ if $\varepsilon(G)$ is even and $(\leq \frac{\varepsilon(G)+1}{2}, \varepsilon(G))$ if $\varepsilon(G)$ is odd.*

**Proof**. The result follows directly from Proposition 2.3 read together with the fact that all vertices in $G$ has $deg_G(v_i) \geq 2$. The aforesaid read together with Definition 1.1 implies that all sections of a minimum Chinese-T-walk has $\ell(S_j) \geq 2$. Finally, by definition it is possible to traverse each edge exactly once. ■

Recall that a star graph (star for brevity) $S_{1,n}$ is obtained by attaching $n$ pendent vertices say, $v_1, v_2, v_3, \ldots, v_n$ to a common central vertex $v_0$. For our purposes let $n \geq 3$.

**Proposition 2.5.** *Consider a star $S_{1,n}$, $n \geq 3$ then:*
*(i) If $n$ is even, a minimum $q^* = 2$ exists with $T$-pair, $(2, 2n-1)$.*
*(ii) If $n$ is even, a minimum $\ell^* = 2(n-1)$ exists with $T$-pair,*
*$(3, 2(n-1))$.*
*(iii) If $n$ is odd, a minimum $q^* = 2$ exists with the $T$-pair,*
*$(2, 2n-1)$.*
*(iv) If $n$ is odd, a minimum $\ell^* = 2n-3$ exists with the $T$-pair, $(3, 2(n-1)$.*

**Proof**. Observe that in any graph a pendent vertex is connected by a pendent edge. Hence, in a minimum Chinese-T-walk a pendent edge can be traversed once if and only if the edge serves either as the origin-edge or as the terminus-edge. Otherwise, the minimum number of times a pendent edge can be traversed is twice.
Part 1. Let $n \geq 4$ and even and let vertex $v_0$ be the origin of the Chinese-T-walk. Without loss of generality the

$$(v_0v_1v_0v_2v_0v_3 \cdots v_0v_{\frac{n}{2}}v_0)\text{-section}$$

is possible since $deg(v_0) = n$. Finally, the

$$(v_0v_{\frac{n}{2}+1}v_0v_{\frac{n}{2}+2}v_0 \cdots v_0v_n)\text{-section}$$

yields the desired Chinese-T($q^*$)-walk with corresponding $T$-pair, $(2, 2n-1)$.
Part 2. Let $n \geq 4$ and even and without loss of generality let vertex $v_1$ be the origin of the Chinese-T-walk. The first section can only be $v_1v_0$ since $deg(v_1) = 1$. Without loss of generality the

$$(v_0v_2v_0v_3v_0v_4 \cdots v_0v_{\frac{n}{2}+1}v_0)\text{-section}$$

is possible since $deg(v_0) = n$. Finally, the

$$(v_0v_{\frac{n}{2}+2}v_0v_{\frac{n}{2}+3}v_0 \cdots v_0v_n)\text{-section}$$

yields the desired Chinese-T($\ell^*$)-walk with corresponding $T$-pair, $(3, 2(n-1))$.
Part 3 and Part 4. The respective proofs follow in similar fashion as in Part 1 and Part 2. ■

For stars Proposition 2.5 show that the minimum $T$-pairs for respectively $q^*$ and $\ell^*$ are not equal. However, in the case of the star it is indicated that $r^* = q^* + \ell = q + \ell^*$. Characterizing graphs for which two distinct walk $W_1, W_2$ exist such that, $(q^*, \ell) \neq (q, \ell^*)$ and $r^* = q^* + \ell = q + \ell^*$ remains open.

**Theorem 2.6.** *A $t$-regular graph $G$ of order $n \geq 3$ with $t$ is even has an optimal Chinese-T($r^*$)-walk $W$ with corresponding $T$-pair, $(\lceil \frac{n}{2} \rceil, \frac{nt}{2})$.*

**Proof**. Since $G$ is $t$-regular, $t$ is even, the graph $G$ has an Euler tour. Hence, for a minimum Chinese-T($r^*$)-walk $W$ read together with Proposition 2.3(ii), the value $q^*$ is given by $\lceil \frac{\varepsilon}{t} \rceil = \lceil \frac{n \times t}{2t} \rceil = \lceil \frac{n}{2} \rceil$. Finally since, $\ell^* = \varepsilon(G) = \frac{nt}{2}$ the $T-$-pair is given by $(\lceil \frac{n}{2} \rceil, \frac{nt}{2})$ and is optimal. ■

Recall that a wheel graph (wheel for brevity) $W_{1,n}$ is obtain by taking a cycle $C_n$ and attaching each vertex $v_i \in V(C_n)$ to a central vertex $v_0$. The edges $v_0v_i$, $1 \leq i \leq n$ are called the *spokes* of the wheel. The edges of the cycle are called the *rim edges* of the wheel. Consider a wheel $W_{1,n}$, $n \geq 4$ and even. If every second rim edge is deleted a Dutch windmill graph, $DW_{1,n}$ is obtained.

**Proposition 2.7.** *Consider a complete graph $K_n$, $n \geq 4$.*
*(i) For $n \geq 5$ and odd the complete graph $K_n$ has an optimal Chinese-T($r^*$)-walk $W$ with corresponding $T$-pair, $(\lceil \frac{n}{2} \rceil, \frac{n(n-1)}{2})$.*
*(ii) For $n \geq 4$ and even the complete graph $K_n$ has an optimal Chinese-T($r^*$)-walk $W$ with corresponding $T$-pair, $(\frac{n+2}{2}, \frac{n^2-2}{2})$.*

**Proof**. As convention we only consider graphs of order $n \geq 2$ so $K_1$ is excluded. Since $K_2$ is a path and $K_3$ is a cycle and both have been dealt with let $n \geq 4$.
(i) For $n \geq 5$ and odd all $K_n$ the result is proved in Theorem 2.6.
(ii) For $n \geq 4$ and even consider $K_n$. Begin by considering the induced subgraph $G = K_{n-1}$ on vertices $v_1, v_2, v_3, \dots, v_{n-1}$ This subgraph $G$ has all degrees even at $n-2$. However, artificially ”increase” each vertex-degree by $+1$. Since $G$ structurally yields an Euler tour and Definition 1.1 permits a maximum section of length $n-1$, (”increase” $+1$ included) it takes exactly $q_1 = \frac{(n-1)(n-2)}{2(n-1)} = \frac{n-2}{2}$ sections say, $W_1 = S_1S_2S_3 \cdots S_{q_1}$ to yield an optimal Chinese-T($r^*$)-walk through $G$. Without loss of generality assume that $v_1$ serves as the origin and the terminus. Since $q_1$ is a divisor of $(n-1)(n-2)$ it follows that no other partial minimum Chinese-T-walk in $K_n$ can improve on the minimality of $W_1$ in order to traverse all edges in $G$. We are left with $(n-1)$ edges to traverse in a minimum number of additional sections with the additional aim to minimize the number of edges to be traversed twice. Consider the wheel $W_{1,n-1}$ on the cycle $C_{n-1} = v_1v_2v_3 \cdots v_{n-1}v_1$ and central vertex $v_n$. Note that all rim edges were traversed in $W_1$. Clearly $v_1$ serves as origin of say $W_2$. Traverse along a section of maximum length $n-1$ in a *Dutch windmill* fashion as follows:

$$S_1 = \underbrace{v_1v_nv_2v_3v_nv_4v_5v_n \cdots \triangleleft}_{\ell=n-1}, \text{ where } \triangleleft \text{ signals the end.}$$

It is easy to see that exactly two sections are required. Hence, $q^* = \frac{n-2}{2} + 2 = \frac{n+2}{2}$. Finally, it follows through enumeration and immediate induction that $\forall n \geq 4$ and even,

$$\ell^* = \underbrace{\frac{(n-1)(n-2)}{2}}_{\varepsilon(K_{n-1})} + \underbrace{(n-1) + \frac{n-2}{2}}_{Dutch\ windmill\ fashion} = \frac{n^2-2}{2}.$$

■

**Claim 2.8.** *The claim is that the second term in the line above, namely*

$$\underbrace{(n-1) + \frac{n-2}{2}}_{Dutch\ windmill\ fashion}$$

*is indeed a minimum. Any reader who doubt the validity of the claim may attempt to disprove it.*

## 3. On trees

Consider a tree $T$ on $n \geq 2$ vertices. Select a path $P_{t+1}$ in $T$ between any pair of distinct pendent vertices. Note that the length of $P_{t+1}$ equals $t$. Label the vertices of this path, $v_1, v_2, v_3, \dots, v_{t+1}$. Label the rest of the vertices from $v_{t+2}$ through to $v_n$. If a vertex $v_j \in V(P_{t+1})$ exists with $deg(v_j) = k \geq 3$ it is said that $k-2$ *branches* sprout from $v_j$. Denote these branches as $T^s_{v_j}$, $s = 1, 2, 3, \dots, k-2$. Note that each branch is a sub-tree hence, has no cycles. It implies that if a Chinese-T-walk traverses a branch from vertex $v_j$, the return to $v_j$ in order to proceed along the path $P_{t+1}$ will require that each edge of the branch be traversed at least twice. It is easy to see that in a minimal Chinese-T-walk it is required and, it is indeed possible to traverse each edge of a branch exactly twice. This observation is called the *principle of twiceness*. By traversing all branches which sprout from $v_j$ before proceeding to vertex $v_{j+1}$ it can be seen that the *principle of twiceness* does not apply to the edges of path $P_{t+1}$. The methodology is called the the *Descriptive Heuristic Method* or $DHM(T)$.

**Theorem 3.1.** *Consider a tree $T$ on $n \geq 2$ vertices. In respect of $\ell^*$ a minimum Chinese-T($\ell^*$)-walk has $T$-pair, $(q, 2(n-1)-t)$ where $diam(T) = t$ or more specifically, $\ell^* = 2(n-1)-t$.*

**Proof**. It follows from the $DHM(T)$ that to obtain a minimum Chinese-T($\ell^*$)-walk the length of the path $P_{t+1}$ in $T$ must be a maximum. For a tree $T$ under consideration select a path $P_{t+1}$ in $T$ such that the length of $P_{t+1}$ equals $diam(T) = t$. Clearly, the origin (by arbitrary choice) say, $v_1$ and the terminus say, $v_{t+1}$ of $P_{t+1}$ will be pendent vertices.
For the only tree on $n = 2$ i.e. $P_2$ (or $K_2$) the result $\ell^* = 1 = 2(2-1)-1$ holds. Similarly for the only tree on $n = 3$ vertices i.e. $P_3$ the result holds. For $n = 4$ the result is equally obvious for the tree, $P_4$. However, the star $S_{1,3}$ must be considered as well. Without loss of generality consider the diam-path $P_3 = v_1v_0v_2$. Note that the path $P_2 = v_0v_3$ sprouts at $v_0$. A minimum (in fact, optimal) $\ell^*$ is obtained by the minimum Chinese-T($\ell^*$)-walk, $W = (v_1v_0)(v_0v_3v_0v_2)$. Hence, $\ell^* = 4 = 2(4-1)-2$. So the result holds for all trees on $n = 4$ vertices. Similarly as reasoned thus far the result holds for the three distinct trees on $n = 5$ vertices (see https://www.graphclasses.org>smallgraphs). Assume the result holds for all distinct trees for each $n$, $6 \leq n \leq k$. Consider any tree $T$ on $n = k+1$ vertices. Remove any pendent vertex say, $v_m$ (with edge thereto) to obtain the tree $T_1$ on $k$ vertices. For $T_1$ the result $\ell^* = 2(k-1)-t$, $diam(T_1) = t$ holds. Replace vertex $v_m$ with pendent edge thereto. If the replacement was at a branch sprouting from the diam-path then a minimum edge-traverse count of $+2$ is required. Hence, $\ell^* = [2(k-1)-t]+2 = 2((k+1)-1)-t$. Note that the salient implication is that, $diam(T_1) = diam(T)$. Hence, the result holds. If $v_m$ sprouted directly from some internal vertex of $P_{t+1}$ a similar argument settles the result. However, if the replacement is to say, the terminus of the diam-path of $T_1$ it implies that $diam(T) = t+1$ in the first instance and a minimum edge-traverse count of $+1$ is required. Hence, $\ell^* = [2(k-1)-t]+1 = 2(k-1)+2-1-t = 2((k+1)-1)-(t+1)$. Clearly, in all cases a minimum edge-traverse count was obtained. This settles the result $\forall\, n \geq 2$. ∎

Finding either the value or an upper-bound for $q$ remains open.

For any tree $T$ the value of $q$ enumerated through the $DHM(T)$ is an upper bound. Hence, $q^* \leq q$. This observation will be illustrated by an example. Consider Figure 1 below.
Let $W_1 = (v_3v_2)(v_2v_1)(v_1v_4v_1v_5v_1v_6v_1)(v_1v_7v_1v_9v_8)$, $T$-pair $= (4, 12)$.
Let $W_2 = (v_1v_2v_3v_2v_1v_4v_1)(v_1v_5v_1v_6v_1v_7v_1)(v_1v_9v_8)$, $T$-pair $= (3, 14)$.
Let $W_3 = (v_4v_1)(v_1v_2v_3v_2v_1v_5v_1)(v_1v_6v_1v_7v_1v_9v_8)$, $T$-pair $= (3, 13)$.
The Chinese-T-walk $W_1$ yields $\ell_1^* = 12$. However, the corresponding $q_1 = 4 > 3$. $r_1^* = 16$
The Chinese-T-walk $W_2$ yields $\ell_2 = 14$. However, the corresponding $q_2^* = 3$. $r_2 = 17$
The Chinese-T-walk $W_3$ yields $\ell_3 = 13$. However, the corresponding $q_3^* = 3$. $r_3^* = 16$
Note that both $W_1, W_3$ have a pendent vertex as origin and $r_1^* = r_3^* = 16$.

**Theorem 3.2.** *Let $G$ be a graph of order $n \geq 2$ which does not have an Euler trail. Then there exist the $T$-pairs $(q^*, \ell)$, $(q, \ell^*)$ and possibly $(q^*, \ell^*)$ such that in the corresponding minimum (possibly optimal) Chinese-T-walks with regards to $q^*$, $\ell^*$ or $r^*$ traverse an edge at most, twice.*

**Proof**. Since any graph $G$ has at least one spanning tree the family of spanning trees is denoted by, $\mathcal{T}(G) = \{T : T \text{ is a spanning tree of } G\}$. We know that the result holds for any $T \in \mathcal{T}(G)$. Select a tree $T \in \mathcal{T}(G)$ with "weighed" degrees, $deg_G(v_i) \mapsto deg_T(v_i)$ and apply the Chinese-T-game rules per the

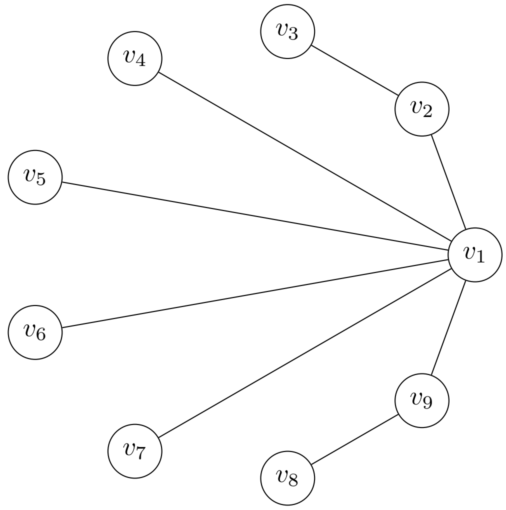


Figure 1: Tree graph $T$.

$DHM(T)$ with the proviso that each section traverses "edge by edge". If at an *edge step* a vertex $v_z \in V(T)$ is reached and there exists an edge $v_zv_w \in V(G)$ not in $T$, then either add the edge to $T$ to obtain graph $H_1$ or await reaching $v_w$ in $T$. Assume without loss of generality that the edge $v_zv_w$ is added. Simply traverse the added edge "across and back". Clearly, for each tree in $\mathcal{T}(G)$ the addition of edges from $E(G)\backslash E(T)$ to obtain successive graphs $H_1, H_2, H_3, \cdots, \lhd$ is always possible. Hence, there exists at least one way to reconstruct graph $G$ such that each edge in $E(G)$ is traversed at most twice. In fact the edges on a selected diam-path of $T$ are traversed once. The same reasoning is valid had the edge $v_zv_w$ been added upon reaching vertex $v_w$.

After repeating the $DHM(T)$ $\forall T \in \mathcal{T}(G)$ in all possible ways of selecting a diam-path, a set $Y$ of all the corresponding minimal $T$-pairs $(q, \ell)$ can be obtained. Certainly, $q^* \leq min\{q : (\leq q, \ell) \in Y\}$, $\ell^* = min\{\ell : (q, \leq \ell) \in Y\}$ and $r^* \leq min\{q + \ell : (\leq q, \leq \ell) \in Y\}$. Hence, in the corresponding minimum Chinese-T-walks an edge is traversed at most, twice. ■

**Corollary 3.3.** *For any graph $G$ with $diam(G) = t$ and without an Euler trail the value $\ell^*$ in a minimum Chinese-T($\ell^*$)-walk is bounded by, $\varepsilon(G) + 1 \leq \ell^* \leq 2\varepsilon(G) - t$.*

**Proof**. Because any graph $G$ has a diameter both $t$ and some diameter path $P_{t+1}$ and a spanning tree $T$ containing $P_{t+1}$ exist. The aforesaid read together with the result for trees and the result of Theorem 3.2 settles this result. ■

Recall that a graph $G$ contain a *Hamilton path* if and only if $G$ has a path which contains each vertex of $G$ exactly once. A Hamilton graph $G$ is said to be *Hamilton connected* if between any pair of distinct vertices of $G$ there exist a Hamilton path. If a Hamilton path in $G$ can be closed in $G$ then it is said that $G$ has a *Hamilton cycle* (or is *Hamiltonian*).

**Theorem 3.4.** *For graphs $G$ of order $n$ which has a Hamilton path and a Hamiltonian graph $H$ of order $m$ there exist for each, two pairs of $T$-pairs i.e. $(q_1, \ell_1)$, $(q_2, \ell_2)$ and $(q_3, \ell_3)$, $(q_4, \ell_4)$ respectively, such that:*
*(i) For $G$, $(q^*, \ell)$, $q^* \leq q_1$ and $(q, \ell^*)$, $\ell^* \leq \ell_2 \leq 2\varepsilon(G) - (n - 1)$.*
*(ii) For $H$, $(q^*, \ell)$, $q^* \leq q_3$ and $(q, \ell^*)$, $\ell^* \leq \ell_4 \leq 2\varepsilon(H) - n$.*

**Proof**. The results follow easily from reasoning similar to that found in the proof of Theorem 3.2. ■

## 4. Conclusion

For a graph in general an upper bound for $q^*$ is conjectured as follows.

**Conjecture 4.1.** *For any graph $G$ the value $q^*$ in an minimum Chinese-T-walk is bound by,*

$$\lceil \frac{\varepsilon(G)}{\Delta(G)} \rceil \leq q^* \leq \lceil \frac{n\varepsilon(G)}{\sum\limits_{v_i \in V(G)} deg_G(v_i)} \rceil + 1.$$

Recall Fleury's algorithm from [1]. Consider the Eulerian graph $G$ in Figure 2. Fleury's algorithm will result in at least ten possible Euler tours by simply selecting any origin from the ten vertices. Two such tours say, $W_1, W_2$ are:

$$W_1 = v_1 v_{10} v_6 v_7 v_8 v_9 v_{10} v_8 v_6 v_5 v_4 v_3 v_2 v_1 v_3 v_5 v_1$$
$$W_2 = v_9 v_8 v_7 v_6 v_8 v_{10} v_6 v_5 v_4 v_3 v_2 v_1 v_3 v_5 v_1 v_{10} v_9.$$

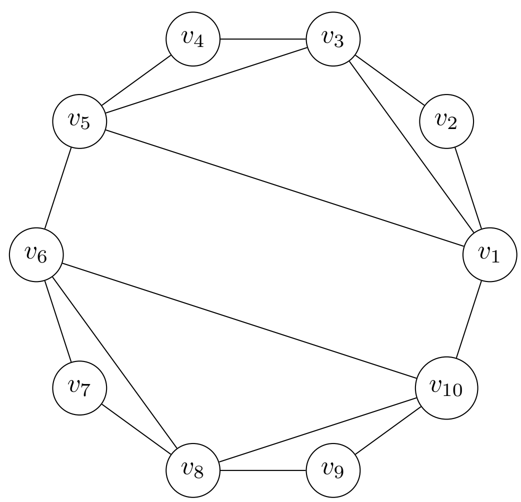

Figure 2: Eulerian graph $G$.

Applying the Chinese-T-game and traversing maximum section length at each step yields the following:

$$W_1 = (v_1 v_{10} v_6 v_7 v_8)(v_8 v_9 v_{10} v_8 v_6)(v_6 v_5 v_4 v_3 v_2)(v_2 v_1 v_3)(v_3 v_5 v_1) \text{ hence, } q_1 = 5$$
$$W_2 = (v_9 v_8 v_7)(v_7 v_6 v_8)(v_8 v_{10} v_6 v_5 v_4)(v_4 v_3 v_2)(v_2 v_1 v_3)(v_3 v_5 v_1 v_{10} v_9) \text{ hence, } q_2 = 6.$$

Firstly, we observe that $W_1$ yields the closer to optimal result since, $\ell^* = 16$ for all possible Euler tours. Since $G$ is not regular and $\Delta(G) = 4$ the lower bound $q^* \geq \lceil \frac{\varepsilon(G)}{\Delta(G)} \rceil$ read together with the fact that $\frac{\varepsilon(G)}{\Delta(G)} = 4$ precisely (4 a divisor of 16) convinces that $q* = 5$ is optimal. In general such deduction is unreliable. Let $W_i$, $i = 1, 2, 3, \ldots, s$ be all the possible Euler tours or Euler trails in a graph $G$ which permits such. Let the $T$-pair of $W_j$ be $(q_j, \ell_j)$. Then $(q^*, \ell^*) = (q^*, \varepsilon(G))$ where, $q^* = min\{q_i$ : over all $W_i, 1 \leq i \leq s\}$.
Problem 1: Can Fleury's algorithm or other appropriate algorithm such as found in Edmonds *et.al.* [2] be adapted to yield the minimum $q^*$ outcome for the Chinese-T-game? See Definition 1.1.
Problem 2. Investigate the Chinese-T-game by setting a min-max section length, $k_1 \leq \ell(S) \leq k_2$, $k_1 \geq 2$.
The classical vertex parameter is the degree of a vertex. However, various other vertex parameters have been published over the years. Clearly, any of these vertex parameters may serve as a bound for the length of a section in a Chinese-T-walk. This remark opens a wide avenue for further research. Theorem 3.2 is regarded as fundamental for further research.

## Dedication

This paper is dedicated to late Theresa Bernadette Kok (née Tomlinson) in acknowledgement of; and with deep gratitude for the profound influence she had on the author's endeavors to become a research mathematician.

*R.I.P. Spokie.*


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### Conflict of interest:

The author declares there is no conflict of interest in respect of this research.


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