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Exponential stability of a porous thermoelastic system with Gurtin Pipkin thermal law and distributed delay time

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Abstract. In this paper, we consider a one-dimensional porous thermoelastic system with herditary heat conduction and a distributed delay time acting only on the porous equation, where the heat conduction is given by Gurtin Pipkin law. Existence and uniqueness of solution are obtained by the use of Hille-Yosida theorem. Then, based on the energy method as well as by constructing a suitable Lyapunov functional, we prove under some assumptions on the derivative of the heat-flux kernel, that the solution of the system decays exponentially without any assumptions on the wave speeds.

AMS Subject Classifications: 35B40, 47D03, 74D05, 74F05.

Keywords: Porous thermo-elastic system, semigroup theory, exponential stability, Gurtin Pipkin law, energy method, distributed delay time.

Contents

1. Introduction

In this paper we are concerned by the following porous thermoelastic system with distributed delay time

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma_1 u_t - \int\limits_{\tau_1}^{\tau_2} \gamma_2(\sigma) u_t(x, t-\sigma)\, d\sigma & in\ (0,\pi)\times(0,\infty) \\ J\varphi_{tt} = \alpha\varphi_{xx} - b u_x - \xi\varphi + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times(0,\infty) \\ c\theta_t = -q_x - \beta u_{xt} - \delta\varphi_t & in\ (0,\pi)\times(0,\infty) \end{cases} \tag{1.1}$$

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with the boundary conditions and the initial data

$$\begin{cases} u\,(0,t) = u\,(\pi,t) = \varphi_x\,(0,t) = \varphi_x\,(\pi,t) = \theta_x\,(0,t) = \theta_x\,(\pi,t) = 0 \ \ , \ \ t > 0 \\ u\,(x,0) = u_0\,(x) \quad , \quad \varphi\,(x,0) = \varphi_0\,(x) \quad , \quad \theta\,(x,0) = \theta_0\,(x) \quad , \quad x \in (0,\pi) \\ u_t\,(x,0) = u_1\,(x) \quad , \quad \varphi_t\,(x,0) = \varphi_1\,(x) \quad , \quad x \in (0,\pi) \\ u_t\,(x,-t) = f_0\,(x,t) \quad , \quad x \in (0,\pi) \quad , \quad t \in (0,\tau_2) \end{cases} \tag{1.2}$$

where $u = u\,(x,t)$ is the transversal displacement, $\varphi = \varphi\,(x,t)$ is the volume fraction, $\theta = \theta\,(x,t)$ is the temperature variation from an equilibrium reference value and $q = q\,(x,t)$ is the heat flux. The coefficients $\rho_1,\ J,\ c,\ \mu,\ \alpha,\ b,\ \xi,\ \tau,\ \gamma_1$ are positive constitutive constants such that

$$\mu\,\xi > b^2 \tag{1.3}$$

The coefficient β and δ are the coupling constants that are different from zero but their signs does not matter in the analysis. The term $\int\limits_{\tau_1}^{\tau_2} \gamma_2\,(\sigma)\,u_t\,(x,t-\sigma)\,d\sigma$ is a distributed delay that acting only on the porous equation and $\gamma_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. The initial data $u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0$ belongs to the suitable functional space.
In order to determine system (1.1)-(1.2), an additional equation relating q and θ must be used.

Over the years, many scientists and researchers have come up with theories about thermoelasticity. In the classical model of heat diffusion or what is known as the classical theory of thermoelasticity, heat flow obeys Fourier's law of thermal conductivity, which states that heat flow is proportional to a temperature gradient. The thermal conductivity equation is given by Fourier's law as

$$q = -\kappa\theta_x \tag{1.4}$$

where $\kappa > 0$ represents the coefficient of thermal conductivity of the material.

In the last three decades much has been written on the analysis of the longtime behavior of porous thermoelastic systems. Casas and Quintanilla [3] proved the exponential decay of the solution of the following system

$$\begin{cases} \rho\,u_{tt} = \mu\,u_{xx} + b\,\varphi_x - \beta\,\theta_x & in\ (0,\pi)\times(0,\infty) \\ J\,\varphi_{tt} = \alpha\,\varphi_{xx} - b\,u_x - \xi\,\varphi + \delta\,\theta - \tau\,\varphi_t & in\ (0,\pi)\times(0,\infty) \\ c\,\theta_t = \kappa\,\theta_{xx} - \beta\,u_{xt} - \delta\,\varphi_t & in\ (0,\pi)\times(0,\infty) \end{cases}$$

In [24, 30] Quintanilla and co-authors showed the slow decay for the solution of the above system when the frictional damping is removed ($\tau = 0$) or replaced by a viscoelastic damping. Moreover, in [30] they established a polynomial rate of decay provided that $\delta\,(\beta\,b - \delta\,\mu) > 0$.

Closely to the porous thermoelastic systems, Muñoz Rivera and Racke [31] studied the Timoshenko type system

$$\begin{cases} \rho_1\,\varphi_{tt} = k(\varphi_x + \psi)_x \\ \rho_1\,\psi_{tt} = b\,\psi_{xx} - k\,(\varphi_x + \psi) + \gamma\,\theta_x \\ c\,\theta_t = \kappa\,\theta_{xx} - \gamma\,\psi_{tx} \end{cases}$$

with different boundary conditions, where ψ represents the rotation angle of the filament, they proved that the solution of the system is exponentially stable in the case of the wave speeds are equal.

It should be noted that Fourier's thermal conductivity equation is an equation of parabolic type, which leads to the physical contradiction of the infinite speed of heat diffusion, in other words any thermal disturbance at a point will instantly transfer to other parts of the body. To overcome this paradox, other theories of thermoelasticity have emerged.

Green and Naghdi [14, 15] proposed a way to eliminate the paradox of infinite velocities, they used an analogy between the concepts and equations of purely thermal theories and purely mechanical theories and came up with three types of constitutive equations for heat flow in a fixed solid cohesive material classified as type I , type II and type III , where Type I leads to the usual thermal conductivity according to Fourier's law. In type II and type III theories, the constitutive equations for the heat flux are given by

$$q = -f_1\,\psi_x \quad , \quad q = -f_1\,\psi_x - f_2\,\theta_x \tag{1.5}$$

respectively, where

$$\psi = \theta_0\,(x) + \int_0^t \theta\,(x,\tau)\,d\tau$$

is the thermal displacement and f_1 , f_2 are positive constants.

In the framework of Green and Naghdi theory, Quintanilla and co-workers [21, 29] considered the following porous thermoelastic system

$$\begin{cases} \rho_1\,u_{tt} = \mu\,u_{xx} + \gamma\,\phi_x - \beta\,\psi_{tx} \\ J\,\phi_{tt} = b\,\phi_{xx} + m\;\psi_{xx} - \xi\phi + d\,\psi_t - \gamma\,u_x - \tau\,\phi_t \\ a\,\psi_{tt} = k\,\psi_{xx} + m\;\phi_{xx} - d\,\psi_t - \beta\,u_{tx} + k^*\,\theta_{xx} \end{cases}$$

where $(x,t) \in (0,\pi) \times (0,\infty)$ with coefficient satisfy $\mu\,\xi > \gamma^2$ and $b\,k > m^2$.Precisely. Leseduarte et al [21] examined the type II case $(k^* = 0)$ with $(\tau \neq 0)$ and Miranville and Quintanilla [29] considered the type III case $(k^* \neq 0)$ with $(\tau = 0)$. Both have proven that the solution is exponentially stable.

In [11, 19, 25, 27, 28] the authors were considered Timoshenko systems with thermoelastic dissipation of type III, the exponential stability was obtained provided that the wave speeds associated to the hyperbolic part of the system are equal. Otherwise, the solution decays polynomially.

In [22] Lord and Shulman propose a second theory to overcome the paradox of infinite velocity due to Fourier's law, They suggest to replace Fourier's law with the following Cattaneo's law of heat conduction

$$\tau_0\,q_t + q + \kappa\,\theta_x = 0 \tag{1.6}$$

where τ_0 is a positive constant represents the time lag in the response of the heat flux to the temperature gradient and is referred to as the thermal relaxation time.

In accordance with this theory, a hyperbolic system was obtained, and as a result, the heat spreads with a finite speed and a new component of the wave speed appears. The heat is transferred by the process of wave propagation rather than the usual diffusion, and this process is known as the second sound, making the first sound the usual sound.

Fernandez Sare and Racke [12] considered a Timoshenko system coupled with the heat equation modeled by Cattaneo's law, they prove that the solution of the system losses the exponential stability in the case of equal wave speeds.

By introducing a new stability number χ_0 that links all the wave speeds (three), Santos et al [35] refined the results found in [12] and demonstrated the exponential stability of the solution in the case of $\chi_0 = 0$ where

$$\chi_0 = \left(\tau - \frac{\kappa\,\rho_1}{\rho_3}\right)\left(\rho_2 - \frac{b\,\rho_1}{\kappa}\right) - \frac{\tau\,\delta^2\,\rho_1}{\kappa\,\rho_3}$$

In the setting of hyperbolic type porous thermoelastic systems, Han and Xu [17] considered the non uniform porous system with second sound thermoelasticity

$$\begin{cases} \rho\,(x)\,u_{tt} = [\mu\,(x)\,u_x\,(x)]_x + [b\,(x)\,\phi\,(x)]_x - [\beta\,(x)\,\theta\,(x)]_x \\ J\,(x)\,\phi_{tt} = [\alpha\,(x)\,\phi_x\,(x)]_x - b\,(x)\,u_x\,(x) - \xi\,(x)\,\phi\,(x) + m\,(x)\,\theta\,(x) - \tau\,(x)\,\phi_t\,(x) \\ c\,(x)\,\theta_t\,(x) = -q_x\,(x) - \beta\,(x)\,u_{tx}\,(x) - m\,(x)\,\phi_t\,(x) \\ q_t\,(x) + \delta q\,(x) + \eta\theta_x\,(x) = 0 \end{cases} \tag{1.7}$$

where $\rho,\ \mu,\ J,\ \alpha,\ b,\ \xi$ and τ are positive function in $[0,1]$ and $\mu\,(x)\;\xi\,(x)\;>\;(b\,(x))^2$ for any $x\;\in\;[0,1]$, they have used the spectral method and proved that the solution decays exponentialy. Messaoudi and Fareh [26] studied the uniform case of (1.7). they used the multiplier method and established an exponential stability result. Fareh and Messaoudi [9] examined the solution of the following system

$$\begin{cases} \rho\, u_{tt} - \mu\, u_{xx} + b\,\phi_x = 0 \\ J\,\phi_{tt} - \alpha\,\phi_{xx} + b\,u_x + \xi\,\phi + \beta\,\theta = 0 \\ c\,\theta_t + q_x + \beta\,\phi_{tx} + \delta\,\theta = 0 \\ \tau_0\, q_t + q + \kappa\,\theta_x = 0 \end{cases}$$

in the case when $\mu\,\xi = b^2$. They introduce the stability number

$$\chi = \beta^2 - \left(\frac{c\,\alpha\,\mu}{\rho} - \frac{\kappa\,\alpha}{\tau_0}\right)\left(\frac{J}{\alpha} - \frac{\rho}{\mu}\right)$$

and showed that the solution is exponentially stable if and only if $\chi = 0$.

It is important to note that the second sound and type III theories cannot adequately explain the memory effect that predominates in specific materials, especially at low temperatures. As a result, a more general fundamental assumption about heat flow to thermal memory is required. In [16] Gurtin and Pipkin prosed that heat flux depends on the integrated history of the weighted temperature gradient against a relaxation function called the heat flux kernel. They developed a general nonlinear theory in which thermal disturbances propagate with a finite speed. According to this theory, the linear constitutive equation for q is given as follows

$$q = -\int_{-\infty}^{t} k\,(t-s)\,\theta_x\,(x,s)\,ds \tag{1.8}$$

where $k\,(s)$ is the heat conductivity relaxation kernal. The presence of the convolution term (1.8) renders the porous system coupled with the heat equation into a fully hyperbolic system, this allows the heat to propagate with finite speed and admits to describe the memory effect of heat conduction. We note that many different constitutive models arise from different choices for $k\,(s)$, in particular, if we take $k\,(s) = \kappa\,\delta\,(s)$, where δ is the Dirac mass weighted at 0, then (1.8) reduced to the Fourier's law (1.4), and if we choose

$$k\,(s) = \frac{\kappa}{\tau_0}\,e^{-\frac{s}{\tau_0}}\quad,\quad \tau_0 > 0$$

we obtained Cattaneo's law (1.6). So (1.8) is a generalized from Fourier's and Cattaneo's law.

In [6] Dell'Oro an Pata extended the result of [35] to the following system

$$\begin{cases} \rho_1\,\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \\ \rho_2\,\psi_{tt} - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) + \delta\,\theta_x = 0 \\ \rho_3\,\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{\infty} k\,(s)\,\theta_{xx}\,(t-s)\,ds + \delta\,\psi_{tx} = 0 \end{cases}$$

they introduce hte stability number

$$\chi_k = \left(\frac{\rho_1}{\rho_3\,\kappa} - \frac{\beta}{k\,(0)}\right)\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right) - \frac{\beta}{k\,(0)}\frac{\rho_1\,\delta^2}{\rho_3\,\kappa\,b}$$

and proved in the case of $\chi_k = 0$ that the solution of the system is exponentially stable. For other models with Gurtin-Pipkin composition, we refer the readers to [2, 4, 5, 8, 10, 18, 33].

In the present paper we consider the porous thermoelastic system (1.1)-(1.2) coupled with the heat equation via the constitutive equation (1.8) and establish an exponential stability result without any restriction on the coefficients. We note that our work is an extension of the results obtained in [7].

The rest of this article is organized as follows: In section 2, we introduce some transformations and state the assumptions needed in our work. In section 3, we use the semigroupe method to prove the well-posedness of problem. Finally, in section 4, we state and prove our stability results. We use c_0 throughout this paper to denote a geniric positive constant.

2. Preliminaries

We note that the presence of the convolution term in the constitutive equation for q renders the family operators mapping the initial value $(u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0)$ into the solution (u, φ, θ) not match the semigroup properties. This is due to the fact that the solution value of θ at time t depends on the whole function up to time t. In order to overcome this difficulty we introduce the new variables

$$\theta^t(x,s) = \theta(x,t-s) \quad , \quad s \geq 0 \tag{2.1}$$

and

$$\eta(x,s) = \eta^t(x,s) = \int_0^s \theta^t(x,\lambda)\, d\lambda \quad , \quad s \geq 0 \tag{2.2}$$

which denote the past history and the summed past history of θ up to t, respectively.

Clearly, $\eta^t(x,s)$ satisfies the following conditions

$$\eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0$$

$$\eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0$$

$$\eta(x,0) = \lim_{s\to 0} \eta^t(x,s) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0$$

and it's easy to prove that

$$\eta_t(x,s) = \theta - \eta_s(x,s) \qquad in\ \ (0,\pi)\times(0,\infty)\times(0,\infty) \tag{2.3}$$

Moreover, we assume that $\lim_{s\to\infty} k(s) = 0$ then a simple computations give us

$$q = -\int_{-\infty}^{t} k(t-s)\,\theta_x(x,s)\, ds = \int_0^\infty k'(s)\,\eta_x^t(x,s)\, ds$$

setting $\kappa(s) = -k'(s)$, system (1.1)-(1.2) and equation (2.2) become

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma_1 u_t - \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, u_t(x,t-\sigma)\, d\sigma & in\ (0,\pi)\times(0,\infty) \\ J\varphi_{tt} = \alpha\varphi_{xx} - b u_x - \xi\varphi + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times(0,\infty) \\ c\theta_t = \displaystyle\int_0^\infty \kappa(s)\,\eta_{xx}^t(x,s)\, ds - \beta u_{xt} - \delta\varphi_t & in\ (0,\pi)\times(0,\infty) \\ \eta_t(x,s) = \theta - \eta_s(x,s) & in\ \ (0,\pi)\times(0,\infty)\times(0,\infty) \end{cases} \tag{2.4}$$

with the boundary conditions and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \;,\; t > 0 \\ \eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \;,\; t > 0 \\ u(x,0) = u_0(x) \quad , \quad \varphi(x,0) = \varphi_0(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x) \quad , \quad \varphi_t(x,0) = \varphi_1(x) \quad , \quad x \in (0,\pi) \\ \eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0 \\ \eta(x,0) = 0 \quad , \quad x \in (0,\pi) \;,\; t > 0 \\ u_t(x,-t) = f_0(x,t) \quad , \quad x \in (0,\pi) \quad , \quad t \in (0,\tau_2) \end{cases} \tag{2.5}$$

Conserning the memory kernel κ, we assume the following set of hypotheses:

$(H1) : \kappa \in C(IR^+) \cap L^1(IR^+)$

$(H2) : \kappa(s) \geq 0 \;,\; \kappa'(s) \leq 0 \;,\; \forall s \geq 0$

$(H3) : \kappa(0) > 0$

$(H4) : \int\limits_0^\infty \kappa(s)\,ds = \kappa_0 = k(0)$

$(H5) : \int\limits_0^\infty s\,\kappa(s)\,ds = 1$

$(H6) : \exists r > 0 \;;\; \kappa'(s) \leq -r\,\kappa(s) \;,\; \forall s \geq 0$

$(H7) : \lim\limits_{s\to\infty} \kappa(s) = 0$

concerning the weight of the delay, we only assume that

$$\int\limits_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\,d\sigma < \gamma_1 \tag{2.6}$$

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (2.4)-(2.5) we can always take solutions where φ and θ are constant, for this reason, we impose that

$$\int\limits_0^\pi \varphi_0(x)\,dx = \int\limits_0^\pi \varphi_1(x)\,dx = \int\limits_0^\pi \theta_0(x)\,dx = 0 \tag{2.7}$$

It is worth noting that condition (2.7) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior we need to impose condition (2.7). In addition as in [1], to be able to use Poincaré's inequality for φ and θ we perform the following transformation

From $(2.4)_2$ and $(2.4)_3$ respectively we have

$$\begin{cases} J\int\limits_0^\pi \varphi_{tt}\,dx + \tau\int\limits_0^\pi \varphi_t\,dx + \xi\int\limits_0^\pi \varphi\;dx - \delta\int\limits_0^\pi \theta\,dx = 0 \\ c\int\limits_0^\pi \theta_t\,dx + \delta\int\limits_0^\pi \varphi_t\,dx = 0 \end{cases} \tag{2.8}$$

If we take $\psi(t) = \int\limits_0^\pi \varphi\;dx$ and $\vartheta(t) = \int\limits_0^\pi \theta\,dx$, we observe that $\psi(0) = \int\limits_0^\pi \varphi_0\;dx$, $\psi'(0) = \int\limits_0^\pi \varphi_1\;dx$ and

$\vartheta(0) = \int\limits_0^\pi \theta_0 \, dx$. Moreover, (ψ, ϑ) is a solution of the following initial value system

$$\begin{cases} J\,\psi'' + \tau\,\psi' + \xi\,\psi - \delta\,\vartheta = 0 \quad , \quad \forall t \geq 0 \\ c\,\vartheta' + \delta\,\psi' = 0 \qquad , \qquad \forall t \geq 0 \\ \psi(0) = \int\limits_0^\pi \varphi_0 \; dx = 0 \\ \psi'(0) = \int\limits_0^\pi \varphi_1 \; dx = 0 \\ \vartheta(0) = \int\limits_0^\pi \theta_0 \, dx = 0 \end{cases}$$

The solution of system is $\psi(t) = \vartheta(t) = 0 \quad , \quad \forall t \geq 0$

Consequently

$$\int\limits_0^\pi \varphi(x,t)\, dx = \int\limits_0^\pi \theta(x,t) \; dx = 0 \quad , \quad \forall t \geq 0$$

Further more, from (2.2) we get

$$\int\limits_0^\pi \eta(x,s) \; dx = 0 \quad , \quad \forall t \geq 0 \quad , \quad \forall s \geq 0$$

3. Well-posedness

In this section, we give the existence and uniqueness of solutions for the system (2.4)-(2.5) using semigroup theory.

First, we introduce as in [32], new dependent variable

$$z(x,\rho,\sigma,t) = u_t(x, t - \rho\,\sigma) \qquad in \; (0,\pi) \times (0,1) \times (\tau_1, \tau_2) \times (0,\infty) \tag{3.1}$$

A simple differentiation shows that z satisfies

$$\sigma\, z_t(x,\rho,\sigma,t) + z_\rho(x,\rho,\sigma,t) = 0 \qquad in \; (0,\pi) \times (0,1) \times (\tau_1, \tau_2) \times (0,\infty) \tag{3.2}$$

Hence problem (2.4) takes the form:

$$\begin{cases} \rho_1\, u_{tt} = \mu\, u_{xx} + b\,\varphi_x - \beta\,\theta_x - \gamma_1\, u_t - \int\limits_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(x,1,\sigma,t)\, d\sigma \quad in \; (0,\pi) \times (0,\infty) \\ J\,\varphi_{tt} = \alpha\,\varphi_{xx} - b\,u_x - \xi\,\varphi + \delta\,\theta - \tau\,\varphi_t \qquad in \; (0,\pi) \times (0,\infty) \\ c\,\theta_t = \int\limits_0^\infty \kappa(s)\, \eta^t_{xx}(x,s)\, ds - \beta\, u_{xt} - \delta\,\varphi_t \qquad in \; (0,\pi) \times (0,\infty) \\ \sigma\, z_t = -z_\rho \qquad in \; (0,\pi) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \\ \eta_t(x,s) = \theta - \eta_s(x,s) \qquad in \; (0,\pi) \times (0,\infty) \times (0,\infty) \end{cases} \tag{3.3}$$

with the boundary and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0 \\ \eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0 \\ u(x,0) = u_0(x) \quad , \quad \varphi(x,0) = \varphi_0(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x) \quad , \quad \varphi_t(x,0) = \varphi_1(x) \quad , \quad x \in (0,\pi) \\ \eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0 \\ \eta(x,0) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0 \\ z(x,\rho,\sigma,0) = f_0(x,\rho\,\sigma) \qquad in \ \ (0,\pi) \times (0,1) \times (0,\tau_2) \end{cases} \tag{3.4}$$

Second, we introduce the vector function $U = (u, v, \varphi, \phi, \theta, z, \eta)^T$, with $v = u_t$, and $\phi = \varphi_t$.

We consider the following Hilbert spaces:

$$\begin{aligned} L^2_*(0,\pi) &= \left\{ w \in L^2(0,\pi) \ \ , \ \ \int_0^\pi w(x)\,dx = 0 \right\}, \\ H^1_*(0,\pi) &= H^1(0,\pi) \cap L^2_*(0,\pi), \\ H^2_*(0,\pi) &= \left\{ w \in H^2(0,\pi) \ ; \ w_x(0) = w_x(\pi) = 0 \right\} \end{aligned}$$

Furthermore, we introduce the weight Hilbert spaces

$$\mathcal{M}_1 = L^2_\kappa\left((0,\infty); H^1_*(0,\pi)\right) = \left\{ w : R_+ \to H^1_*(0,\pi) \ ; \ \int_0^\infty \kappa(s) \left\| w_x(s) \right\|_2^2 ds < \infty \right\}$$

and

$$\mathcal{H} = H^1_\kappa\left((0,\infty) \ , \ H^1_*(0,\pi)\right) = \{\eta \ / \ \eta, \eta_s \in \mathcal{M}_1\}$$

We define the enegy space by

$$\begin{aligned} \mathbb{H} = &H^1_0(0,\pi) \times L^2(0,\pi) \times H^1_*(0,\pi) \times L^2_*(0,\pi) \times H^1_*(0,\pi) \\ &\times L^2((0,\pi) \times (0,1) \times (\tau_1,\tau_2)) \times \mathcal{M}_1 \end{aligned}$$

Then $\mathbb{H}$, along with the inner product

$$\begin{aligned} \left\langle U, \tilde{U} \right\rangle_{\mathbb{H}} &= \rho_1 \int_0^\pi v\tilde{v}dx + J \int_0^\pi \phi\tilde{\phi}dx + c \int_0^\pi \theta\tilde{\theta}dx + \alpha \int_0^\pi \varphi_x\tilde{\varphi}_x dx \\ &+ \frac{\mu}{2} \int_0^\pi \left(u_x + \frac{b}{\mu}\varphi\right)\left(\tilde{u}_x + \frac{b}{\mu}\tilde{\varphi}\right) dx + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right) \int_0^\pi u_x\,\tilde{u}_x\,dx \\ &+ \frac{\xi}{2} \int_0^\pi \left(\varphi + \frac{b}{\xi}u_x\right)\left(\tilde{\varphi} + \frac{b}{\xi}\tilde{u}_x\right) dx + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right) \int_0^\pi \varphi\,\tilde{\varphi}\,dx \\ &+ \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\ \left|\gamma_2(\sigma)\right|\ z\,\tilde{z}\,d\sigma\,d\rho\,dx + \int_0^\infty \kappa(s) \int_0^\pi \eta_x\,\tilde{\eta}_x\,dx\,ds \end{aligned} \tag{3.5}$$

is a Hilbert space for any $U = (u, v, \varphi, \phi, \theta, z, \eta)^T \in \mathbb{H}$ and $U = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\phi}, \tilde{\theta}, \tilde{z}, \tilde{\eta})^T \in \mathbb{H}$.

The system (3.3)-(3.4) can be rewritten as follows:

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathcal{A}\,U(t) \ , \ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, f_0, \eta_0)^T, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{\mu}{\rho_1} u_{xx} + \dfrac{b}{\rho_1}\varphi_x - \dfrac{\beta}{\rho_1}\theta_x - \frac{\gamma_1}{\rho_1}\ v - \dfrac{1}{\rho_1}\displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\ z(x,\ 1,\ \sigma,\ t)\ d\sigma \\ \phi \\ \dfrac{\alpha}{J}\varphi_{xx} - \dfrac{b}{J}u_x - \dfrac{\xi}{J}\varphi + \dfrac{\delta}{J}\theta - \dfrac{\tau}{J}\phi \\ \dfrac{1}{c}\displaystyle\int_0^{\infty} \kappa(s)\ n_{xx}(x,\ s)\ ds - \dfrac{\beta}{c}v_x - \dfrac{\delta}{c}\phi \\ -\dfrac{1}{\sigma}z_\rho \\ \theta - \eta_s \end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) = \big\{ & U \in \mathbb{H}/\, u \in H^2(0,\pi) \cap H_0^1(0,\pi)\ ;\ \varphi\,,\, \theta \in H_*^2(0,\pi) \cap H_*^1(0,\pi)\ ; \\ & v \in H_0^1(0,\pi)\ ;\ \phi \in H_*^1(0,\pi)\ ;\ z\,,\, z_\rho \in L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2))\ ; \\ & \eta \in \mathcal{H}\ ;\ \int_0^{\infty} \kappa(s)\ n_{xx}(x,\ s)\ ds \in L^2(0,\pi)\ ;\ \eta(x,0) = 0 \big\}. \end{aligned}$$

Now we have the following existence and uniqueness result

Theorem 3.1. *Let $U_0 \in \mathbb{H}$ and assume that* (1.3) *holds. Then, there exists a unique solution $U \in C(\mathbb{R}_+, \mathbb{H})$ for problem* (3.3)-(3.4). *Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathbb{H}).$$

Proof. We use the semi-group approach. So we prove that $\mathcal{A}$ is a maximal dissipative operator.

First, we prove that $\mathcal{A}$ is dissipative. Let $U \in D(\mathcal{A})$, then we have

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathbb{H}} = & -\gamma_1 \int_0^{\pi} v^2\, dx - \tau \int_0^{\pi} \phi^2\, dx + \int_0^{\pi}\int_0^{\infty} \kappa(s)\ \eta_s\, \eta_{xx}\, ds\, dx \\ & - \int_0^{\pi} v \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(x,\ 1,\ \sigma,\ t)\, d\sigma\, dx - \int_0^{\pi}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\ z_\rho\, z\, d\sigma\, d\rho\, dx \end{aligned} \tag{3.6}$$

Using integration by parts and the fact that $z(x,0,t) = v(x,t)$, the last term in the right-hand side of (3.6) gives

$$\begin{aligned} -\int_0^{\pi}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\ z_\rho\, z\, d\sigma\, d\rho\, dx = & -\frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x,\ 1,\ \sigma,\ t)\, d\sigma\, dx \\ & + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^{\pi} v^2\, dx \end{aligned}$$

Also, using Young's inequality we get

$$
\begin{aligned}
&-\int_0^\pi v \int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,\, 1,\, \sigma,\, t)\, d\sigma\, dx \\
&\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right) \int_0^\pi v^2\, dx + \frac{1}{2}\int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x,\, 1,\, \sigma,\, t)\, d\sigma\, dx
\end{aligned}
$$

Furthermore, using integation by part and bringing in mind $(H7)$ we have

$$
\int_0^\pi \int_0^\infty \kappa(s)\ \eta_s\, \eta_{xx}\, ds\, dx = \frac{1}{2}\int_0^\infty \kappa'(s) \int_0^\pi n_x^2\, dx\, ds
$$

Consequently, using $(H2)$,(3.6) and (2.6) yields

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathbb{H}} \leq &-\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, d\sigma\right) \int_0^\pi v^2\, dx \\
&+ \frac{1}{2}\int_0^\infty \kappa'(s) \int_0^\pi n_x^2\, dx\, ds - \tau \int_0^\pi \phi^2\, dx \leq 0
\end{aligned}
$$

Therefore, the operator $\mathcal{A}$ is dissipative. Next, we prove that the operator $\lambda I - \mathcal{A}$ is surjective. For any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathbb{H}$, we prove that there exists a unique $U \in D(\mathcal{A})$ such that

$$
(\lambda I - \mathcal{A})\, U = F \tag{3.7}
$$

The problem (3.7) , leads to solve the following system

$$
\begin{cases}
\lambda u - v = f_1 \in H_0^1(0,\pi) \\
(\lambda \rho_1 + \gamma_1)\, v - \mu\ u_{xx} - b\ \varphi_x + \beta\ \theta_x + \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\ z(x,\ 1,\ \sigma,\ t)\ d\sigma \\
= \rho_1\, f_2 \in L^2(0,\pi) \\
\lambda \varphi - \phi = f_3 \in H_*^1(0,\pi) \\
(\lambda J + \tau)\ \phi - \alpha\ \varphi_{xx} + b\ u_x + \xi\ \varphi - \delta\ \theta = J\, f_4 \in L_*^2(0,\pi) \\
\lambda\, c\, \theta - \displaystyle\int_0^\infty \kappa(s)\, \eta_{xx}(x,s)\, ds + \beta\, v_x + \delta\, \phi = c\, f_5 \in H_*^1(0,\pi) \\
\lambda\, \sigma\, z + z_\rho = \sigma\, f_6 \in L^2((0,\pi) \times (0,1) \times (0,\infty)) \\
\lambda\, \eta - \theta + \eta_s = f_7 \in \mathcal{M}_1
\end{cases} \tag{3.8}
$$

Suppose u and φ are given with the appropriate regularity. Then, the first and the third equations in (3.8) yield

$$
v = \lambda u - f_1 \in H_0^1(0,\pi) \tag{3.9}
$$

and

$$
\phi = \lambda \varphi - f_3 \in H_*^1(0,\pi) \tag{3.10}
$$

respectively.

The sixth equation in (3.8) together with (3.9) and the fact that $z(x,0) = v(x,t)$ gives

$$z(x,\rho,\sigma,t) = \lambda u(x,t)\, e^{-\lambda\,\sigma\,\rho} - f_1\, e^{-\lambda\,\sigma\,\rho} + \sigma\, e^{-\lambda\,\sigma\,\rho} \int_0^\rho e^{\lambda\,\sigma\,y}\, f_6(x,y,\sigma,t)\, dy \tag{3.11}$$

The last equation in (3.8) under the condition $\eta(0) = 0$ gives

$$\eta(x,s) = \frac{1}{\lambda}\theta(x,t)\left(1 - e^{-\lambda\, s}\right) + \int_0^s e^{\lambda(w-s)} f_7(w)\, dw \tag{3.12}$$

Using integration by parts, it can easily be shown that the second, fourth and fifth equations in (3.8) satisfy the following:

$$\begin{cases} (\lambda\,\rho_1 + \gamma_1)\displaystyle\int_0^\pi v\,\tilde{u}\, dx + \mu\int_0^\pi u_x\,\tilde{u}_x\, dx + b\int_0^\pi \varphi\,\tilde{u}_x\, dx - \beta\int_0^\pi \theta\,\tilde{u}_x\, dx \\ + \displaystyle\int_0^\pi \tilde{u}\int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,1,\sigma,t)\, d\sigma\, dx = \rho_1\int_0^\pi f_2\,\tilde{u}\, dx \\ (\lambda\, J + \tau)\displaystyle\int_0^\pi \phi\,\tilde{\varphi}\, dx + \alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\, dx + b\int_0^\pi u_x\,\tilde{\varphi}\, dx + \xi\int_0^\pi \varphi\,\tilde{\varphi}\, dx - \delta\int_0^\pi \theta\,\tilde{\varphi}\, dx \\ = J\displaystyle\int_0^\pi f_4\,\tilde{\varphi}\, dx \\ c\displaystyle\int_0^\pi \theta\,\tilde{\theta}\, dx + \frac{1}{\lambda}\int_0^\pi \tilde{\theta}_x\int_0^\infty \kappa(s)\,\eta_x\, ds\, dx + \frac{\beta}{\lambda}\int_0^\pi v_x\,\tilde{\theta}\, dx + \frac{\delta}{\lambda}\int_0^\pi \phi\,\tilde{\theta}\, dx = \frac{c}{\lambda}\int_0^\pi f_5\,\tilde{\theta}\, dx \end{cases} \tag{3.13}$$

Furthermore, by using (3.9)-(3.12) , we have the following corresponding weak formulation for the second, fourth and fifth equation in (3.8):
Finding $(u,\varphi,\theta) \in H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ such that for all $\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) \in H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ the following holds:

$$B\left((u,\varphi,\theta);\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\right) = l\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) \tag{3.14}$$

where $B : \left[H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B\left((u,\varphi,\theta);\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\right) &= \mu_0\int_0^\pi u\,\tilde{u}\, dx + \mu\int_0^\pi u_x\,\tilde{u}_x\, dx + \mu_1\int_0^\pi \varphi\,\tilde{\varphi}\, dx + \alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\, dx \\ &+ c\int_0^\pi \theta\,\tilde{\theta}\, dx + c_\kappa\int_0^\pi \theta_x\,\tilde{\theta}_x\, dx + b\int_0^\pi (\varphi\,\tilde{u}_x + u_x\,\tilde{\varphi})\, dx \\ &+ \beta\int_0^\pi \left(u_x\tilde{\theta} - \theta\,\tilde{u}_x\right) dx + \delta\int_0^\pi \left(\varphi\,\tilde{\theta} - \theta\,\tilde{\varphi}\right) dx \end{aligned}$$

and $l : H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi) \to \mathbb{R}$ is the linear form given by

$$l\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) = \int_0^\pi g_1\,\tilde{u}\,dx + \int_0^\pi g_2\,\tilde{\varphi}\,dx + \int_0^\pi g_3\,\tilde{\theta}\,dx + \int_0^\pi g_4\,\tilde{\theta}_x\,dx.$$

where

$$\begin{aligned}
\mu_0 &= \lambda^2\rho_1 + \lambda\gamma_1 + \lambda\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,e^{-\lambda\,\sigma}d\sigma \quad > 0\\
\mu_1 &= \lambda^2 J + \xi + \lambda\,\tau \quad > 0\\
c_\kappa &= \frac{1}{\lambda^2}\int_0^\infty \kappa(s)\left(1 - e^{-\lambda\,s}\right)ds \quad > 0\\
g_1 &= \frac{\mu_0}{\lambda}f_1 + \rho_1 f_2 - \int_{\tau_1}^{\tau_2}\sigma\,\gamma_2(\sigma)\,e^{-\lambda\,\sigma}\int_0^1 e^{\lambda\,\sigma\,y} f_6(x,y,\sigma,t)\,dy\,d\sigma \ \in L^2(0,\pi)\\
g_2 &= (\lambda\,J + \tau)\,f_3 + J\,f_4 \ \in L^2(0,\pi)\\
g_3 &= \frac{\beta}{\lambda}f_{1_x} + \frac{\delta}{\lambda}f_3 + \frac{c}{\lambda}f_5 \ \in L^2(0,\pi)\\
g_4 &= -\frac{1}{\lambda}\int_0^\infty \kappa(s)\int_0^s e^{\lambda(w-s)} f_{7_x}(w)\,dw\,ds \ \in L^2(0,\pi)
\end{aligned}$$

Now, for $\mathcal{V} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ equipped with the norm

$$\|(u,\varphi,\theta)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2$$

we have

$$\begin{aligned}
|B((u,\varphi,\theta);(u,\varphi,\theta))| &= \mu_0\int_0^\pi u^2\,dx + \mu\int_0^\pi u_x^2\,dx + \mu_1\int_0^\pi \varphi^2\,dx + \alpha\int_0^\pi \varphi_x^2\,dx\\
&\quad + c\int_0^\pi \theta^2\,dx + c_\kappa\int_0^\pi \theta_x^2\,dx + 2b\int_0^\pi u_x\,\varphi\,dx
\end{aligned}$$

On the other hand , we can write

$$\begin{aligned}
\mu\,u_x^2 + \mu_1\,\varphi^2 + 2b\,u_x\,\varphi &= \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \mu_1\left(\varphi + \frac{b}{\mu_1}u_x\right)^2\right]\\
&\quad + \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right)u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right]
\end{aligned}$$

then, using (1.3) we deduce that

$$\mu\,u_x^2 + \mu_1\,\varphi^2 + 2b\,u_x\,\varphi \geq \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right)u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right]$$

consiquently

$$|B((u,\varphi,\theta);(u,\varphi,\theta))| \geq M\,\|(u,\varphi,\theta)\|_V^2$$

where $M = min\left\{\frac{1}{2}\left(\mu - \frac{b^2}{\mu_1}\right)\ ;\ \frac{1}{2}\left(\mu_1 - \frac{b^2}{\mu}\right)\ ;\ \alpha\ ;\ c\ ;\ \mu_0\ ;\ c_\kappa\right\}$. Thus, B is coercive. Moreover, we can easily see that B and l are bounded. Consequently, by Lax-Milgram Lemmam we conclude that there exists a unique $(u, \varphi, \theta) \in \mathcal{V}$ which satisfies (3.14).

Substituting u in (3.9) and (3.11), respectively, we obtain

$$v \in\ H_0^1(0,\pi) \quad , \quad z \in L^2\left((0,\pi)\times(0,1)\times(\tau_1,\tau_2)\right)$$

and z in $(3.8)_6$ we find $z_\rho \in L^2\left((0,\pi)\times(0,1)\times(\tau_1,\tau_2)\right)$

then, inserting φ in (3.10) and we get

$$\phi \in\ H_*^1(0,\pi)$$

Similarly, inserting θ in (3.12) and bearing in mind $(3.8)_7$, we obtain

$$\eta \in \mathcal{H} \quad , \quad \eta(x,0) = 0$$

Moreover, if we take $\left(\tilde{\varphi}, \tilde{\theta}\right) \equiv (0,0) \in H_*^1(0,\pi) \times H_*^1(0,\pi)$, then (3.14) reduces to

$$\mu \int_0^\pi u_x\,\tilde{u}_x\,dx + b\int_0^\pi \varphi\,\tilde{u}_x\,dx - \beta\int_0^\pi \theta\,\tilde{u}_x\,dx = -\int_0^\pi \left(-g_1 + \mu_0\,u\right)\tilde{u}\,dx \quad , \ \forall \tilde{u} \in H_0^1(0,\pi)$$

That is

$$\mu\,u_{xx} = -g_1 + \mu_0\,u - b\,\varphi_x + \beta\,\theta_x \quad , \quad in\ L^2(0,\pi)$$

which implies

$$u \in H^2(0,\pi) \cap H_0^1(0,\pi)$$

Then, we choose $\left(\tilde{u}, \tilde{\theta}\right) \equiv (0,0) \in H_0^1(0,\pi) \times H_*^1(0,\pi)$, then (3.14) become

$$\alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\,dx = -\int_0^\pi \left(\mu_1\,\varphi + b\,u_x - \delta\,\theta - g_2\right)\tilde{\varphi}\,dx \quad , \ \forall \tilde{\varphi} \in H_*^1(0,\pi) \tag{3.15}$$

Here, we can not use the regularity theorem, because $\tilde{\varphi} \in H_*^1(0,\pi)$. So, we take $\psi \in H_0^1(0,\pi)$ and we set

$$\tilde{\varphi}(x) = \psi(x) - \int_0^\pi \psi(x)\ dx$$

It's clear that $\tilde{\varphi} \in H_*^1(0,\pi)$. Then, a substitution in (3.15) leads to

$$\alpha\int_0^\pi \varphi_x\,\psi_x\,dx = -\int_0^\pi r\,\psi\,dx \quad , \ \forall \psi \in H_0^1(0,\pi)$$

where,

$$r = \mu_1\,\varphi + b\,u_x - \delta\,\theta - g_2$$

That is

$$\alpha\,\varphi_{xx} = \mu_1\,\varphi + b\,u_x - \delta\,\theta - g_2 \quad , \ in\,L^2(0,\pi) \tag{3.16}$$

which implies

$$\varphi \in H^2(0,\pi)$$

On the other hand, from (3.15) and using integration by parts we get

$$\alpha\left[\varphi_x\,\tilde{\varphi}\right]_0^\pi-\alpha\int_0^\pi\varphi_{xx}\,\tilde{\varphi}\,dx+\int_0^\pi\left(\mu_1\,\varphi+b\,u_x-\delta\,\theta-g_2\right)\,\tilde{\varphi}\,dx=0\ \ ,\ \ \forall\tilde{\varphi}\in H_*^1\left(0,\pi\right)$$

and from (3.16) we obtain

$$\varphi_x\left(\pi\right)\,\varphi\left(\pi\right)-\varphi_x\left(0\right)\,\varphi\left(0\right)=0$$

Since $\tilde{\varphi}\in H_*^1\left(0,\pi\right)$ is arbitrary then,

$$\varphi_x\left(\pi\right)\,=\varphi_x\left(0\right)\,=0$$

Consequently

$$\varphi\in H_*^2\left(0,\pi\right)\cap H_*^1\left(0,\pi\right)$$

Similary if we take $\left(\tilde{u},\tilde{\varphi}\right)\equiv\left(0,0\right)\in H_0^1\left(0,\pi\right)\times H_*^1\left(0,\pi\right)$, we find

$$\theta\in H_*^2\left(0,\pi\right)\cap H_*^1\left(0,\pi\right)$$

Finally, from $(3.8)_5$ we get

$$\int_0^\infty\kappa\left(s\right)\eta_{xx}\left(x,s\right)ds\in L^2\left(0,\pi\right)$$

Hence, there exists a unique $U\in D\left(\mathcal{A}\right)$ such that (3.7) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. With this , we conclude that $\mathcal{A}$ is a maximal dissipative operator. Consequently, $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathbb{H}$. Therefore, the well-posedness result follows from the Hille-Yosida theorem. (see [34]) ■

4. Exponential decay

In this section, we state and prove technical lemmas needed for the proof of our stability result.

Lemma 4.1. *Let* $\left(u,\varphi,\theta,z,\eta\right)$ *be a solution of* (3.3) - (3.4) *. Then, the energy functional* $E\left(t\right)$*, defined by*

$$\begin{aligned}E\left(t\right)=&\frac{1}{2}\int_0^\pi\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\\&+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,\left|\gamma_2\left(\sigma\right)\right|\,z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^\infty\kappa\left(s\right)\int_0^\pi\eta_x^2\left(x,s\right)dx\,ds\end{aligned}\tag{4.1}$$

satisfies

$$E'\left(t\right)\leq-\left(\gamma_1-\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|d\sigma\right)\int_0^\pi u_t^2\,dx+\frac{1}{2}\int_0^\infty\kappa'\left(s\right)\int_0^\pi\eta_x^2\left(x,s\right)dx\,ds-\tau\int_0^\pi\varphi_t^2\,dx\tag{4.2}$$

Proof. Multiplying $(3.3)_1$, $(3.3)_2$, $(3.3)_3$ by u_t , φ_t , θ respectively, integrating over $(0,\pi)$, and Multiplying$(3.3)_4$ by $\left|\gamma_2\left(\sigma\right)\right|\,z$, integrating over $(0,\pi)\times(0,1)\times(\tau_1,\tau_2)$ then,using integration by part and taking into account the boundary conditions and summing them up, we obtain

$$\frac{d}{2\,dt}\left\{\int_0^\pi \left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\right.$$
$$\left.+\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\gamma_1\int_0^\pi u_t^2\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx$$
$$+\int_0^\pi\theta\int_0^\infty\kappa(s)\,\eta_{xx}(x,s)\,ds\,dx-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$

Using $(3.3)_5$, we obtain

$$\frac{d}{2\,dt}\left\{\int_0^\pi \left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\right.$$
$$\left.+\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\gamma_1\int_0^\pi u_t^2\,dx \tag{4.3}$$
$$-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$
$$+\int_0^\pi\int_0^\infty\kappa(s)\,\eta_t\,\eta_{xx}(x,s)\,ds\,dx+\int_0^\pi\int_0^\infty\kappa(s)\,\eta_s\,\eta_{xx}(x,s)\,ds\,dx \tag{4.4}$$

integrating by part the last two terms of (4.4) we get

$$E(t)=\frac{1}{2}\int_0^\pi\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx$$
$$+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^\infty\kappa(s)\int_0^\pi\eta_x^2(x,s)\,dx\,ds \tag{4.5}$$

and

$$E'(t)=-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$
$$-\gamma_1\int_0^\pi u_t^2\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\frac{1}{2}\int_0^\infty\kappa(s)\,\frac{\partial}{\partial s}\int_0^\pi\eta_x^2(x,s)\,dx\,ds \tag{4.6}$$

On the other hand we have $z\left(x,0,\sigma,t\right)=u_t\left(x,t\right)$, then

$$
\begin{aligned}
&-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|\,z_\rho\,z\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx\\
&=-\frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|\,z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|\,d\sigma\right)\int_0^{\pi}u_t^2\,dx
\end{aligned}
\tag{4.7}
$$

using integration by part and bringing in mind $(H7)$ we find

$$
\begin{aligned}
&-\frac{1}{2}\int_0^{\infty}\kappa\left(s\right)\frac{\partial}{\partial s}\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds\\
&=-\frac{1}{2}\underset{s\to b}{lim}\left[\kappa\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\right]_{b=0}^{b=\infty}+\frac{1}{2}\int_0^{\infty}\kappa'\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds\\
&=\frac{1}{2}\int_0^{\infty}\kappa'\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds
\end{aligned}
\tag{4.8}
$$

Then, using Young's inequality on the first term in (4.6) we have

$$
\begin{aligned}
-\int_0^{\pi}u_t\int_{\tau_1}^{\tau_2}\gamma_2\left(\sigma\right)\,z\left(x,1,\sigma,t\right)d\sigma\,dx\leq{}&\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|\,d\sigma\right)\int_0^{\pi}u_t^2\,dx\\
&+\frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|\,z^2\left(x,1,\sigma,t\right)d\sigma\,dx
\end{aligned}
\tag{4.9}
$$

Inserting (4.7) , (4.8) and (4.9) in (4.6), we get (4.2) ■

Remark 4.2. *The energy function $E\left(t\right)$ defined in* (4.1) *is nonnegative. In fact,*

$$
\begin{aligned}
\mu\,u_x^2+\xi\,\varphi^2+2b\,u_x\,\varphi={}&\frac{1}{2}\left[\mu\left(u_x+\frac{b}{\mu}\varphi\right)^2+\xi\left(\varphi+\frac{b}{\xi}u_x\right)^2\right]\\
&+\frac{1}{2}\left[\left(\mu-\frac{b^2}{\xi}\right)u_x^2+\left(\xi-\frac{b^2}{\mu}\right)\varphi^2\right]
\end{aligned}
$$

from (1.3) *we deduce that*

$$
\mu\,u_x^2+\xi\,\varphi^2+2b\,u_x\,\varphi\geq\frac{1}{2}\left[\left(\mu-\frac{b^2}{\xi}\right)u_x^2+\left(\xi-\frac{b^2}{\mu}\right)\varphi^2\right]
$$

consequently

$$
\begin{aligned}
E\left(t\right)\;>\;&\frac{1}{2}\int_0^{\pi}\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\frac{1}{2}\left(\mu-\frac{b^2}{\xi}\right)u_x^2+\alpha\,\varphi_x^2+\frac{1}{2}\left(\xi-\frac{b^2}{\mu}\right)\varphi^2\right)dx\\
&+\frac{1}{2}\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\,\left|\gamma_2\left(\sigma\right)\right|\,z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^{\infty}\kappa\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds
\end{aligned}
$$

then $E\left(t\right)$ is nonnegative.

Remark 4.3. *From $(H2)$ we conclude that the energy functional $E(t)$ is decreasing and bounded above by $E(0)$*

Lemma 4.4. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_1(t)=\rho_1\int_0^{\pi}u_t u\,dx+\frac{\gamma_1}{2}\int_0^{\pi}u^2dx\ ,\ t\geq 0,$$

satisfies

$$\begin{aligned}I_1{}'(t)\leq &-\frac{\mu}{2}\int_0^{\pi}u_x^2\,dx+\rho_1\int_0^{\pi}u_t^2\,dx+c_0\int_0^{\pi}\left(\varphi^2+\theta^2\right)dx\\ &+c_0\int_0^{\pi}\int_{\tau_1}^{\tau_2}\left|\gamma_2(\sigma)\right|\ z^2(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.10}$$

Proof. By differentiating $I_1(t)$, using $(3.3)_1$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned}I_1{}'(t)=&-\mu\int_0^{\pi}u_x^2\,dx-b\int_0^{\pi}\varphi\,u_x\,dx+\beta\int_0^{\pi}\theta\,u_x\,dx+\rho_1\int_0^{\pi}u_t^2\,dx\\ &-\int_0^{\pi}u\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\ z(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.11}$$

Young's , Poincaré and Cauchy Schwarz inequalities lead to

$$-b\int_0^{\pi}\varphi\,u_x\,dx\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3\,b^2}{2\,\mu}\int_0^{\pi}\varphi^2dx\tag{4.12}$$

$$\beta\int_0^{\pi}\varphi\,u_x\,dx\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3\,\beta^2}{2\,\mu}\int_0^{\pi}\theta^2dx\tag{4.13}$$

and

$$\begin{aligned}&-\int_0^{\pi}u\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\ z(x,1,\sigma,t)\ d\sigma\,dx\\ &\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3}{2\,\mu}\left(\int_{\tau_1}^{\tau_2}\left|\gamma_2(\sigma)\right|\,d\sigma\right)\int_0^{\pi}\int_{\tau_1}^{\tau_2}\left|\gamma_2(\sigma)\right|\ z^2(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.14}$$

Substituting (4.12) , (4.13) and (4.14) in (4.11), we get (4.10). ∎

Lemma 4.5. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_2(t)=J\int_0^{\pi}\varphi\,\varphi_t\,dx+\frac{\tau}{2}\int_0^{\pi}\varphi^2dx+\frac{b\,\rho_1}{\mu}\int_0^{\pi}\varphi\int_0^{x}u_t(y)\,dy\,dx\ ,\ t\geq 0,$$

satisfies

$$I_2{}'(t) \le -\alpha \int_0^\pi \varphi_x^2 dx - \frac{\chi}{4}\int_0^\pi \varphi^2 dx + c_0 \int_0^\pi \left(u_t^2 + \varphi_t^2 + \theta^2\right) dx \tag{4.15}$$

$$+ c_0 \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx \tag{4.16}$$

where $\chi = \xi - \dfrac{b^2}{\mu}$

Proof. By differentiating $I_2(t)$, using $(3.3)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} I_2{}'(t) = {} & -\alpha \int_0^\pi \varphi_x^2 dx - \chi \int_0^\pi \varphi^2 dx + J \int_0^\pi \varphi_t^2 dx + \left(\delta - \frac{b\,\beta}{\mu}\right)\int_0^\pi \varphi\,\theta\, dx \\ & + \frac{b\,\rho_1}{\mu}\int_0^\pi \varphi_t \int_0^x u_t(y)\, dy\, dx - \frac{b\,\gamma_1}{\mu}\int_0^\pi \varphi \int_0^x u_t(y)\, dy\, dx \\ & - \frac{b}{\mu}\int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(y,1,\sigma,t)\, d\sigma\, dy\, dx \end{aligned} \tag{4.17}$$

Using Young's and Cauchy Schwarz inequalities, we get

$$\delta - \frac{b\,\beta}{\mu}\int_0^\pi \varphi\,\theta\, dx \le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\delta - \frac{b\,\beta}{\mu}\right)^2 \int_0^\pi \theta^2 dx \tag{4.18}$$

$$\frac{b\,\rho_1}{\mu}\int_0^\pi \varphi_t \int_0^x u_t(y)\, dy\, dx \le \frac{b\,\rho_1}{2\mu}\int_0^\pi \varphi_t^2 dx + \frac{b\,\rho_1\,\pi^2}{2\mu}\int_0^\pi u_t^2 dx \tag{4.19}$$

$$-\frac{b\,\gamma_1}{\mu}\int_0^\pi \varphi \int_0^x u_t(y)\, dy\, dx \le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\frac{b\,\gamma_1\,\pi}{\mu}\right)^2 \int_0^\pi u_t^2 dx \tag{4.20}$$

and

$$\begin{aligned} & -\frac{b}{\mu}\int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(y,1,\sigma,t)\, d\sigma\, dy\, dx \\ & \le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\frac{b\,\pi}{\mu}\right)^2 \left(\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, d\sigma\right) \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx \end{aligned} \tag{4.21}$$

Inserting (4.18)-(4.21) in (4.17), we obtain (4.16). ■

Lemma 4.6. *Let* $(u, \varphi, \theta, z, \eta)$ *be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_3(t) = -c\,\rho_1 \int_0^\pi \theta \int_0^x u_t(y)\, dy\, dx\ ,\ t \ge 0,$$

satisfies, for any $\varepsilon_1 > 0$, the following estimate

$$I_3{}'(t) \le -\frac{\rho_1\,|\beta|}{2}\int_0^\pi u_t^2\,dx + \varepsilon_1\int_0^\pi \left(u_x^2+\varphi^2\right)\,dx + c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^\pi \theta^2\,dx$$
$$+\frac{\rho_1\,\kappa_0}{|\beta|}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\;z^2(x,1,\sigma,t)\,d\sigma\,dx \tag{4.22}$$

Proof. Differentiating the functional $I_3(t)$ using $(3.3)_1, (3.3)_3$ and integrating by parts together with the boundary conditions,, we obtain

$$I_3{}'(t) = -\rho_1\,\beta\int_0^\pi u_t^2\,dx - \mu\,c\int_0^\pi \theta\,u_x\,dx - b\,c\int_0^\pi \theta\,\varphi\,dx + \beta\,c\int_0^\pi \theta^2\,dx - c\,\gamma_1\int_0^\pi u_t\,\theta\,dx$$
$$+c\int_0^\pi \theta\int_0^x\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\;z(y,1,\sigma,t)\;d\sigma\,dy\,dx + \rho_1\int_0^\pi u_t\int_0^\infty \kappa(s)\,\eta_x(x,s)\,ds\,dx \tag{4.23}$$

Using Young's and Cauchy Schwarz inequalities,

$$-\mu\,c\int_0^\pi \theta\,u_x\,dx \le \varepsilon_1\int_0^\pi u_x^2\,dx + \frac{\mu^2c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx \tag{4.24}$$

$$-b\,c\int_0^\pi \theta\,\varphi\,dx \le \varepsilon_1\int_0^\pi \varphi^2\,dx + \frac{b^2c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx \tag{4.25}$$

$$\int_0^\pi \theta\int_0^x\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\;z(y,1,\sigma,t)\;d\sigma\,dy\,dx$$
$$\le \frac{c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\;z^2(x,1,\sigma,t)\,d\sigma\,dx \tag{4.26}$$

$$\rho_1\int_0^\pi u_t\int_0^\infty \kappa(s)\,\eta_x(x,s)\,ds\,dx \le \frac{\rho_1\,|\beta|}{4}\int_0^\pi u_t^2\,dx + \frac{\rho_1\,\kappa_0}{|\beta|}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.27}$$

$$-c\,\gamma_1\int_0^\pi u_t\,\theta\,dx \le \frac{|\beta|\,\rho_1}{4}\int_0^\pi u_t^2dx + \frac{(c\,\gamma_1)^2}{|\beta|\,\rho_1}\int_0^\pi \theta^2dx \tag{4.28}$$

Substituting (4.24)-(4.28) in (4.23), we get (4.22). ■

Lemma 4.7. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_4(t) = -\frac{c}{\kappa_0}\int_0^\pi \theta\int_0^\infty \kappa(s)\;\eta(x,s)\,ds\,dx\;,\;\; t\ge 0,$$

satisfies, for any $\varepsilon_2, \varepsilon_3 > 0$, the following estimate

$$I_4{}'(t) \leq -\frac{c}{2}\int_0^\pi \theta^2\,dx + \varepsilon_2 \int_0^\pi \varphi_t^2\,dx + \varepsilon_3 \int_0^\pi u_t^2\,dx - \frac{c\,k(0)}{2\kappa_0^2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2\,dx\,ds$$
$$+ c_0\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.29}$$

Proof. By differentiating $I_4(t)$, using $(3.3)_3$, $(3.3)_5$ and integrating by parts together with the boundary conditions,, we obtain

$$I_4{}'(t) = -c\int_0^\pi \theta^2\,dx + \frac{c}{\kappa_0}\int_0^\pi \theta \int_0^\infty \kappa(s)\ \eta_s(x,s)\,ds\,dx + \frac{1}{\kappa_0}\int_0^\pi \left(\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\right)^2 dx$$
$$-\frac{\beta}{\kappa_0}\int_0^\pi u_t \int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\,dx + \frac{\delta}{\kappa_0}\int_0^\pi \varphi_t \int_0^\infty \kappa(s)\ \eta(x,s)\,ds\,dx \tag{4.30}$$

Young's , Poincaré and Cauchy Schwarz inequalities lead to

$$\frac{\delta}{\kappa_0}\int_0^\pi \varphi_t \int_0^\infty \kappa(s)\ \eta(x,s)\,ds\,dx \leq \varepsilon_2 \int_0^\pi \varphi_t^2\,dx + \frac{\delta^2}{4\varepsilon_2\,\kappa_0}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.31}$$

$$-\frac{\beta}{\kappa_0}\int_0^\pi u_t \int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\,dx \leq \varepsilon_3 \int_0^\pi u_t^2\,dx + \frac{\beta^2}{4\varepsilon_3\,\kappa_0}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.32}$$

$$\frac{1}{\kappa_0}\int_0^\pi \left(\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\right)^2 dx \leq \int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.33}$$

and

$$\frac{c}{\kappa_0}\int_0^\pi \theta \int_0^\infty \kappa(s)\ \eta_s(x,s)\,ds\,dx \leq \frac{c}{2}\int_0^\pi \theta^2\,dx - \frac{c\,k(0)}{2\kappa_0^2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.34}$$

Estimate (4.29) follows by substituting (4.31)-(4.34) into (4.30). ■

Lemma 4.8. *Let $(u, \varphi, \theta, z, \eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_5(t) = \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\, e^{-\sigma\,\rho}\,|\gamma_2(\sigma)|\, z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx \qquad t \geq 0$$

satisfies the estimate

$$I_5{}'(t) \leq -m_1 \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, z^2(x,1,\sigma,t)\,d\sigma\,dx + \gamma_1 \int_0^1 u_t^2\,dx$$
$$-m_1 \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\ |\gamma_2(\sigma)|\, z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx \qquad t \geq 0 \tag{4.35}$$

Proof. By differentiating $I_5(t)$, using $(3.3)_4$, integrating by parts and using the fact that $z\left(x,0,\sigma,t\right)=u_t\left(x,t\right)$ gives, we obtain

$$I_5{}'\left(t\right)=-\int_0^{\pi}\int_{\tau_1}^{\tau_2} e^{-\sigma}\left|\gamma\left(\sigma\right)\right| z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\left(\int_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|d\sigma\right)\int_0^{\pi} u_t^2\,dx$$
$$-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\,e^{-\sigma\,\rho}\,\left|\gamma\left(\sigma\right)\right| z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx$$

using the fact that $e^{-\sigma}\leq e^{-\sigma\,\rho}\leq 1$ we get for all $\rho\in[0,1]$

$$I_5{}'\left(t\right)\leq-\int_0^{\pi}\int_{\tau_1}^{\tau_2} e^{-\sigma}\left|\gamma\left(\sigma\right)\right| z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\left(\int_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|d\sigma\right)\int_0^{\pi} u_t^2\,dx$$
$$-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\,e^{-\sigma}\,\left|\gamma\left(\sigma\right)\right| z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx$$

Since $-e^{-\sigma}$ is an increasing function, we have $-e^{-\sigma}\leq -e^{-\tau_2}$ for all $\sigma\in\left[\tau_1,\tau_2\right]$. Finally , setting $m_1=e^{-\tau_2}$ and bringing in mind (2.6) we get (4.35) ∎

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t)=N\,E\left(t\right)+I_1\left(t\right)+N_1I_2\left(t\right)+\frac{2}{\left|\beta\right|\,\rho_1}I_3\left(t\right)+N_2I_4\left(t\right)+N_3I_5\left(t\right) \tag{4.36}$$

where N, N_1, N_2, N_3 are positive constants.

Lemma 4.9. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.3) - (3.4)*. Then, there exist two positive constants* λ_1 *and* λ_2 *such that the Lyapunov functional* (4.36) *satisfies*

$$\lambda_1 E\left(t\right)\leq\mathcal{L}(t)\leq\lambda_2 E\left(t\right),\ \forall t\geq 0, \tag{4.37}$$

and

$$\mathcal{L}'(t)\leq-\varsigma_1 E(t)+\varsigma_2\int_0^{\infty}\kappa\left(s\right)\left\|\eta_x\right\|^2 ds\ \ ;\ \ \varsigma_1\ ,\ \varsigma_2>0. \tag{4.38}$$

Proof. From (4.36), we have

$$\left|\mathcal{L}(t)-NE\left(t\right)\right|\leq\rho_1\int_0^{\pi}\left|u_t u\right|dx+\frac{\gamma_1}{2}\int_0^{\pi}u^2dx+N_1J\int_0^{\pi}\left|\varphi_t\varphi\right|dx+N_1\frac{\tau}{2}\int_0^{\pi}\varphi^2dx$$
$$N_1\frac{b\,\rho_1}{\mu}\int_0^{\pi}\varphi\int_0^{x}u_t\left(y\right)dy\,dx+\frac{2c}{\left|\beta\right|}\int_0^{\pi}\left|\theta\int_0^{x}u_t\left(y\right)dy\right|dx$$
$$+\frac{N_2\,c}{\kappa_0}\int_0^{\pi}\left|\theta\int_0^{\infty}\kappa\left(s\right)\eta\left(x,s\right)ds\right|dx$$
$$+N_3\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma e^{-\sigma\rho}\left|\gamma\left(\sigma\right)\right|z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx$$

By using the Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t) - NE\,(t)| \leq \varsigma E\,(t)\,,\ \varsigma > 0,$$

which yields

$$(N-\varsigma)\,E\,(t) \leq \mathcal{L}\,(t) \leq (N+\varsigma)\,E\,(t)\,,$$

by choosing N (depending on N_1, N_2, and N_3) sufficiently large we obtain (4.37).

Now, By differentiating $\mathcal{L}\,(t)$, exploiting (4.2), (4.10), (4.16), (4.22), (4.29), (4.35) and setting $\varepsilon_1 = \dfrac{\mu\,\rho_1\,|\beta|}{8}$, $\varepsilon_2 = \dfrac{1}{N_2}$, $\varepsilon_3 = \dfrac{\rho_1}{N_2}$, we get

$$\begin{aligned}
\mathcal{L}'\,(t) \leq & -\left[\left(\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2\,(\sigma)|\,d\sigma\right) N + 1\right) - 2\rho_1 - c_0 N_1 - \gamma_1 N_3\right] \int_0^{\pi} u_t^2 dx \\
& - (N\,\tau - c_0 N_1 - 1) \int_0^{\pi} \varphi_t^2 dx - \left(\frac{N_1\chi}{4} - c_0 - \frac{\mu}{4}\right) \int_0^{\pi} \varphi^2 dx - \frac{\mu}{4} \int_0^{\pi} u_x^2 dx \\
& - \left(\frac{c\,N_2}{2} - c_0 - c_0 N_1 - \frac{2c_0}{\rho_1\,|\beta|}\left(1 + \frac{8}{\mu\,\rho_1\,|\beta|}\right)\right) \int_0^{\pi} \theta^2 dx - \alpha\,N_1 \int_0^{\pi} \varphi_x^2 dx \\
& - m_1 N_3 \int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2} \sigma\,|\gamma_2\,(\sigma)|\,z^2\,(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx \\
& - \left(m_1 N_3 - c_0 - c_0 N_1 - \frac{\mu\,\pi^2}{4}\int_{\tau_1}^{\tau_2} |\gamma_2\,(\sigma)|\,d\sigma\right) \int_0^{\pi}\int_{\tau_1}^{\tau_2} |\gamma_2\,(\sigma)|\,z^2\,(x,1,\sigma,t)\,d\sigma\,dx \\
& + \left(\frac{N}{2} - \frac{c\,k\,(0)}{2\kappa_0^2} N_2\right) \int_0^{\infty} \kappa'\,(s)\,\|\eta_x\|^2 ds \\
& + \left(\frac{2\kappa_0}{\beta^2} + c_0 N_2\left(1 + N_2 + \frac{N_2}{\rho_1}\right)\right) \int_0^{\infty} \kappa\,(s)\,\|\eta_x\|^2 ds
\end{aligned} \tag{4.39}$$

Now, we select our parameters appropriately as follows:
First, we choose N_1 large enough so that

$$\alpha_1 = \frac{N_1\chi}{4} - c_0 - \frac{\mu}{4} > 0.$$

Next, we select N_2 large enough so that

$$\alpha_2 = \frac{c\,N_2}{2} - c_0 - c_0 N_1 - \frac{2c_0}{\rho_1\,|\beta|}\left(1 + \frac{8}{\mu\,\rho_1\,|\beta|}\right) > 0.$$

We take N_3 large such that

$$m_1 N_3 - c_0 - c_0 N_1 - \frac{\mu\,\pi^2}{4}\int_{\tau_1}^{\tau_2} |\gamma_2\,(\sigma)|\,d\sigma > 0$$

Finally, we choose N large enough so that (4.37) remains valid, further

$$\alpha_3 = N\,\tau - c_0 N_1 - 1 > 0 \quad , \quad \frac{N}{2} - \frac{c\,k\,(0)}{2\kappa_0^2} N_2 > 0$$

and

$$\alpha_4 = \left(\left(\gamma_1 - \int\limits_{\tau_1}^{\tau_2} |\gamma_2(\sigma)| \, d\sigma \right) N + 1 \right) - 2\rho_1 - c_0 N_1 - \gamma_1 N_3 > 0.$$

Let $\alpha_5 = \frac{\mu}{4}$, $\alpha_6 = \alpha\, N_1$, $\alpha_7 = m_1\, N_3$, $\alpha_8 = \frac{2\kappa_0}{\beta^2} + c_0 N_2 \left(1 + N_2 + \frac{N_2}{\rho_1}\right)$

Ultimately, (4.39) turns out to be

$$\mathcal{L}'(t) \leq -\omega \left[\int\limits_0^{\pi} \left(u_t^2 + \varphi_t^2 + \theta^2 + u_x^2 + \varphi_x^2 + \varphi^2\right) dx \right]$$
$$- \omega \int\limits_0^{\pi} \int\limits_0^{1} \int\limits_{\tau_1}^{\tau_2} \sigma \, |\gamma_2(\sigma)| \, z^2(x, \rho, \sigma, t) \, d\sigma \, d\rho \, dx + \alpha_8 \int\limits_0^{\infty} \kappa(s) \, \|\eta_x\|^2 ds$$

Meanwhile, by revisiting the energy functional (4.1) and utilizing Young's inequality we find (4.38) ■

Now, we can state and prove the following stability result

Theorem 4.10. *Assume that* (1.3) *holds and* κ *satisfies* $(H1)$ - $(H7)$. *Then system* (3.3)-(3.4) *is exponentially stable. In other words there exist two positive constants* υ_1 *and* υ_2 *such that*

$$E(t) \leq \upsilon_2 \, e^{-\upsilon_1 t} \quad , \quad \forall t \geq 0 \tag{4.40}$$

Proof. Multiplying (4.1) by r, using $(H6)$, we end up with

$$\mathcal{Y}'(t) \leq -r\,\varsigma_1 E(t) \quad , \quad \forall t \geq 0 \tag{4.41}$$

where$\mathcal{Y}(t) = r\mathcal{L}(t) + 2\varsigma_2 E(t)$.Using(4.37),it's readily follows,for some $a_0, b_0 > 0$

$$a_0 E(t) \leq Y(t) \leq b_0 E(t) \quad , \quad \forall t \geq 0 \tag{4.42}$$

Consequently, inequality (4.41) becomes

$$Y'(t) \leq -\,\upsilon_1 Y(t) \quad , \quad \forall t \geq 0 \tag{4.43}$$

where $\upsilon_1 = \dfrac{r\,\varsigma_1}{b_0}$. A simple integration of (4.43) over $(0, t)$ induces

$$Y(t) \leq Y(0) \, e^{-\,\upsilon_1 t} \quad , \quad \forall t \geq 0 \tag{4.44}$$

Accordingly, by merging (4.42) and (4.44), we get (4.40). which leads to the conclusion of our stability result. ■

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