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Some coefficient properties of a certain family of regular functions associated with lemniscate of Bernoulli and Opoola differential operator

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Abstract. In this exploration, we introduce a certain family of regular (or analytic) functions in association with the right-half of the Lemniscate of Bernoulli and the well-known Opoola differential operator. For the regular function f studied in this work, some estimates for the early coefficients, Fekete-Szegö functionals and second and third Hankel determinants are established. Another established result is the sharp upper estimate of the third Hankel determinant for the inverse function f^{-1} of f.

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Contents

1. Introductory Statements

Firstly, we represent by $\mathcal{A}$, the family of *normalized and regular functions* whose form is of the Taylor's series

$$f(z) = z + \sum_{x=2}^{\infty} a_x z^x, \quad f(0) = f'(0) - 1 = 0 \tag{1.1}$$

and $z \in \Sigma := \{z \in \mathbb{C}, \text{ such that } |z| < 1\}$. Also, represented by $\mathcal{S}$ is the family of functions $f \in \mathcal{A}$ that are also univalent in Σ. A famous subfamily of $\mathcal{S}$ is the family $\mathcal{ST}$ of starlike functions. A function $f \in \mathcal{S}$ is said to be in $\mathcal{ST}$ if the condition $\mathcal{R}e(z(f'/f)) > 0$ holds. For function class $\mathcal{S}$, the Koebe one-quarter theorem, see [10],

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is a famous theorem that affirms that the range of every function $f \in \mathcal{S}$ includes the disk $\{w : |w| < 0.25\}$. For this purpose, $f \in \mathcal{S}$ has the inverse function f^{-1} where

$$f^{-1}(f(z)) = z, \quad z \in \Sigma,$$

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f),\ r_0(f) \geq 0.25,$$

and some computations show that

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

We represent the family of regular functions of the form

$$\wp(z) = 1 + \sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma \tag{1.3}$$

by $\mathcal{P}$ where $\mathcal{P}$ is called the family of functions with positive real parts in Σ. A generalization of (1.3) is the function

$$\wp_\sigma(z) = 1 + (1-\sigma)\sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma,\ 0 \leq \sigma < 1, \tag{1.4}$$

known as the function with positive real parts of order σ. Let $\mathcal{P}(\sigma)$ represent the family of functions $\wp_\sigma(z)$.

Let "$\prec$" represent subordination. Then for $f, F \in \mathcal{A}$, $f(z) \prec F(z)$ if there exists a Schwarz function

$$s(z) = \sum_{x=1}^{\infty} s_x z^x, \quad z \in \Sigma$$

such that $s(0) = 0$, $|s(z)| = |z| < 1$, and $f(z) = F(s(z))$. Suppose $F(z)$ is univalent in Σ, then

$$f(z) \prec F(z) \text{ if and only if } f(0) = F(0) \text{ and } f(\Sigma) \subset F(\Sigma).$$

Recently, the direction of research in theory of geometric functions shows that the study of some prescribed domains $\wp(\Sigma)$ is inexhaustible. In fact, special cases of functions $\wp(z)$ have greatly motivated many researchers to study various kinds of natural image domains of $\wp(\Sigma)$. Some of these domains can be found in [7, 9, 12, 13, 15, 16, 18, 21, 25–27, 29, 31] and the citations therein. *Precisely*, Sokól and Stankiewicz [32] reported the subfamily $\mathcal{SL}(\ell b) \subset \mathcal{ST}$ satisfying the condition

$$\varphi(z) = z(f'/f) \prec \ell b(z) = \sqrt{1+z}, \quad z \in \Sigma \tag{1.5}$$

such that function φ lies in the domain bounded by the *right half of the lemniscate of Bernoulli* which is geometrically represented by $|\varphi^2 - 1| < 1$, $\forall z \in \Sigma$. One can find some descriptive diagrams and more properties of domain $|\varphi^2 - 1| < 1$ in [32]. The work of Lockwood [20] is a treatise of curves available for further research.

The differential operator $\mathcal{D}^{n,\beta}_{\tau,\mu} : \mathcal{A} \longrightarrow \mathcal{A}$ was announced by Opoola [23], see also [4, 17, 27]. For $f \in \mathcal{A}$ of the form (1.1),

$$\begin{aligned}
&\mathcal{D}^{0,\beta}_{\tau,\mu} f(z) = f(z)\\
&\mathcal{D}^{1,\beta}_{\tau,\mu} f(z) = (1 + (\beta - \mu - 1)\tau)f(z) - z\tau(\beta - \mu) + z\tau f'(z) = \mathcal{J}_\tau(f(z))\\
&\mathcal{D}^{2,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{1,\beta}_{\tau,\mu} f(z))\\
&\mathcal{D}^{3,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{2,\beta}_{\tau,\mu} f(z))
\end{aligned}$$

and

$$\mathcal{D}^{n,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{n-1,\beta}_{\tau,\mu} f(z))$$

which can be simplified as

$$\mathcal{D}_{\tau,\mu}^{n,\beta} f(z) = z + \sum_{x=2}^{\infty} (1 + (x + \beta - \mu - 1)\tau)^n a_x z^x, \quad z \in \Sigma \tag{1.6}$$

for parameters in (2.2). It is clear that from (1.6),

1. $\mathcal{D}_{\tau,\mu}^{0,\beta} f(z) = \mathcal{D}_{0,\mu}^{n,\beta} f(z) = \mathcal{D}_{0,\mu}^{0,\beta} f(z) = f(z)$.
2. $\mathcal{D}_{1,\beta}^{n,\beta} f(z) = \mathcal{D}_{1,\mu}^{n,\mu} f(z) = \mathcal{D}^n f(z)$ is the famous Sălăgean differential operator, see [3, 30].
3. $\mathcal{D}_{\tau,\beta}^{n,\beta} f(z) = \mathcal{D}_{\tau,\mu}^{n,\mu} f(z) = \mathcal{D}_{\tau}^{n} f(z)$ is the famous Al-Oboudi differential operator, see [2].

2. A New Family of Regular Functions

The function f in $\mathcal{A}$ is in the family $\mathcal{B}_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b)$ if it satisfies the condition

$$(1 - e^{-2i\delta}\gamma^2 z^2) \frac{\mathcal{D}_{\tau,\mu}^{n+1,\beta} f(z)}{z} \prec \ell b(z) \tag{2.1}$$

for

$$n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ 0 \le \mu \le \beta;\ \beta, \tau \ge 0,\ \delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ 0 \le \gamma \le 1,\ z \in \Sigma, \tag{2.2}$$

$\ell b(z)$ and $\mathcal{D}_{\tau,\mu}^{n+1,\beta} f(z)$ are functions declared in (1.5) and (1.6), respectively. We however demonstrate that the following are special cases of $\mathcal{B}_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b)$. Let $\widetilde{\wp}_0(z) = (1+z)/(1-z)$ and $\widetilde{\wp}_\sigma(z) = (1 + (1-2\sigma)z)/(1-z)$ be the extremal functions, respectively in $\mathcal{P}$ and $\mathcal{P}(\sigma)$, then

1. $\mathcal{B}_{\tau,\mu}^{0,\beta}(0, 0, \widetilde{\wp}_0) = R$, the family of bounded turning functions presented in [1].
2. $\mathcal{B}_{\tau,\mu}^{0,\beta}(0, 0, \widetilde{\wp}_\sigma) = R(\sigma)$, the family of bounded functions of order σ presented in [33] and
3. $\mathcal{B}_{\tau,\mu}^{0,\beta}(0, 1, \widetilde{\wp}) = H$, the family of functions presented in [11].

In this investigation, a new subfamily of regular functions is defined and some estimates for early coefficients, Fekete-Szegö functional (for both real and complex parameters), and the second, and third Hankel determinants for the functions $f \in \mathcal{A}$ are established. We also established the upper estimate for the third Hankel determinant for the inverse function f^{-1} of $f \in \mathcal{A}$. We are inspired by the works in [18].

3. Lemmas

The lemmas that follow shall be needed.

Lemma 3.1 ([6]). *If $\wp(z) \in \mathcal{P}$ and $\alpha \in \mathbb{R}$, then*

$$\left| p_2 - \alpha \frac{p_1^2}{2} \right| \le \begin{cases} 2(1-\alpha) & when \quad \alpha \le 0, \\ 2 & when \ 0 \le \alpha \le 2, \\ 2(\alpha - 1) & when \quad \alpha \ge 2. \end{cases}$$

Lemma 3.2 ([6]). *If $\wp(z) \in \mathcal{P}$ and $\beta \in \mathbb{C}$, then*

$$\left| p_2 - \beta \frac{p_1^2}{2} \right| \le 2 \max\{1, |1 - \beta|\}.$$

Lemma 3.3 ([14]). *If $\wp(z) \in \mathcal{P}$, $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{N}$, then*

$$|p_{x+y} - \alpha p_x p_y| \le \begin{cases} 2 & when \quad 0 \le \alpha \le 1, \\ 2|2\alpha - 1| & elsewhere. \end{cases}$$

Lemma 3.4 ([10]). *If $\wp(z) \in \mathcal{P}$, then $|p_x| \le 2$ and $x \in \mathbb{N}$.*

4. Main Results

Henceforth, it is assumed that all parameters are as declared in (2.2) unless otherwise stated. Our results are therefore as follows.

4.1. Coefficient Estimates

Theorem 4.1. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then*

$$|a_2| \leq \frac{1}{2\phi_2} \tag{4.1}$$
$$|a_3| \leq \frac{13+8\gamma^2}{8\phi_3} \tag{4.2}$$
$$|a_4| \leq \frac{25+8\gamma^2}{16\phi_4} \tag{4.3}$$
$$|a_5| \leq \frac{1603+832\gamma^2+512\gamma^4}{512\phi_5} \tag{4.4}$$

where

$$\phi_x = (1+(x+\beta-\mu-1)\tau)^{n+1}. \tag{4.5}$$

Proof. Let $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then the definition of subordination permits us to represent (2.1) as

$$(1-e^{-2i\delta}\gamma^2 z^2)\frac{\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)}{z} = \ell b(s(z))$$

or

$$(1-e^{-2i\delta}\gamma^2 z^2)(\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)) = z[1+s(z)]^{1/2}. \tag{4.6}$$

For brevity, we use ϕ_x in (4.5) so that simple computation shows that (4.6) expands as

$$\begin{aligned} z + \phi_2 a_2 z^2 + (\phi_3 a_3 - e^{-2i\delta}\gamma^2)z^3 + (\phi_4 a_4 - e^{-2i\delta}\gamma^2\phi_2 a_2)z^4 + (\phi_5 a_5 - e^{-2i\delta}\gamma^2\phi_3 a_3)z^5 + \cdots \\ = z + \frac{1}{4}p_1 z^2 + \frac{1}{4}\left(p_2 - \frac{17}{8}p_1^2\right)z^3 + \frac{1}{4}\left(\frac{13}{32}p_1^3 - \frac{5}{4}p_1p_2 + p_3\right)z^4 \\ + \frac{1}{4}\left(\frac{419}{2048}p_1^4 + \frac{105}{96}p_1^2p_2 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + p_4\right)z^5 + \cdots \end{aligned} \tag{4.7}$$

so that the comparison of the coefficients yields

$$a_2 = \frac{p_1}{4\phi_2} \tag{4.8}$$
$$a_3 = \frac{(p_2 - \frac{17}{8}p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3} \tag{4.9}$$
$$a_4 = \frac{(p_3 - \frac{5}{4}p_1p_2 + \frac{13}{32}p_1^3) + e^{-2i\delta}\gamma^2 p_1}{4\phi_4} \tag{4.10}$$

and

$$a_5 = \frac{(p_4 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + \frac{35}{32}p_1^2p_2 + \frac{419}{2048}p_1^4) + (p_2 - \frac{17}{8}p_1^2)e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4}{4\phi_5}. \tag{4.11}$$

Application of triangle inequality and Lemma 3.4 in (4.8) yields our result in (4.1). Also, from (4.9),

$$|a_3| \leq \frac{|p_2 - \frac{17}{8}p_1^2| + 4|e^{-2i\delta}|\gamma^2}{4\phi_3}$$

and the application of Lemma 3.1 yields the result in (4.2). From (4.10), we have the presentation

$$|a_4| \leq \frac{|p_3 - \frac{5}{4}p_1p_2| + \frac{13}{32}|p_1^3| + |e^{-2i\delta}|\gamma^2|p_1|}{4\phi_4}$$

which by the application of Lemmas 3.1 and 3.4 yields our result in (4.3). To obtain estimate for a_5 we have from (4.11) that

$$a_5 = \frac{(p_4 - \frac{5}{4}p_1p_3) - \frac{5}{8}p_2(p_2 - \frac{7}{2}\frac{p_1^2}{2}) + \frac{419}{2048}p_1^4 + (p_2 - \frac{17}{4}\frac{p_1^2}{2})e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4}{4\phi_5}.$$

and

$$|a_5| \leq \frac{|p_4 - \frac{5}{4}p_1p_3| + \frac{5}{8}|p_2||p_2 - \frac{7}{2}\frac{p_1^2}{2}| + \frac{419}{2048}|p_1^4| + |p_2 - \frac{17}{4}\frac{p_1^2}{2}||e^{-2i\delta}|\gamma^2 + 4|e^{-4i\delta}|\gamma^4}{4\phi_5}$$

which by the application of Lemmas 3.1, 3.3 and 3.4 yields our result in (4.4). ■

4.2. Estimates for Fekete-Szegö Functional

Another commonly studied property of the coefficient problems of $f \in \mathcal{A}$ is the Fekete-Szegö functional

$$\mathcal{FS}(\varepsilon, f) = \left|a_3 - \varepsilon a_2^2\right|, \quad \varepsilon \in \mathbb{R} \tag{4.12}$$

announced in [8]. Interested reader may see [4, 5, 17–19, 24] and the citations therein for more properties, applications, and background details.

Theorem 4.2. *If $f \in \mathcal{B}_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b)$, then for real parameter ε,*

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{1-\alpha+2\gamma^2}{2\phi_3} & when \qquad \varepsilon \leq -\frac{17\phi_2^2}{2\phi_3} \\ \frac{1+2\gamma^2}{2\phi_3} & when \ -\frac{17\phi_2^2}{2\phi_3} \leq \varepsilon \leq -\frac{9\phi_2^2}{2\phi_3} \\ \frac{\alpha-1+2\gamma^2}{2\phi_3} & when \qquad \varepsilon \geq -\frac{9\phi_2^2}{2\phi_3} \end{cases} \tag{4.13}$$

where

$$\alpha = \frac{17\phi_2^2 + 2\varepsilon\phi_3}{4\phi_2^2}. \tag{4.14}$$

Proof. Let $\varepsilon \in \mathbb{R}$. If we substitute (4.8) and (4.9) into (4.12) we will arrive at

$$|a_3 - \varepsilon a_2^2| = \left|\frac{(p_2 - \frac{17}{8}p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3} - \frac{\varepsilon p_1^2}{16\phi_2^2}\right|$$

so that

$$|a_3 - \varepsilon a_2^2| \leq \frac{1}{4\phi_3}\left|p_2 - \left(\frac{17\phi_2^2 + 2\varepsilon\phi_3}{4\phi_2^2}\right)\frac{p_1^2}{2}\right| + \left|\frac{e^{-2i\delta}\gamma^2}{\phi_3}\right|$$

or

$$|a_3 - \varepsilon a_2^2| \leq \frac{1}{4\phi_3}\left|p_2 - \alpha\frac{p_1^2}{2}\right| + \frac{\gamma^2}{\phi_3}$$

where α is defined in (4.14). The application of Lemma 3.1 means that for α that satisfies conditions $\alpha \leq 0$, $0 \leq \alpha \leq 2$ and $\alpha \geq 2$, we have the results in (4.13). ■

Theorem 4.3. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then for complex parameter ξ,*

$$\left|a_3 - \xi a_2^2\right| \le \frac{1}{2\phi_3}\max\{1, |1-\beta|\} + \frac{\gamma^2}{\phi_3} \tag{4.15}$$

where

$$\beta = \frac{17\phi_2^2 + 2\xi\phi_3}{4\phi_2^2} \tag{4.16}$$

Proof. Let $\xi \in \mathbb{C}$. If we substitute (4.8) and (4.9) into (4.12) we will arrive at the inequality

$$|a_3 - \xi a_2^2| \le \frac{1}{4\phi_3}\left|p_2 - \left(\frac{17\phi_2^2 + 2\xi\phi_3}{4\phi_2^2}\right)\frac{p_1^2}{2}\right| + \left|\frac{e^{-2i\delta}\gamma^2}{\phi_3}\right|$$

or

$$|a_3 - \xi a_2^2| \le \frac{1}{4\phi_3}\left|p_2 - \beta\frac{p_1^2}{2}\right| + \frac{\gamma^2}{\phi_3}$$

where β is defined in (4.16). The application of Lemma 3.2 produce the result in (4.15). ■

4.3. Estimates for some Hankel Determinants

The yth-Hankel determinant

$$\mathcal{HD}_{y,x}(f) = \begin{vmatrix} 1 & a_{x+1} & a_{x+2} & \cdots & a_{x+y-1} \\ a_{x+1} & a_{x+2} & \cdots & \cdots & a_{x+y} \\ a_{x+2} & a_{x+3} & \cdots & \cdots & a_{x+y+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{x+y-1} & a_{x+y} & \cdots & \cdots & a_{x+2(y-1)} \end{vmatrix} \tag{4.17}$$

$(x, y \in \mathbb{N})$ was introduced by Pommerenke [28]. (4.17) has its elements from the coefficients of f in (1.1). Observe that from (4.17), we can establish that

$$|\mathcal{HD}_{2,1}(f)| = |a_3 - a_2^2|, \tag{4.18}$$

$$|\mathcal{HD}_{2,2}(f)| = |a_2a_4 - a_3^2|, \tag{4.19}$$

$$\mathcal{HD}_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{4.20}$$

hence,

$$|\mathcal{HD}_{3,1}(f)| \le |a_3||\mathcal{HD}_{2,2}(f)| + |a_4||\mathcal{G}_2(f)| + |a_5||\mathcal{HD}_{2,1}(f)|. \tag{4.21}$$

where

$$|\mathcal{G}_x(f)| = |a_xa_{x+1} - a_{x+2}|, \quad x = \{2,3,4,\ldots\}. \tag{4.22}$$

Even though the functionals in (4.12) and (4.18) have different historical background, yet it can be observed that the functionals are related since $|\mathcal{HD}_{2,1}(f)| = \mathcal{FS}(1,f)$.

For the inverse functions f^{-1} in (1.2), Obradovic and Tuneski [22] established that

$$|\mathcal{HD}_{3,1}(f^{-1})| = |\mathcal{HD}_{3,1}(f) - (a_3 - a_2^2)^3| \tag{4.23}$$

and obtained some estimates for some subfamilies of $\mathcal{S}$. Interested reader may see [4, 5, 17–19] and the citations therein for some properties and applications; and more background details on Hankel determinants.

Theorem 4.4. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{2,1}(f)| \le \frac{1+2\gamma^2}{2\phi_3} \tag{4.24}$$

Proof. Substituting $\xi = 1$ in (4.15) yields (4.24). ■

Theorem 4.5. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then*

$$|\mathcal{HD}_{2,2}(f)| \le -4A + 8B - 2C + 8D - E + 4F + \frac{\gamma^4}{\phi_2^2} \tag{4.25}$$

where

$$\left.\begin{aligned} A = \tfrac{1}{16\phi_2\phi_4}, \quad B = \tfrac{289\phi_2\phi_4 - 26\phi_3^2}{1024\phi_2\phi_3^2\phi_4}, \quad C = \tfrac{1}{16\phi_3^2}, \\ D = \tfrac{17\phi_4 - 5\phi_3^2}{64\phi_2\phi_3^2\phi_4}, \quad E = \tfrac{1}{2\phi_3^2}\gamma^2 \quad \textit{and} \quad F = \tfrac{17\phi_2\phi_4 + \phi_3^2}{16\phi_2\phi_3^2\phi_4}\gamma^2 \end{aligned}\right\} \tag{4.26}$$

Proof. Substituting (4.8), (4.9) and (4.10) into (4.19) simplifies to

$$\begin{aligned} \mathcal{HD}_{2,2}(f) = & \frac{1}{16\phi_2\phi_4}p_1p_3 - \frac{289\phi_2\phi_4 - 26\phi_3^2}{1024\phi_2\phi_3^2\phi_4}p_1^4 - \frac{1}{16\phi_3^2}p_2^2 + \frac{17\phi_4 - 5\phi_3^2}{64\phi_2\phi_3^2\phi_4}p_1^2p_2 \\ & - \frac{1}{2\phi_3^2}e^{-2i\delta}\gamma^2 p_2 + \frac{17\phi_2\phi_4 + \phi_3^2}{16\phi_2\phi_3^2\phi_4}e^{-2i\delta}\gamma^2 p_1^2 - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2} \end{aligned}$$

and for brevity we get

$$\mathcal{HD}_{2,2}(f) = Ap_1p_3 - Bp_1^4 - Cp_2^2 + Dp_1^2p_2 - Ee^{-2i\delta}p_2 + Fe^{-2i\delta}p_1^2 - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}$$

for A, B, C, D, E and F in (4.26). Now some rearrangement and simplifications yield

$$|\mathcal{HD}_{2,2}(f)| = \left|Ap_1\left(p_3 - \frac{B}{A}p_1^3\right) - Cp_2\left(p_2 - \frac{2D}{C}\frac{p_1^2}{2}\right) - Ee^{-2i\delta}\left(p_2 - \frac{2F}{E}\frac{p_1^2}{2}\right) - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}\right|$$

so that

$$|\mathcal{HD}_{2,2}(f)| \le |Ap_1|\left|p_3 - \frac{B}{A}p_1^3\right| + |Cp_2|\left|p_2 - \frac{2D}{C}\frac{p_1^2}{2}\right| + |Ee^{-2i\delta}|\left|p_2 - \frac{2F}{E}\frac{p_1^2}{2}\right| + \left|\frac{e^{-4i\delta}\gamma^4}{\phi_2^2}\right|$$

and the appropriate application of Lemmas 3.1, 3.3 and 3.4 yields (4.25). ■

Theorem 4.6. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then*

$$|\mathcal{G}_2(f)| \le -2G + 4H + 8I + 2J \tag{4.27}$$

where

$$G = \frac{1}{4\phi_4}, \quad H = \frac{\phi_4 + 5\phi_2\phi_3}{16\phi_2\phi_3\phi_4}, \quad I = \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}, \quad \textit{and} \quad J = \frac{\phi_2\phi_3 - \phi_4}{4\phi_2\phi_3\phi_4}\gamma^2. \tag{4.28}$$

Proof. Substituting (4.8), (4.9) and (4.10) into (4.22) simplifies to

$$\mathcal{G}_2(f) = a_2a_3 - a_4 = -\frac{1}{4\phi_4}p_3 + \frac{\phi_4 + 5\phi_2\phi_3}{16\phi_2\phi_3\phi_4}p_1p_2 - \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}p_1^3 - \frac{\phi_2\phi_3 - \phi_4}{4\phi_2\phi_3\phi_4}e^{-2i\delta}\gamma^2 p_1$$

and for brevity we get

$$\mathcal{G}_2(f) = -Gp_3 + Hp_1p_2 - Ip_1^3 - Je^{-2i\delta}p_1$$

for G, H, I and J in (4.28). Now some rearrangement and simplifications yield

$$|\mathcal{G}_2(f)| = \left|-G\left(p_3 - \frac{H}{G}p_1p_2\right) - Ip_1^3 - Je^{-2i\delta}p_1\right|$$

so that

$$|\mathcal{G}_2(f)| \le |-G|\left|p_3 - \frac{H}{G}p_1p_2\right| + |Ip_1^3| + |Je^{-2i\delta}p_1|$$

and the appropriate application of Lemmas 3.3 and 3.4 yields (4.27). ■

Theorem 4.7. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{3,1}(f)| \leq \left(\frac{13+8\gamma^2}{8\phi_3}\right)\left[-4A+8B-2C+8D-E+4F+\frac{\gamma^4}{\phi_2^2}\right] + \left(\frac{25+8\gamma^2}{16\phi_4}\right)\left[-2G+4H+8I+2J\right] + \left(\frac{1603+832\gamma^2+512\gamma^4}{512\phi_5}\right)\left[\frac{1+2\gamma^2}{2\phi_3}\right] \quad (4.29)$$

where $A, B, C, \ldots, J$ are defined in (4.26) *and* (4.28).

Proof. Substitute (4.2), (4.3), (4.4), (4.24), (4.25) and (4.27) into (4.21) yields (4.29). ■

Theorem 4.8. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{3,1}(f^{-1})| \leq \frac{13+8\gamma^2}{4\phi_3}\left[-4A+8B-2C+8D-E+4F+\frac{\gamma^2}{\phi_2^2}\right] + \frac{1+2\gamma^2}{2\phi_3}\left[4L-2K+4N-2M+8P-4R+16Q+\frac{\gamma^4}{\phi_5}\right] + \left[\frac{25+8\gamma^2}{16\phi_4}\right]^2 + \left[\frac{1}{2\phi_2}\right]^6 \quad (4.30)$$

where $A, B, C, \ldots, J$ are defined in (4.26) *and* (4.28)*, and*

$$\left.\begin{aligned} K=\tfrac{1}{4\phi_5},\ L=\tfrac{5}{16\phi_5},\ M=\tfrac{1}{4\phi_5}\gamma^2,\ N=\tfrac{17\phi_2^2\phi_3-6\phi_5}{32\phi_2^2\phi_3\phi_5}\gamma^2,\\ R=\tfrac{5}{32\phi_5},\ P=\tfrac{6\phi_5+35\phi_2^2\phi_3}{128\phi_2^2\phi_3\phi_5},\ Q=\tfrac{816\phi_5-419\phi_2^2\phi_3}{8192\phi_2^2\phi_3\phi_5}.\end{aligned}\right\} \quad (4.31)$$

Proof. Substituting (4.20) into (4.23) yields

$$\begin{aligned}\mathcal{HD}_{3,1}(f^{-1}) &= \Big(a_3(a_2a_4-a_3^2)+a_4(a_2a_3-a_4)+a_5(a_3-a_2^2)\Big)-\Big(a_3-a_2^2\Big)^3\\ &= 2a_2a_3a_4-2a_3^3-a_4^2+a_3a_5-a_2^2a_5+3a_2^2a_3^2-3a_2^4a_3+a_2^6\\ &= 2a_3(a_2a_4-a_3^2)+a_5(a_3-a_2^2)+3a_2^2a_3(a_3-a_2^2)-a_4^2+a_2^6\\ &= 2a_3(a_2a_4-a_3^2)+(a_3-a_2^2)(3a_2^2a_3+a_5)-a_4^2+a_2^6\end{aligned}$$

so that

$$|\mathcal{HD}_{3,1}(f^{-1})| \leq 2|a_3||a_2a_4-a_3^2|+|a_3-a_2^2||3a_2^2a_3+a_5|+|a_4|^2+|a_2|^6 \quad (4.32)$$

or

$$|\mathcal{HD}_{3,1}(f^{-1})| \leq 2|a_3||\mathcal{HD}_{2,2}(f)|+|\mathcal{HD}_{2,1}(f)||3a_2^2a_3+a_5|+|a_4|^2+|a_2|^6. \quad (4.33)$$

Observe that by using (4.8), (4.9) and (4.11),

$$3a_2^2a_3+a_5 = \frac{1}{4\phi_5}p_4-\frac{5}{16\phi_5}p_1p_3+\frac{1}{4\phi_5}e^{-2i\delta}\gamma^2p_2-\frac{17\phi_2^2\phi_3-6\phi_5}{32\phi_2^2\phi_3\phi_5}e^{-2i\delta}\gamma^2p_1^2 - \frac{5}{32\phi_5}p_2^2+\frac{6\phi_5+35\phi_2^2\phi_3}{128\phi_2^2\phi_3\phi_5}p_1^2p_2+\frac{816\phi_5-419\phi_2^2\phi_3}{8192\phi_2^2\phi_3\phi_5}p_1^4+\frac{e^{-4i\delta}\gamma^4}{\phi_5}$$

so that for brevity,

$$3a_2^2a_3+a_5 = Kp_4-Lp_1p_3+Me^{-2i\delta}p_2-Ne^{-2i\delta}p_1^2-Rp_2^2+Pp_1^2p_2+Qp_1^4+\frac{e^{-4i\delta}\gamma^4}{\phi_5}$$

for K, L, M, N, R, P and Q in (4.31). Now some rearrangement and simplifications yield

$$|3a_2^2a_3+a_5| = \left|K\left(p_4-\frac{L}{K}p_1p_3\right)+Me^{-2i\delta}\left(p_2-\frac{2N}{M}\frac{p_1^2}{2}\right)-Rp_2\left(p_2-\frac{2P}{R}\frac{p_1^2}{2}\right)+Qp_1^4+\frac{e^{-4i\delta}\gamma^4}{\phi_5}\right|$$

so that

$$|3a_2^2a_3+a_5| = |K|\left|p_4-\frac{L}{K}p_1p_3\right|+|Me^{-2i\delta}|\left|p_2-\frac{2N}{M}\frac{p_1^2}{2}\right|+|Rp_2|\left|p_2-\frac{2P}{R}\frac{p_1^2}{2}\right|+|Qp_1^4|+\left|\frac{e^{-4i\delta}\gamma^4}{\phi_5}\right|$$

and the appropriate application of Lemmas 3.4, 3.1 and 3.3 yields

$$|3a_2^2a_3+a_5| \le 4L-2K+4N-2M+8P-4R+16Q+\frac{\gamma^4}{\phi_5}. \tag{4.34}$$

Now substituting (4.1), (4.2), (4.3), (4.24), (4.25) and (4.34) into (4.33) yields (4.30). ■

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