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Nonlinear partial completely continuous operators in a partially ordered Banach space and nonlinear hyperbolic partial differential equations

BAPURAO C. DHAGE*[1]
[1] *Kasubai, Gurukul Colony, Thodga Road, Ahmedpur - 413515, Dist. Latur, Maharashtra, India.*

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Abstract. We prove a hybrid fixed point theorem for partial completely continuous operators in a partially ordered metric space and derive an applicable hybrid fixed point result in an ordered Banach space as a special case. As an application, we discuss a nonlinear hyperbolic partial differential equation for approximation result of local solutions by constructing the algorithms. Finally, an example is indicated to elaborate the hypotheses and abstract result of this paper.
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Contents

1. Introduction

Relaxing the convexity condition of the well-known Schauder fixed point theorem in a Banach space, the present author in Dhage [6] proved the following hybrid fixed point theorem in a partially ordered Banach space.

Theorem 1.1. *Let S be a non-empty, partially compact subset of a regular partially ordered Banach space $\big(X, \|\cdot\|, \preceq\big)$ and let every chain C in S be a Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 1.1 yields an applicable hybrid fixed point theorem in an ordered Banach space having numerous applications to nonlinear analysis. See Dhage [4], Dhage *et al.* [9–12], Dhage and Dhage [7], Dhage *et al.* [8] and references therein. Note that Theorem 1.1 removes the convexity condition from Schauder fixed pint theorem and replaced it by monotonicity condition of the operator in question. However, as a result we obtain an additional feature that it gives the algorithms which can be used to obtain the approximation of solution to the nonlinear problems. Now, the problem with the above hybrid fixed point theorem is that it is difficult to find the partially compact subset of an ordered Banach space always. To overcome this difficulty, here we relax the condition of existence of a partially compact subset and replace it by partial complete continuity of the operator $\mathcal{T}$ on S which is the main motivation of the present paper.

*****Corresponding author.** Email address: bcdhage@gmail.com (Bapurao C. Dhage)

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2. A Hybrid Fixed Point Principle

Before going to the main hybrid fixed point result, we give some preliminary definitions which we need in what follows. The details appear in Dhage [4, 5] and references therein.

Let $(E, d, \preceq)$ be a partially ordered metric space and let $S \subset E$. E is called **regular** if a monotone nondecreasing (resp. monotone nonincreasing) sequence $\{x_n\}$ in E converges to x_*, then $x_n \preceq x_*$ (resp. $x_* \preceq x_n$) for all $n \in \mathbb{N}$. The metric d and the order relation $\preceq$ are said to be **compatible** in S if a monotone sequence $\{x_n\}$ in S has a convergent subsequence, then the original sequence $\{x_n\}$ is convergent and converges to the same limit point. S is called a **Janhavi set** if d and $\preceq$ are compatible in it. S is called **partial bounded** (resp. partially closed, partially compact) if every chain C in S is bounded (resp. closed, compact).

A mapping $\mathcal{T} : S \to S$ is called **monotone nondecreasing** (resp. monotone nonincreasing) if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ (resp. $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$). $\mathcal{T}$ is **monotone** if it is either monotone nondereasing or monotone nonincreasing. $\mathcal{T}$ is called **partial bounded** (resp. partially totally bounded or partially precompact) if $\mathcal{T}(S)$ is partially bounded (resp. partially totally bounded or partially precompact for partially bounded S). T is **partially continuous** if $\{x_n\} \subset S$ converges to x_* with $x_n \preceq x_*$, then $\mathcal{T}x_n \to \mathcal{T}x$. T is called **partial completely continuous** if it is partially continuous and partially totally bounded.

Now we are equipped with all the necessary details to state our main result if this section.

Theorem 2.1. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered complete metric space $(E, d, \preceq)$ and let every chain C in S be Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partial completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Proof. Assume first that we have an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ and define a sequence $\{x_n\}_{n=0}^{\infty}$ of points in S by

$$x_{n+1} = \mathcal{T}x_n, \; n = 0, 1, 2, \ldots. \tag{2.1}$$

From the monotonic nndecreasing nature of $\mathcal{T}$, it follows that $\{x_n\}_{n=0}^{\infty}$ is a nondecreasing sequence of point in S, i.e., we have

$$x_0 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.2}$$

Consequently, $\{x_n\}_{n=0}^{\infty}$ is a chain in S. Denote $C = \{x_n\}_{n=0}^{\infty}$. Then, C is bounded and by the construction of $\{x_n\}_{n=0}^{\infty}$, we have

$$\begin{aligned} C &= \{x_0, x_1, x_2, \ldots\} \\ &= \{x_0\} \cup \{x_1, x_2, \ldots\} \\ &= \{x_0\} \cup \mathcal{T}(C). \end{aligned} \tag{2.3}$$

As $\mathcal{T}$ is partially completely continuous, we have that $\overline{\mathcal{T}(C)}$ is compact. From (2.3), $\overline{C}$ is also a compact set in S. As a result, $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ converging to a point, say, ξ^*. By hypothesis, $C = \{x_n\}_{n=0}^{\infty}$ is a Janhavi set in S, so the original sequence $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to ξ^*. Since $(E, \preceq, d)$ is a regular, we have that $x_n \to \xi^*$ and that $x_n \preceq \xi^*$ for all $n \in \mathbb{N}$. Finally, from partial continuity of $\mathcal{T}$, it follows that

$$\xi^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \mathcal{T}\xi^*.$$

Similarly, if $x_0 \succeq \mathcal{T}x_0$, it can be shown using analogous arguments that $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations converges monotone nonincreasingly to ξ^* Thus, in both the cases $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. This completes the proof. □

Corollary 2.2. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered Banach space $\big(X, \|\cdot\|, \preceq\big)$ and let every chain C in S be Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partial completely continuous and monotone nondecreasing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

If the Banach X is partially ordered by an order cone K in X, then in this case, we simply say that X is an ordered Banach space and we denote it by (X, K). The details of order cones and related fixed point theorems appear in the monographs Guo and Lakshmikantham [13] and Granas and Dugundji [14]. Then, we have the following useful results proved in Dhage [4, 5].

Lemma 2.3 (Dhage [4, 5])**.** *Every ordered Banach space (X, K) is regular.*

Lemma 2.4 (Dhage [4, 5])**.** *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.3 and 2.4 we obtain an applicable hybrid fixed point theorem in the area of nonlinear analysis and applications.

Theorem 2.5. *Let S be a non-empty, partially closed and partially bounded subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.5 is an improvement of the following hybrid fixed point theorem of Dhage *et al.* [9] which is comparatively more convenient for applications to nonlinear equations.

Theorem 2.6 (Dhage *et al.* [9])**.** *Let S be a non-empty and partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Remark 2.7. *We mention that Theorem 2.5 is an ordered Banach space version of the Schauder fixed point theorem wherein the convexity argument is altogether omitted and replaced by the monotonicity of the operator in question. The advantage of this approach over Schauder is that we obtain an algorithm which goes to the fixed point when applied repeatedly.*

3. Hyperbolic Partial Differential Equations

Given the closed and bunded intervals $J_a = [0, a]$ and $J_b = [0, b]$ in the real line $\mathbb{R}$, for some real numbers $a > 0$ and $b > 0$, consider the nonlinear IVP of hyperbolic partial differential equation (in short HPDE)

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u(x, y)), \ (x, y) \in J_a \times J_b, \tag{3.1}$$

satisfying the initial conditions

$$u(x, 0) = \phi(x) \quad \text{and} \quad u(0, y) = \psi(y), \tag{3.2}$$

where $f : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$, $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous functions.

Definition 3.1. *By a solution of the HPDE* (3.1)-(3.2) *we mean a function $u \in C(J_a \times J_b, \mathbb{R})$ that satisfies the equations in* (3.1)-(3.2)*, where $C(J_a \times J_b, \mathbb{R})$ is the space of continuous real-valued functions defined on $J_a \times J_b$. If a solution u exists in a neighbourhood of a point $z \in C(J_a \times J_b, \mathbb{R})$, then we say that it is a local or neighbourhood solution of the HPDE* (3.1)-(3.2) *defined on $J_a \times J_b$.*

The HPDE (3.1)-(3.2) is fundamental in the theory of nonlinear hyperbolic partial differential equations and widely discussed in the literature for existence of solution. See Lakshmikantham and Pandit [15] and references therein. But to the knowledge of present author no approximation result is proved for local solution without the assumption of Lipschitz condition on the function f or without the assumption of existence of both lower as well as upper solution for the HPDE (3.1)-(3.2) on $J_a \times J_b$. Therefore, the approximation result of this section seem to be new to the theory of hyperbolic partial differential equations.

We put the HPDE (3.1)-(3.2) in the Banach space $C(J_a \times J_b, \mathbb{R})$. We introduce a supremum norm $\|\cdot\|$ in $C(J_a \times J_b, \mathbb{R})$ defined by

$$\|u\| = \sup_{(x,y)\in J_a\times J_b} |u(x,y)|. \tag{3.3}$$

and an order relation $\preceq$ in $C(J_a \times J_b, \mathbb{R})$ by the cone K given by

$$K = \{u \in C(J_a \times J_b, \mathbb{R}) \mid u(x,y) \geq 0 \ \forall\ (x,y) \in J_a \times J_b\}. \tag{3.4}$$

Thus,

$$u \preceq v \iff v - u \in K, \tag{3.5}$$

or equivalently,

$$u \preceq v \iff u(x,y) \leq v(x,y) \ \forall\ (x,y) \in J_a \times J_b.$$

It is known that the Banach space $C(J_a \times J_b, \mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J_a \times J_b, \mathbb{R}), K\big)$. We denote the open and closed spheres centred at $z_0 \in C(J_a \times J_b, \mathbb{R})$ of radius r by

$$B_r(z_0) = \{u \in C(J_a \times J_b, \mathbb{R}) \mid \|u - z_0\| < r\} = B(z_0, r),$$

and

$$B_r[z_0] = \{u \in C(J, \mathbb{R}) \mid \|u - z_0\| \leq r\} = \overline{B(z_0, r)},$$

respectively.

Remark 3.2. *It is clear that an open ball $B(z_0, r)$ in $C(J_a \times J_b, \mathbb{R})$ centred at a point $z_0 \in C(J_a \times J_b, \mathbb{R})$ of radius $r > 0$ is a neighbourhood of the point z_0, so if a solution u^* of the HPDE (3.1)-(3.2) lies in a closed ball $\overline{B(z_0, r)}$ in $C(J_a \times J_b, \mathbb{R})$, then it is a* local solution *in view of the fact that $B(z_0, r) \subset \overline{B(z_0, r)} \subset B(z_0, r + \epsilon)$ for every $\epsilon > 0$. Note that the idea of local or nbhd-solution is different from the usual notion of a local solution as mentioned in Coddington [1].*

4. Local Approximation Results

We consider the following definition in the sequel.

Definition 4.1. *A function $f : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$ is said to be $L^1_{\mathbb{R}}$-Carathéodory if*

(i) the map $(x,y) \mapsto f(x,y,u)$ is jointly measurable for each $u \in \mathbb{R}$,

(ii) the map $u \mapsto f(x,y,u)$ is continuous for each $(x,y) \in J_a \times J_b$, and

(iii) there exists a function $h \in L^1(J_a \times J_b, \mathbb{R})$ such that

$$|f(x,y,u)| \leq h(x,y) \text{ a.e. } (x,y) \in J_a \times J_b,$$

for all $u \in \mathbb{R}$.

Lemma 4.2 (Granas and Dugundji [14]). *If $f(x, y, u)$ is $L^1_{\mathbb{R}}$-Carathéodory, then the function $(x, y) \mapsto f(x, y, u(x, y))$ is jointly measurable for each $u \in C(J_a \times J_b, \mathbb{R})$.*

We need the following hypotheses in what follows.

(H_1) The function f is $L^1_{\mathbb{R}}$-Carathéodory.

(H_2) $f(x, y, u)$ is nondecreasing in u for each $(x, y) \in J_a \times J_b$.

(H_3) $f(x, y, z_0(x, y)) \geq 0$ for all $(x, y) \in J_a \times J_b$, where $z_0(x, y) = \psi(y) + \phi(x) - \phi(0)$.

Now, by using the theory of partial differentiation and integration, we obtain the following useful result.

Lemma 4.3. *If $h \in L^1(J_a \times J_b, \mathbb{R})$, then the IVP of ordinary second order linear hyperbolic partial differential equation*

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= h(x, y), \ \ (x, y) \in J_a \times J_b, \\ u(x, 0) = \phi(x) \quad &\textit{and} \quad u(0, y) = \psi(y), \end{aligned} \tag{4.1}$$

is equivalent to the integral equation

$$u(x, y) = z_0(x, y) + \int_0^x \int_0^y h(s, t)\, ds\, dt, \ \ , (x, y) \in J_a \times J_b, \tag{4.2}$$

where $z_0(x, y) = \psi(y) + \phi(x) - \phi(0)$ is a continuous function on $J_a \times J_b$.

Theorem 4.4. *Suppose that the hypotheses (H_1), (H_2) and (H_3) hold. Furthermore, if $\|h\|_{L^1} \leq r$, then the HPDE (3.1)-(3.2) has a local solution u^* in $B_r[z_0]$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} u_0(x, y) &= z_0(x, y), \ \ (x, y) \in J_a \times J_b, \\ u_{n+1}(x, y) &= z_0(x, y) + \int_0^x \int_0^y f(s, t, u_n(s, t))\, ds\, dt, \ \ (x, y) \in J_a \times J_b, \end{aligned}\right\} \tag{4.3}$$

where $n = 0, 1, \ldots$; is monotone nondecreasing and converges to u^*.

Proof. Set $X = C(J_a \times J_b, \mathbb{R})$. Clearly, X is an ordered Banach space ordered by the cone K defined by (2.2). Let u_0 be a function on $J_a \times J_b$ such that $u_0 \equiv z_0$ on $J_a \times J_b$. Define a closed ball $B_r[z_0]$ in X, where $r \geq \|h\|_{L^1}$. By Lemma 4.2, the HPDE (3.1)-(3.2) is equivalent to the nonlinear hybrid integral equation (HIE)

$$u(x, y) = z_0(x, y) + \int_0^x \int_0^y f(s, t, u(s, t))\, ds\, dt, \ \ , (x, y) \in J_a \times J_b. \tag{4.4}$$

Now, define an operator $\mathcal{T}$ on $B_r[u_0]$ into X by

$$\mathcal{T}u(x, y) = z_0(x, y) + \int_0^x \int_0^y f(s, t, u(s, t))\, ds\, dt, \ \ , (x, y) \in J_a \times J_b. \tag{4.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.5 on $B_r[u_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r[z_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r[z_0]$ into itself. Let $u \in B_r[z_0]$ be arbitrary element. Then, by hypothesis (H_1),

$$\begin{aligned} |\mathcal{T}u(x,y) - z_0(x,y)| &= \left| \int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt \right| \\ &\le \int_0^x \int_0^y \big| f(s,t,u(s,t)) \big|\,ds\,dt \\ &\le \int_0^x \int_0^y h(s,t)\,ds\,dt \\ &\le \|h\|_{L^1}. \end{aligned}$$

Taking the supremum over x and y in the above inequality yields

$$\|\mathcal{T}u - z_0\| \le \|h\|_{L^1} = r$$

which implies that $\mathcal{T}u \in B_r[z_0]$ for all $u \in B_r[z_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $u, v \in B_r[z_0]$ be any two elements such that $u \succeq v$ on $J_a \times J_b$. Then,

$$\begin{aligned} \mathcal{T}u(x,y) &= z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt \\ &\ge z_0(x,y) + \int_0^x \int_0^y f(s,t,v(s,t))\,ds\,dt \\ &= \mathcal{T}v(x,y) \end{aligned}$$

for all $(x,y) \in J_a \times J_b$. So, $\mathcal{T}u \succeq \mathcal{T}v$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r[z_0]$ and let $\{u_n\}$ be a sequence of points in C converging to a point $u \in C$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}u_n(x,y) &= \lim_{n\to\infty} \left[z_0(x,y) + \int_0^x \int_0^y f(s,t,u_n(s,t))\,ds\,dt \right] \\ &= z_0(x,y) + \lim_{n\to\infty} \int_0^x \int_0^y f(s,t,u_n(s,t))\,ds\,dt \\ &= z_0(x,y) + \int_0^x \int_0^y \Big[\lim_{n\to\infty} f(s,t,u_n(s,t)) \Big]\,ds\,dt \\ &= z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt \\ &= \mathcal{T}u(x,y) \end{aligned}$$

for all $(x,y) \in J_a \times J_b$. Therefore, $\mathcal{T}u_n \to \mathcal{T}u$ pointwise on $J_a \times J_b$.

Next, we shows that $\mathcal{T}u_n$ is an equicontinuous sequence of functions on on the compact $J_a \times J_b$. Let $(x_1,y_1), (x_2,y_2) \in J_a \times J_b$ be arbitrary. Without loss of generality, we assume that $x_1 \le x_2$ and $y_1 \le y_2$. Then, by definition of $\mathcal{T}$, we have that

$$\begin{aligned}
&|\mathcal{T}u_n(x_1,y_1) - \mathcal{T}u_n(x_2,y_2)| \\
&\quad \leq |z_0(x_1,y_1)| - z_0(x_2,y_2)| + \int_{x_1}^{x_2}\int_{y_1}^{y_2} |f(s,t,u_n(s,t))|\,ds\,dt \\
&\quad \leq |z_0(x_1,y_1)| - z_0(x_2,y_2)| + \int_{x_1}^{x_2}\int_{y_1}^{y_2} h(s,t)\,ds\,dt \\
&\quad \to 0 \quad \text{as} \quad (x_1,y_1) \to (x_2,y_2),
\end{aligned} \tag{4.6}$$

uniformly for all n, $n = 1, 2, \ldots$. This shows that $\mathcal{T}u_n$ is an equicontinuous sequence of functions on $J_a \times J_b$. As a result, we have that $\mathcal{T}u_n \to \mathcal{T}u$ uniformly on $J_a \times J_b$. Hence $\mathcal{T}$ is partially continuous operator on $B_r[z_0]$.

Step IV: *$\mathcal{T}$ is partially totally bounded.*

Firstly, we show that $\mathcal{T}$ is partially uniformly bounded. Let C be a chain in $B_r[z_0]$. Then, by monotonicity of $\mathcal{T}$, the set $\mathcal{T}(C)$ is again a chain in $\mathcal{T}\big(B_r[z_0]\big)$. Let $v \in \mathcal{T}(C)$ be arbitrary. Then, there is a point $u \in C$ such that $v(x,y) = \mathcal{T}u(x,y)$. Now, by hypothesis ($H_1$),

$$\begin{aligned}
|v(x,y)| &= \big|\mathcal{T}u(x,y)\big| \\
&\leq |z_0(x,y)| + \int_0^x\int_0^y |f(s,t,u(s,t))|\,ds\,dt \\
&\leq \|z_0\| + \int_0^x\int_0^y h(s,t)\,ds\,dt \\
&\leq \|z_0\| + \|h\|_{L^1}
\end{aligned} \tag{4.7}$$

for all $(x,y) \in J_a \times J_b$. Taking the supremum over (x,y), we obtain $\|v\| \leq \|z_0\| + \|h\|_{L^1}$ for all $v \in \mathcal{T}(C)$. This shows that $\mathcal{T}$ is a partially uniformly bounded on $B_r[z]$. Next, proceeding as in the step III, it can be proved that $\mathcal{T}(C)$ is an equicontinuous chain of points in $\mathcal{T}\big(B_r[z_0]\big)$. As $\mathcal{T}(C)$ is uniformly bounded and equicontinuous set, it is precompact. Consequently $\mathcal{T}$ is partially precompact or partially totally bounded operator on $B_r[z_0]$. Now $\mathcal{T}$ is partially continuous and partially totally bounded, so it is partially completely continuous on $B_r[z_0]$.

Step V: *The element $u_0 = z_0 \in B_r[z_0]$ satisfies the order relation $u_0 \preceq \mathcal{T}u_0$.*

Since (H_3) holds, one has

$$\begin{aligned}
u_0(x,y) &= z_0(x,y) + \int_0^x\int_0^y f(s,t,u_0(s,t))\,ds\,dt \\
&\leq u_0(x,y) + \int_0^x\int_0^y f(s,t,z_0(s,t)\,ds\,dt \\
&= z_0(x,y) + \int_0^x\int_0^y f(s,u_0(s,t))\,ds\,dt \\
&= \mathcal{T}u_0(x,y)
\end{aligned}$$

for all $(x,y) \in J_a \times J_b$. As a result, we have $u_0 \preceq \mathcal{T}u_0$ on $J_a \times J_b$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.5, and so $\mathcal{T}$ has a fixed point u^* in $B_r[z_0]$ and the sequence $\{\mathcal{T}^n u_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to u^*. This further implies that the HIE (3.4) and consequently the HPDE (3.1)-(3.2) has a local solution u^* and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) converges monotone nondecreasingly to u^*. This completes the proof. □

Remark 4.5. *The conclusion of Theorems 4.4 also remains true if we replace the hypothesis (H_3) with the following one.*

(H_4) The function f satisfies $f(x, y, z_0(x, y)) \leq 0$ for all $(x, y) \in J_a \times J_b$.

In this case, the HPDE (3.1)-(3.2) has a local solution x^* defined on $J_a \times J_b$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) is monotone nonincreasing and converges to the solution u^*.

Remark 4.6. *If the initial condition (3.2) is such that $z_0(x, y) > 0$ for all $(x, y) \in J_a \times J_b$, then under the conditions of Theorem 4.4, the HPDE (3.1)-(3.2) has a local positive solution u^* defined on $J_a \times J_b$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) converges monotone nondecreasingly to u^*.*

Finally, we give an example to illustrate the abstract ideas involved in our main approximation result, Theorems 4.4.

Example 4.7. *Given a closed and bounded interval $J_1 = [0, 1]$ in $\mathbb{R}$, consider the IVP of nonlinear second order HPDE,*

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= (x + y) \tanh u(x, y), \\ u(x, 0) &= \frac{x}{2} \ \text{ and } \ u(0, y) = \frac{y}{2}, \end{aligned}\right\} \tag{4.8}$$

for all $(x, y) \in [0, 1] \times [0, 1]$.

Here, $f(x, y, u) = (x + y) \tanh u$, $\phi(x) == \frac{x}{2}$ and $\psi(y) == \frac{y}{2}$ for $(x, y) \in [0, 1] \times [0, 1]$ and $u \in \mathbb{R}$. We show that f satisfies all the conditions of Theorem 4.4. Clearly, f is $L^!_r$-Carathéodory on $[0, 1] \times [0, 1] \times \mathbb{R}$ with $h(x, y) = x + y$, and so the hypothesis (H_1) is satisfied. Also the function $f(x, y, u)$ is nondecreasing in u for each $(x, y) \in [0, 1] \times [0, 1]$. Therefore, hypothesis (H_2) is satisfied. Next, we have $z_0(x, y) = \frac{x}{2} + \frac{y}{2}$. Therefore, $f(x, y, z_0(x, y)) = (x + y) \tanh \left(\frac{x+y}{2}\right) \geq 0$ for each $(x, y) \in [0, 1] \times [0, 1]$, and so the hypothesis (H_3) holds. Now, by an application of Theorem 4.4, the HPDE (4.8) has a local solution u^* in the closed ball $B_1[z_0]$ of $C([0, 1] \times [0, 1], \mathbb{R})$ which is positive in view of Remark 4.6. Furthermore, the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} u_0(t) &= \frac{x + y}{2}, \quad (x, y) \in [0, 1] \times [0, 1], \\ u_{n+1}(t) &= \frac{x + y}{2} + \int_0^x \int_0^y (t + s) \tanh u_n(s, t)\, ds\, dt, \quad (s, t) \in [0, 1] \times [0, 1], \end{aligned}$$

converges monotone nondecreasingly to u^*.

5. The Comparison

We observe that the existence of solutions of the HPDE (3.1) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H_1) and restricted domain of intervals of the problem, but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (3.1) under the considered hypotheses (H_1)-(H_3) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $J_a \times J_b \times \mathbb{R}$. Similarly, we can not apply algebraic Tarski's fixed point theorem [16] or its extension obtained in Dhage [3] to HPDE (3.1) for proving the existence of solution, because the ordered Banach space $\big(C(J_a \times J_b, \mathbb{R}), \preceq\big)$ is not a complete lattice (see Davis [2]). Therefore, all these arguments show that our hybrid fixed point principle, Theorem 2.1 is very much advantageous to get more information about the solution of nonlinear equations in the subject of nonlinear analysis.

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