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Existence of principal eigensurfaces for cooperative $(p_1, \ldots, p_n)$-biharmonic systems

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Abstract. In this work, we establish existence of principal eigensurface as well as simplicity result for a biharmonician system subject to Navier boundary condition using eigencurve approach.

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Contents

1. Introduction

This manuscript deals with two eigensurface problems

$$\begin{cases} \Delta_{p_i}^2 u_i + H_i(\beta, u_i) - V(x)|u_i|^{\alpha_i - 1} u_i \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j + 1} = \lambda m_i(x)|u_i|^{p_i - 2} u_i, & \text{in } \Omega \\ u_i = \Delta u_i = 0, & \text{on } \partial\Omega \\ 1 \leq i \leq n \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta_{p_i}^2 u_i + H_i(\beta, u_i) - V(x)|u_i|^{\alpha_i - 1} u_i \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j + 1} - \lambda m_i(x)|u_i|^{p_i - 2} u_i = \mu |u_i|^{p_i - 2} u_i, & \text{in } \Omega \\ u_i = \Delta u_i = 0, & \text{on } \partial\Omega \\ 1 \leq i \leq n \end{cases} \tag{1.2}$$

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where λ and μ are real parameters, $\Omega \subset \mathbb{R}^N$ (with $N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$ and for $i \in \{1, \cdots, n\}$: α_i, p_i are numbers satisfying $\alpha_i \geq 0$, $p_i > 1$ with $\sum_{i=1}^{n} \frac{\alpha_i+1}{p_i} = 1$; $\Delta^2_{p_i} u_i = \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)$ is the p_i-biharmonic operator and $H_i(\beta, u_i) = 2\beta \cdot \nabla(|\Delta u_i|^{p_i-2}\Delta u_i) + |\beta|^2|\Delta u_i|^{p_i-2}\Delta u_i$. Let us suppose throughout this work that the functions V and m_i lie in $L^\infty(\Omega)$ and are nonnegatives. In this paper, we are essentially interested in the existence of non-trivial solutions of (1.1) that is $((u_1, \cdots, u_n), \beta, \lambda) \in W(\Omega) \setminus \{(0, \cdots, 0)\} \times \mathbb{R}^N \times \mathbb{R}$ with

$$W(\Omega) = \prod_{i=1}^{n}(W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)).$$

The Cartesian product of n Sobolev spaces $W(\Omega)$ is a reflexive Banach space (see [1, 9] for details) endowed with the norm

$$\|(u_1, u_2, \ldots, u_n)\| = \sum_{i=1}^{n} \|\Delta u_i\|_{p_i}$$

where $\|.\|_p$ stands for the Lebesgue norm in L^p for all $p \in (1, \infty]$. When the eigensurface parameter β is neglected in (1.1), the authors in [4] have recently established the existence of weak solutions via mountain pass theorem as well as the positivity and simplicity results for semitrivial and strictly principal eigenvalues of the problem. So the question of considering the presence of such a term becomes natural. This justifies the scalar version of (1.1) in the case $V \equiv 0$, namely

$$\begin{cases} \text{Find} \quad (u, \beta, \Gamma) \in (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \setminus \{0\} \times \mathbb{R}^N \times \mathbb{R} \text{ such that} \\ \Delta(|\Delta u|^{p-2}\Delta u) + 2\beta \cdot \nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = \Gamma a(x)|u|^{p-2}u \text{ in } \Omega \\ u = \Delta u = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

investigated in [10] where the authors proved the existence of a sequence of positive eigensurfaces $(\Gamma_n^p(., a))_n$. Later on, the first eigensurface $\Gamma_1^p(., a)$ is then characterized and it is shown that if $a \geq 0$ a.e. in Ω, then $\Gamma_1^p(., a)$ is simple and principal (see [11]). But the question of the impact of a weight V on the structure of the spectrum as we consider in the current work is left open so far. For some additional results on the spectrum of p-biharmonic operator, we refer to [2, 3, 5, 8, 12, 14]. Considering the Poisson equation subject to Dirichlet boundary condition

$$\begin{cases} -\Delta u = f(x) \quad \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \quad \partial\Omega \end{cases} \tag{1.3}$$

where $f \in L^p(\Omega)$, it is known from [9] that (1.3) has a unique solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and the inverse Λ of Laplace operator possesses the following properties that are of interest for our results:

Proposition 1.1. *[7, 15].*

1. *(Continuity) There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{W^{2,p}} \leq c_p\|f\|_p$$

holds for all $p \in (1, \infty)$ and $f \in L^p(\Omega)$.

2. *(Continuity) Given $k \in \mathbb{N}^*$, there exists a constant $c_{p,k} > 0$ such that*

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k}\|f\|_{W^{k,p}}$$

holds for all $p \in (1, \infty)$ and $f \in W^{k,p}(\Omega)$.

3. *(Symmetry) The identity*
$$\int_\Omega \Lambda u.v dx = \int_\Omega u.\Lambda v dx$$
holds for $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *with* $p' = \frac{p}{p-1}$ *and* $p \in (1, \infty)$.

4. *(Regularity) Given* $f \in L^\infty(\Omega)$*, we have* $\Lambda f \in C^{1,\nu}(\bar{\Omega})$ *for all* $\nu \in (0,1)$. *Moreover, there exists* $c_\nu > 0$ *such that*
$$\|\Lambda f\|_{C^{1,\nu}(\Omega)} \leq c_\nu \|f\|_\infty.$$

5. *(Regularity and Hopf-type maximum principle) Let* $f \in C(\bar{\Omega})$ *and* $f \geq 0$ *then* $w = \Lambda f \in C^{1,\nu}(\bar{\Omega})$*, for all* $\nu \in (0,1)$ *and* w *satisfies:* $w > 0$ *in* Ω, $\frac{\partial w}{\partial n} < 0$ *on* $\partial\Omega$.

6. *(Order preserving property) Given* $f, g \in L^p(\Omega)$ *if* $f \leq g$ *in* Ω*, then* $\Lambda f < \Lambda g$ *in* Ω.

The objective of this work is to examine the extent to which the results, on one hand in [4] on principal eigenvalue of biharmonic system and on the other hand in [6], hold for eigensurface problems in a context of cooperative $(p_1, \cdots, p_n)$-biharmonic systems such as(1.1) and (1.2).

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and preliminary results before completing the section with the statement of our main results regarding (1.1) and (1.2). The proofs of our main results are given in Section 3.

2. Preliminaries and Main Results

We begin with some well-known transformation (see for example [10]) that helps rewriting both problems (1.1) and (1.2) into a different form as follows: For all $\beta \in \mathbb{R}^N$, we have

$$\begin{aligned}
\Delta\left(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\right) &= \nabla\left[\nabla\left(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\right)\right] \\
&= \nabla\left[\nabla\left(e^{\beta\cdot x}\right)|\Delta u|^{p-2}\Delta u + e^{\beta\cdot x}\nabla\left(|\Delta u|^{p-2}\Delta u\right)\right] \\
&= e^{\beta\cdot x}\left[\Delta_p^2 u + 2\beta\cdot\nabla\left(|\Delta u|^{p-2}\Delta u\right) + |\beta|^2|\Delta u|^{p-2}\Delta u\right] \\
\Delta\left(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\right) &= e^{\beta\cdot x}[\Delta_p^2 u + H(\beta, u)],
\end{aligned}$$

and setting $\Delta_p^{2,\beta} u = \Delta\left(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\right)$, one can see that (1.1) (resp. (1.2)) is equivalent to

$$\begin{cases}
\text{Find} \quad ((u_1,\cdots,u_n),\lambda) \in W(\Omega)\setminus\{(0,\cdots,0)\}\times\mathbb{R} \text{ such that} & \\
\Delta_{p_i}^{2,\beta} u_i - V(x)e^{\beta\cdot x}|u_i|^{\alpha_i-1}u_i \prod\limits_{j=1,j\neq i}^{n}|u_j|^{\alpha_j+1} = \lambda m_i(x)e^{\beta\cdot x}|u_i|^{p_i-2}u_i, & \text{in } \Omega \\
u_i = \Delta u_i = 0, & \text{on } \partial\Omega \\
1 \leq i \leq n &
\end{cases} \tag{2.1}$$

(resp.

$$\begin{cases}
\text{Find} \quad ((u_1,\cdots,u_n),\mu) \in W(\Omega)\setminus\{(0,\cdots,0)\}\times\mathbb{R} \text{ such that} & \\
\Delta_{p_i}^{2,\beta} u_i - V(x)e^{\beta\cdot x}|u_i|^{\alpha_i-1}u_i \prod\limits_{j=1,j\neq i}^{n}|u_j|^{\alpha_j+1} - \lambda m_i(x)e^{\beta\cdot x}|u_i|^{p_i-2}u_i = \mu e^{\beta\cdot x}|u_i|^{p_i-2}u_i, & \text{in } \Omega \\
u_i = \Delta u_i = 0, & \text{on } \partial\Omega \\
1 \leq i \leq n &
\end{cases} \tag{2.2}$$

). For the sake of clarity, we introduce a series of definitions and framework in order to well understand the presentation of our main results.

1. The set of couples $(\beta, \lambda) \in \mathbb{R}^N \times \mathbb{R}$ (resp. $(\beta, \mu) \in \mathbb{R}^N \times \mathbb{R}$) such that there exists a solution $((u_1, \cdots, u_n), \beta, \lambda) \in W(\Omega) \setminus \{(0, \cdots, 0)\} \times \mathbb{R}^N \times \mathbb{R}$ (resp. $((u_1. \cdots, u_n), \beta, \mu) \in W(\Omega) \setminus \{(0, \cdots, 0)\} \times \mathbb{R}^N \times \mathbb{R}$) of (1.1) (resp. (1.2)) is called the third-order spectrum of the $(p_1, \cdots, p_n)$-biharmonic operator plus potential. The couple (β, λ) (resp. (β, μ)) is then called a third-order eigenvalue and $(u_1, \cdots, u_n)$ is said to be an associated eigenfunction of (1.1) (resp. (1.2)). Moreover, a set of third-order eigenvalues of the form $(\beta, f(\beta))$, for $\beta \in \mathbb{R}^N$ and some function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$, is called an eigensurface.

2. For $\beta \in \mathbb{R}^N$, $((u_1, \cdots, u_n), \lambda) \in W(\Omega) \times \mathbb{R}$ is a (weak) solution to problem (2.1) if
$$\int_\Omega e^{\beta \cdot x} |\Delta u_i|^{p_i - 2} \Delta u_i \Delta \varphi_i dx - \int_\Omega V e^{\beta \cdot x} \prod_{j=1, j \neq i}^{n} |u_j|^{\alpha_j + 1} |u_i|^{\alpha_i - 1} u_i \varphi_i dx = \lambda \int_\Omega m_i e^{\beta \cdot x} |u|^{p_i - 2} u_i \varphi_i dx,$$
for $1 \leq i \leq n$ and for all $(\varphi_1, \cdots, \varphi_n) \in W(\Omega)$.

3. For $\beta \in \mathbb{R}^N$, $((u_1, \cdots, u_n), \mu) \in W(\Omega) \times \mathbb{R}$ is a (weak) solution to problem (2.2) if
$$\int_\Omega e^{\beta \cdot x} |\Delta u_i|^{p_i - 2} \Delta u_i \Delta \varphi_i dx - \int_\Omega V e^{\beta \cdot x} \prod_{j=1, j \neq i}^{n} |u_j|^{\alpha_j + 1} |u_i|^{\alpha_i - 1} u_i \varphi_i dx - \lambda \int_\Omega m_i e^{\beta \cdot x} |u|^{p_i - 2} u_i \varphi_i dx$$
$$= \mu \int_\Omega e^{\beta \cdot x} |u_i|^{p_i - 2} u_i \varphi_i dx, \text{ for } 1 \leq i \leq n, \tag{2.3}$$
for all $(\varphi_1, \cdots, \varphi_n) \in W(\Omega)$.

4. If $((u_i, \cdots, u_n), \lambda) \in W(\Omega) \times \mathbb{R}$ (resp. $((u_1, \cdots, u_n), \mu) \in W(\Omega) \times \mathbb{R}$) is a (weak) solution to problem (2.1) (resp. (2.2)), $(u_1, \cdots, u_n)$ shall be called an eigenfunction of the problem (2.1) (resp. (2.2)) associated to the eigenvalue λ (resp. μ). Let us agree to say that an eigenvalue of (2.1) or (2.2) is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction $(u_1, \cdots, u_n)$ such that $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\}$ (resp. there exist $\emptyset \neq J_n \subset \{1, \cdots, n\}$ such that $u_k \equiv 0$, $\forall k \in J_n$ and $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\} \setminus J_n$).

5. If $((u_1, \cdots, u_n), \mu(\beta, \lambda)) \in W(\Omega) \times \mathbb{R}$ (resp. $((u_1, \cdots, u_n), \lambda(V, \beta, m_1, \cdots, m_n)) \in W(\Omega) \times \mathbb{R}$) is a weak solution to problem (2.2) (resp. (2.1)), $(u_1, \cdots, u_n)$ shall be called an eigenfunction of the problem (2.2) (resp. (2.1)) associated to the eigenvalue $\mu(\beta, \lambda)$ (resp. $\lambda(V, \beta, m_1, \cdots, m_n)$). So $(u_1, \cdots, u_n)$ shall be called an eigenfunction of the problem (1.1) (resp. (1.2)) associated to the eigensurface $\lambda(V, ., m_1, \cdots, m_n)$ (resp. $\mu(., \lambda)$). We can say that an eigensurface of (1.1) or (1.2) is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction $(u_1, \cdots, u_n)$ such that $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\}$ (resp. there exist $\emptyset \neq J_n \subset \{1, \cdots, n\}$ such that $u_k \equiv 0$, $\forall k \in J_n$ and $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\} \setminus J_n$).

Let then introduce the following energy functional
$$J_{\lambda, \beta} : W(\Omega) \longrightarrow \mathbb{R}$$
$$(u_1, \cdots, u_n) \longmapsto J_{\lambda, \beta}(u_1, \cdots, u_n) = E_\beta(u_1, \cdots, u_n) - V_\beta(u_1, \cdots, u_n) - \lambda M_\beta(u_1, \cdots, u_n),$$
where
$$E_\beta(u_1, \cdots, u_n) = \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} \int_\Omega e^{\beta \cdot x} |\Delta u_i|^{p_i} dx$$
$$V_\beta(u_1, \cdots, u_n) = \int_\Omega V e^{\beta \cdot x} \prod_{i=1}^{n} |u_i|^{\alpha_i + 1} dx, \quad M_\beta(u_1, \cdots, u_n) = \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} M_{i, \beta}(u_i)$$

with

$$M_{i,\beta}(u_i) = \int_\Omega m_i e^{\beta\cdot x}|u_i|^{p_i}dx, \quad \forall(u_1,\cdots,u_n)\in W(\Omega).$$

Equation (2.3) is therefore equivalent to

$$\nabla J_{\lambda,\beta}(u_1,\cdots,u_n) = \mu\nabla I_\beta(u_1,\cdots,u_n)$$

where

$$I_\beta(u_1,\cdots,u_n) = \sum_{i=1}^n \frac{\alpha_i+1}{p_i}\int_\Omega e^{\beta\cdot x}|u_i|^{p_i}dx \quad \forall(u_1,\cdots,u_n)\in W(\Omega).$$

We state our main results in the following and the first reads as:

Theorem 2.1. (i) *For any fix $\lambda\in\mathbb{R}$, there is an eigensurface of* (1.2) *given by*

$$\mu_1(\beta,\lambda) := \inf\{J_{\lambda,\beta}(u) : u\in\mathcal{M}_\beta\} \tag{2.4}$$

for $\beta\in\mathbb{R}^N$ where

$$\mathcal{M}_\beta = \{u=(u_1,\cdots,u_n)\in W(\Omega) : I_\beta(u_1,\cdots,u_n)=1\}.$$

Furthermore, $\mu_1(\beta,\lambda)$ is the smallest eigensurface of (1.2).

(ii) *For $\beta\in\mathbb{R}^N$, the function $\lambda\in\mathbb{R}\mapsto\mu_1(\beta,\lambda)$ is concave and differentiable, strictly decreasing and goes to $-\infty$ as λ tends to ∞ with $\mu_1^{'}(\beta,\lambda) = -M_\beta(u_{1,0},\cdots,u_{n,0})$ where $(u_{1,0},\cdots,u_{n,0})$ is some eigenfunction of* (2.2) *associated to $\mu_1(\beta,\lambda)$.*

(iii) *If $((u_1,\cdots,u_n),\beta,\mu(\lambda))$ is a solution of* (1.2) *then $-\Delta u_i\in C(\bar\Omega)$ and $u_i\in C^{1,\nu}(\bar\Omega)$, for $1\le i\le n$ and for all $\nu\in(0,1)$.*

For $V=m_i\equiv 0, \forall i\in\{1,\cdots,n\}$, let us set

$$\mu_*(\beta) := \inf\left\{\sum_{i=1}^n\frac{\alpha_i+1}{p_i}\int_\Omega e^{\beta\cdot x}|\Delta u_i|^{p_i}dx : (u_1,\cdots,u_n)\in\mathcal{M}_\beta\right\}.$$

Note that $\mu_*(\cdot)>0$ and if $\mu_1(.,\lambda(.))\equiv 0$ then $\beta\longmapsto\lambda(\beta)$ is an eigensurface of problem (1.1). So, our second main result is based on the following assumption **(A)**: $\|V\|_\infty<\mu_*(\cdot)$ and gives the existence of an eigensurface for (1.1).

Theorem 2.2. *Assume that hypothesis* **(A)** *holds. The following conclusions hold.*

(i) *For all $\beta\in\mathbb{R}^N$, $\mu_1(\beta,0)>0$.*

(ii) *There exists an eigensurface of* (1.1).

(iii) *If $\mu_1(.,\lambda(.))\equiv 0$ then the eigensurface $\lambda(\cdot)$ is a semitrivial principal eigensurface or strictly principal eigensurface of problem* (1.1). *Moreover, the eigensurface $\lambda(\cdot)$ is simple.*

We complete this section with our last main result which establishes the lowest positive eigensurface for (1.1) and its simplicity.

Theorem 2.3. *Assume that hypothesis* **(A)** *holds. The lowest positive eigensurface of problem* (1.1) *is defined by*

$$\lambda_1(V,\beta,m_1,\cdots,m_n)=\min_{(u_1\cdots,u_n)\in\mathcal{S}_\beta}E_{\beta,V}(u_1,\cdots,u_n),$$

for all $\beta\in\mathbb{R}^N$ *and*

$$E_{\beta,V}(u_1,\cdots,u_n)=E_\beta(u_1,\cdots,u_n)-V_\beta(u_1,\cdots,u_n),$$
$$\mathcal{S}_\beta=\{(u_1,\cdots,u_n)\in W(\Omega):\quad M_\beta(u_1,\cdots,u_n)=1\}.$$

Furthermore,

(i) $\lambda_1(V,.,m_1,\cdots,m_n)$ *is a semitrivial principal eigensurface or strictly principal eigensurface.*

(ii) $\lambda_1(V,.,m_1,\cdots,m_n)$ *is simple.*

(iii) $\lambda_1(V,.,m_1,\cdots,m_n)\leq\min\{\Gamma_i^{p_i}(.,m_i),i\in\{1,\cdots,n\}\}$ *with*

$$\forall\beta\in\mathbb{R}^N,\quad\Gamma_i^{p_i}(\beta,m_i)=\inf_{K\subset\mathcal{B}_i}\sup_{v\in K}\int_\Omega e^{\beta.x}|\Delta v|^p dx \text{ where}$$
$$\mathcal{B}_i=\{K\subset\mathcal{N}_\beta\text{: } K \text{ is compact, symmetric and genus of } K\geq i\},$$
$$\mathcal{N}_\beta=\left\{v\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega):\ \int_\Omega ae^{\beta.x}|v|^p dx=1\right\}.$$

3. Proofs of the Main Results

3.1. Proof of Theorem 2.1

Lemma 3.1. *Let* $(\omega_1,\cdots,\omega_n)\in[L^\infty(\Omega)]^n$. *If* $\omega_1,\cdots,\omega_n>0$ *on* Ω *then there exist* $n+1$ *positive constants* $c_{1,\beta},\cdots,c_{n+1,\beta}$ *such that*

$$\sum_{i=1}^n\|\Delta u_i\|_{p_i}^{p_i}\leq c_{n+1,\beta}J_{\lambda,\beta}(u_1,\cdots,u_n)+\sum_{i=1}^n c_{i,\beta}\int_\Omega\omega_i e^{\beta\cdot x}|u_i|^{p_i}dx$$

for every $(u_1,\cdots,u_n)\in W(\Omega)$.

Proof. From [4, Lemma 2.3], we conclude the existence of $n+1$ positive constants $k_1,\cdots,k_n$ such that

$$\sum_{i=1}^n\left(\|\Delta u_i\|_{p_i}^{p_i}\right)\leq k_{n+1}J_{\lambda,0_{\mathbb{R}^N}}(u_1,\cdots,u_n)+\sum_{i=1}^n\left(k_i\int_\Omega\omega_i|u_i|^{p_i}dx\right).$$

Consider the function $f_\beta:x\in\overline{\Omega}\longmapsto e^{\beta\cdot x}$. Then, there exist two positive constants $k_{1,\beta}=\min\limits_{x\in\overline{\Omega}}f_\beta(x)$ and $k_{2,\beta}=\max\limits_{x\in\overline{\Omega}}f_\beta(x)$ satisfying

$$k_{1,\beta}\int_\Omega w_i|u_i|^{p_i}dx\leq\int_\Omega w_ie^{\beta\cdot x}|u_i|^{p_i}dx\leq k_{2,\beta}\int_\Omega w_i|u_i|^{p_i}dx,\ \text{for } 1\leq i\leq n$$

and

$$k_{1,\beta}J_{\lambda,0_{\mathbb{R}^N}}(u_i,\cdots,u_n)\leq J_{\lambda,\beta}(u_1,\cdots,u_n)\leq k_{2,\beta}J_{\lambda,0_{\mathbb{R}^N}}(u_1,\cdots,u_n).$$

As a consequence, we obtain

$$k_{1,\beta}\sum_{i=1}^n\|\Delta u_i\|_{p_i}^{p_i}\leq k_{n+1}J_{\lambda,\beta}(u_1,\cdots,u_n)+\sum_{i=1}^n k_i\int_\Omega\omega_ie^{\beta\cdot x}|u_i|^{p_i}dx.$$

One can therefore take $c_{i,\beta}=k_i\times k_{1,\beta}^{-1}$, $1\leq i\leq n+1$ to complete the proof. ∎

Proof of Theorem 2.1. (i) Let $\beta \in \mathbb{R}^N$ and apply Lemma 3.1 for $\omega_i \equiv 1$, $1 \leq i \leq n$. Then

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} \left(\|\Delta u_i\|_p^p\right) &\leq c_{n+1,\beta} J_{\lambda,\beta}(u_1, \cdots, u_n) + \sum_{i=1}^{n} \left(c_{i,\beta} \int_\Omega e^{\beta \cdot x} |u_i|^{p_i} dx \right) \\ &\leq c_{n+1,\beta} J_{\lambda,\beta}(u_1, \cdots, u_n) + c_{0,\beta} \sum_{i=1}^{n} \left(\frac{\alpha_i + 1}{p_i} \int_\Omega e^{\beta \cdot x} |u_i|^{p_i} dx \right) \\ &= c_{n+1,\beta} J_{\lambda,\beta}(u_1, \cdots, u_n) + c_{0,\beta}, \forall (u_1, \cdots, u_n) \in \mathcal{M}_\beta \end{aligned}$$

where $c_{0,\beta} = \max\{\frac{p_i c_{i,\beta}}{\alpha_i+1}, 1 \leq i \leq n\}$, so that $J_{\lambda,\beta}$ is bounded below on $\mathcal{M}_\beta$. Furthermore any sequence $(u_{1,k}, \cdots, u_{n,k})$ that minimizes $J_{\lambda,\beta}$ on $\mathcal{M}_\beta$ is bounded in $W(\Omega)$. Thus there exists $(u_{1,0}, \cdots, u_{n,0}) \in W(\Omega)$ such that, up to a subsequence, $(u_{1,k}, \cdots, u_{n,k})$ converges weakly to $(u_{1,0}, \cdots, u_{n,0})$ in $W(\Omega)$ and strongly in $\prod_{i=1}^{n} L^{p_i}(\Omega)$. As a result,

$$J_{\lambda,\beta}(u_{1,0}, \cdots, v_{n,0}) \leq \lim_{k \longrightarrow \infty} J_{\lambda,\beta}(u_{1,k}, \cdots, v_{n,k}) = \mu_1(\beta, \lambda), \quad (u_{1,0}, \cdots, u_{n,0}) \in \mathcal{M}_\beta$$

and consequently $J_{\lambda,\beta}(u_{1,0}, \cdots, u_{n,0}) = \mu_1(\beta, \lambda)$. By the Lagrange multipliers rule, $\mu_1(\beta, \lambda)$ is an eigenvalue for (2.2) and $(u_{1,0}, \cdots, u_{n,0})$ is an associated eigenfunction. Moreover for any eigenvalue $\mu(\beta, \lambda)$ for (2.2) associated to $(u_{\lambda,1}, \cdots, u_{\lambda,n}) \in W(\Omega) \setminus \{(0, \cdots, 0)\}$, one has $J_{\lambda,\beta}(u_{\lambda,1}, \cdots, u_{\lambda,n}) = \mu(\lambda) I_\beta(u_{\lambda,1}, \cdots, u_{\lambda,n})$ with $I_\beta(u_{\lambda,1}, \cdots, u_{\lambda,n}) > 0$. Consequently

$$\mu_1(\beta, \lambda) \leq J_{\lambda,\beta} \left(\frac{u_{\lambda,1}}{I_\beta(u_{\lambda,1}, \cdots, u_{\lambda,n})^{\frac{1}{p_1}}}, \cdots, \frac{u_{\lambda,n}}{I_\beta(u_{\lambda,1}, \cdots, u_{\lambda,n})^{\frac{1}{p_n}}} \right) = \frac{J_{\lambda,\beta}(u_{\lambda,1}, \cdots, u_{\lambda,n})}{I_\beta(u_{\lambda,1}, \cdots, u_{\lambda,n})} = \mu(\beta, \lambda).$$

We then conclude that $\mu_1(\beta, \lambda)$ is the smallest eigenvalue of (S_λ). Consequently $\mu_1(., \lambda)$ the smallest eigensurface for (1.2).

(ii) This follows a proper modification of the proof of Proposition 2.5 in [4], so we omit it.

(iii) An easy adaptation of Lemma 3.2 in [13]. ■

3.2. Proof of Theorem 2.2

We collect in the following some essential results adapted from [4, 7, 15] which are of need to handle the rest of our statements.

1. For all $p > 1$, $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $v \in L^p(\Omega)$, one has $v = -\Delta u \Longleftrightarrow u = \Lambda v$.
2. If $p > 1$, and N_p denote the Nemytskii operator defined by

$$N_p(u)(x) = \begin{cases} |u(x)|^{p-2} u(x) & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0, \end{cases}$$

we have the following

$$\forall v \in L^p(\Omega), \quad \forall w \in L^{p'}(\Omega) \quad N_p(v) = w \Longleftrightarrow v = N_{p'}(w)$$

where $p' = \frac{p}{p-1}$.

3. If $(u_1, \cdots, u_n)$ is an eigenfunction of (1.2) associated with $\mu(.)$ then for $\beta \in \mathbb{R}^N$, $(u_1, \cdots, u_n)$ is an eigenfunction of (2.2) associated with $\mu(\beta)$ and the functions $\varphi_i = -\Delta u_i$, $1 \leq i \leq n$ check

$$e^{\beta \cdot x} N_{p_j}(\varphi_j) = \Lambda \left([\mu(\lambda) + \lambda m_j] e^{\beta \cdot x} N_{p_j}(\Lambda \varphi_j) + V e^{\beta \cdot x} \prod_{i=1, i \neq j}^{n} |\Lambda \varphi_i|^{\alpha_i + 1} |\Lambda \varphi_j|^{\alpha_j - 1} \Lambda \varphi_j \right), \text{ for } 1 \leq j \leq n.$$

Furthermore,

(a) $((u_{1,0}, \cdots, u_{n,0}), \beta, \mu(\lambda))$ is a solution of (1.2) if and only if $((\varphi_{1,0}, ..., \varphi_{n,0}), \beta, \mu(\lambda))$ verifies

$$\begin{cases} \text{Find } ((\varphi_1, \cdots, \varphi_n), \mu(\lambda)) \in L(\Omega) \text{ such that} \\ e^{\beta\cdot x} N_{p_j}(\varphi_j) = \Lambda([\mu(\lambda) + \lambda m_j] e^{\beta\cdot x} N_{p_j}(\Lambda\varphi_j) + V e^{\beta\cdot x} \prod_{i=1, i\neq j}^{n} |\Lambda(\varphi_i)|^{\alpha_i+1} |\Lambda(\varphi_j)|^{\alpha_j-1} \Lambda(\varphi_j)), \\ 1 \leq j \leq n, \end{cases} \tag{3.1}$$

where $L(\Omega) = \left(\left[\prod_{p_i}^{n} L^{p_i}(\Omega)\right] \setminus \{(0, \cdots, 0)\}\right) \times \mathbb{R}$ and $\varphi_{j,0} = -\Delta(u_{j,0})$.

(b) $((\varphi_{1,0}, \cdots, u_{n,0}), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of (3.1) if and only if $((\varphi_{1,0}, \cdots, \varphi_{n,0}), \lambda) \in L(\Omega)$ satisfies

$$\begin{cases} \text{Find } ((\varphi_1, \cdots, \varphi_n), \lambda) \in L(\Omega) \text{ such that} \\ e^{\beta\cdot x} N_{p_j}(\varphi_j) = \Lambda\left(\lambda m_j e^{\beta\cdot x} N_{p_j}(\Lambda\varphi_j) + V e^{\beta\cdot x} \prod_{i=1, i\neq j}^{n} |\Lambda(\varphi_i)|^{\alpha_i+1} |\Lambda(\varphi_j)|^{\alpha_j-1} \Lambda(\varphi_j)\right), \\ 1 \leq j \leq n. \end{cases}$$

Based on all stated above, we can give a new characterization of (2.4) as follows.

$$\mu_1(\beta, \lambda(\beta)) := \inf\left\{F_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) : (\varphi_1, \cdots, \varphi_n) \in \prod_{i=1, i\neq j}^{n} L^{p_i}(\Omega),\ R_\beta(\varphi_1, \cdots, \varphi_n) = 1\right\}$$

where

$$F_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) = \sum_{i=1}^{n}\left(\frac{\alpha_i+1}{p_i}\left[\int_\Omega e^{\beta\cdot x}|\varphi_i|^{p_i}dx - \lambda\int_\Omega m_i e^{\beta\cdot x}|\Lambda\varphi_i|^{p_i}dx\right]\right) - \int_\Omega V e^{\beta\cdot x}\prod_{i=1}^{n}|\Lambda\varphi_i|^{\alpha_i+1}dx$$

and

$$R_\beta(\varphi_1, \cdots, \varphi_n) = \sum_{i=1}^{n}\frac{\alpha_i+1}{p_i}\int_\Omega e^{\beta\cdot x}|\Lambda\varphi_i|^{p_i}dx.$$

Lemma 3.2. $((\varphi_{1,1}, \cdots, \varphi_{1,n}), \mu_1(\beta, \lambda)) \in L(\Omega)$ *is a solution of problem* (3.1) *if and only if*

$$G_{\beta,\lambda}(\varphi_{1,1}, ..., \varphi_{1,n}) = 0 = \min_{(\varphi_1,...,\varphi_n)\in L^*(\Omega)} G_{\beta,\lambda}(\varphi_1, ..., \varphi_n) \tag{3.2}$$

where

$$G_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) = F_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) - \mu_1(\beta, \lambda) R_\beta(\varphi_1, \cdots, \varphi_n) \text{ and } L^*(\Omega) = \left[\prod_{i=1}^{n} L^{p_i}(\Omega)\right] \setminus \{(0, \cdots, 0)\}.$$

Proof. Assume that $((\varphi_{1,1}, \cdots, \varphi_{1,n}), \mu_1(\beta, \lambda)) \in L(\Omega)$ is a solution of (3.1). Then $F_{\beta,\lambda}(\varphi_{1,1}, \cdots, \varphi_{1,n}) = \mu_1(\beta, \lambda) R_\beta(\varphi_{1,1}, \cdots, \varphi_{1,n})$ and $G_{\beta,\lambda}(\varphi_{1,1}, \cdots, \varphi_{1,n}) = F_{\beta,\lambda}(\varphi_{1,1}, \cdots, \varphi_{1,n}) - \mu_1(\beta, \lambda) R_\beta(\varphi_{1,1}, \cdots, \varphi_{1,n}) = 0$. Let us set $\bar{\varphi}_i = \dfrac{\varphi_i}{[R_\beta(\varphi_1, \cdots, \varphi_n)]^{\frac{1}{p_i}}}$ for every $(\varphi_1, \cdots, \varphi_n) \in L^*(\Omega)$ and $1 \leq i \leq n$. Thus $R_\beta(\bar{\varphi}_1, \cdots, \bar{\varphi}_n) = 1$ and we first infer that $\mu_1(\beta, \lambda) \leq F_{\beta,\lambda}(\bar{\varphi}_1, \cdots, \bar{\varphi}_n) = \dfrac{F_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n)}{R_\beta(\varphi_1, \cdots, \varphi_n)}$ and secondly, $G_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) = F_{\beta,\lambda}(\varphi_1, \cdots, \varphi_n) - \mu_1(\lambda) R_\beta(\varphi_1, \cdots, \varphi_n) \geq 0$ for all $(\varphi_1, \cdots, \varphi_n) \in L^*(\Omega)$ so that (3.2) is satisfied. Conversely, assume that (3.2) holds. We then have $\nabla G_{\beta,\lambda}(\varphi_{1,1}, \cdots, \varphi_{1,n}) = (0, \cdots, 0)$ that is $\langle\dfrac{\partial G_{\beta,\lambda}}{\partial \varphi_i}(\varphi_{1,1}, \cdots, \varphi_{1,n}), \Psi_i\rangle = 0$, for $1 \leq i \leq n$, for all $(\Psi_1, \cdots, \Psi_n) \in \prod_{i=1}^{n} L^{p_i}(\Omega)$. This implies that $((\varphi_{1,1}, \cdots, \varphi_{1,n}), \mu_1(\lambda)) \in L(\Omega)$ is a solution of (3.1). ■

Lemma 3.3. *If* **(A)** *holds and* $((\varphi_{1,1},\cdots,\varphi_{1,n}),\mu_1(\beta,\lambda))\in L_0(\Omega)=\left(\left[\prod_{p_i}^{n}L^{p_i}(\Omega)\right]\setminus\{(0,\cdots,0)\}\right)\times\{0\}$ *is a solution of problem* (3.1) *then* $((|\varphi_{1,1}|,\cdots,|\varphi_{1,n}|),\mu_1(\beta,\lambda))\in L_0(\Omega)$ *is also a solution of problem* (3.1).

Proof. The ideas of the proof are similar to those of [4, Lemma 3.5]. ∎

Proof of Theorem 2.2. Assume that **(A)** holds.

(i) Let $\beta\in\mathbb{R}^N$. We have $V_\beta(u_1,\cdots,u_n)\le\|V\|_\infty I_\beta(u_1,\cdots,u_n)$, $\forall(u_1,\cdots,u_n)\in W(\Omega)$ and also $E_\beta(u_1,\cdots,u_n)-\|V\|_\infty I_\beta(u_1,\cdots,u_n)\le E_\beta(u_1,\cdots,u_n)-V_\beta(u_1,\cdots,u_n),\forall(u_1,\cdots,u_n)\in W(\Omega)$. Then

$$\mu_*(\beta)\le E_\beta(u_1,\cdots,u_n)-V_\beta(u_1,\cdots,u_n)+\|V\|_\infty,\quad\forall(u_1,\cdots,u_n)\in\mathcal{M}_\beta$$

and $\mu_*(\beta)-\|V\|_\infty\le\inf\{E_\beta(u_1,\cdots,u_n)-V_\beta(u_1,\cdots,u_n),(u_1,\cdots,u_n)\in\mathcal{M}_\beta\}\le\mu_1(\beta,0)$. This ensures that $\mu_1(\beta,0)>0$ for all $\beta\in\mathbb{R}^N$.

(ii) It follows from Theorem 2.1-(ii) that there is a unique $\lambda(\cdot)>0$ such that $\mu_1(.,\lambda(.))\equiv0$ that is (1.1) admits an eigensurface.

(iii) Suppose that $\lambda(\cdot)$ is an eigensurface of problem (1.1) associated to $(u_1,\cdots,u_n)\in W(\Omega)\setminus\{(0,\cdots,0)\}$. To prove that either $\lambda(\cdot)$ is a semitrivial principal eigensurface or strictly principal eigensurface of (1.1), we proceed in two steps.

Step 1. Suppose that for all $i\in\{1,\cdots,n\}$, $u_i\not\equiv0$ and let $\beta\in\mathbb{R}^N$. It follows from Lemma 3.3 that both elements $((\varphi_1,\cdots,\varphi_n),\mu_1(\beta,\lambda(\beta)))\in L_0(\Omega)$ and $((|\varphi_1|,\cdots,|\varphi_n|),\mu_1(\beta,\lambda(\beta)))\in L_0(\Omega)$ are solutions of (3.1) with $\varphi_i=-\Delta u_i\not\equiv0$, for $1\le i\le n$. Since $|\varphi_i|\ge0$, then $\Lambda(|\varphi_i|)>0$, for $1\le i\le n$. Therefore for $1\le i\le n$, $N_{p_i}(\Lambda|\varphi_i|)>0$; $\prod_{j=1,i\neq j}^{n}\left(\Lambda(|\varphi_j|)|^{\alpha_i+1}|\Lambda(|\varphi_i|)|^{\alpha_i}\right)>0$ and

$$\begin{cases}|\varphi_i|=N_{p_i'}\left(e^{-\beta\cdot x}\Lambda\left[\lambda(\beta)m_ie^{\beta\cdot x}N_{p_i}(\Lambda|\varphi_i|)+Ve^{\beta\cdot x}\prod_{j=1,i\neq j}^{n}\left(\Lambda(|\varphi_j|)|^{\alpha_i+1}|\Lambda(|\varphi_i|)|^{\alpha_i}\right)\right]\right)>0\\ \text{for } 1\le i\le n.\end{cases}$$

Hence $((\varphi_1,\cdots,\varphi_n),\mu_1(\beta,\lambda(\beta)))$ satisfies (3.1) with $\varphi_i>0$ or $\varphi_i<0$. Since by regularity result Theorem 2.1-(iii), $\varphi_i\in C(\bar\Omega)$, we conclude from Proposition 1.1 that $u_i=\Lambda\varphi_i>0$ or $u_i=\Lambda\varphi_i<0$. This expresses that $\lambda(.)$ is a strictly principal eigensurface of (1.1).

Step 2. Suppose there are $i,j\in\{1,\cdots,n\}$ such that $[u_i\equiv0$ and $u_j\not\equiv0]$, then one proves similarly that $[u_i\equiv0$ and $u_j>0$ in Ω or $u_j<0$ in $\Omega]$ so that $\lambda(.)$ is a semitrivial principal eigensurface of (1.1) by Section 2. The rest of the proof is devoted for the simplicity of $\lambda(\cdot)$ and we just consider the case $\lambda(\cdot)$ is a strictly principal eigensurface as the second one follows readily. Indeed, let $(u_{1,1},\cdots,u_{1,n})$ and $(u_{2,1},\cdots,u_{2,n})$ be two eigenfunctions of ((1.1)) associated with $\lambda(.)$. Then, $((\varphi_{1,1},\cdots,\varphi_{1,n}),0)$, $((\varphi_{2,1},\cdots,\varphi_{2,n}),0)\in L_0(\Omega)$ as well as $((|\varphi_{1,1}|,\cdots,|\varphi_{1,n}|),0)$, $((|\varphi_{2,1}|,\cdots,|\varphi_{2,n}|),0)\in L_0(\Omega)$, are solutions of (3.1) where $\varphi_{j,i}=-\Delta u_{j,i}$ with $\varphi_{j,i}>0$ or $\varphi_{j,i}<0$, for $j\in\{1,2\}$ and $i\in\{1,\cdots,n\}$. For $x_0\in\Omega$, let set $k_i=\dfrac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}$, $w_{1,i}(x)=\max\{\varphi_{2,i}(x),k_i\varphi_{1,i}(x)\}$ for all $x\in\Omega$. From an extended version of [6, Lemma 9], we derive that $((w_{1,1},...,w_{1,n}),0)$ is a solution of (3.1). Hence, $N_{p_i}(\varphi_{1,i})$, $N_{p_i}(\varphi_{2,i})$, $N_{p_i}(w_{1,i})\in C^{1,\nu}(\bar\Omega)$ and $\dfrac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}\in C^1(\Omega)$. For any unit vector $e=(0,\cdots,e_j,\cdots,0)$ with $j\in\{1,\cdots,N\}$ and $t\in\mathbb{R}$, we have

$$\begin{cases}N_{p_i}(\varphi_{2,i})(x_0+te)-N_{p_i}(\varphi_{2,i})(x_0)\le N_{p_i}(w_{1,i})(x_0+te)-N_{p_i}(w_{1,i})(x_0)\\ N_{p_i}(k\varphi_{1,i})(x_0+te)-N_{p_i}(k\varphi_{1,i})(x_0)\le N_{p_i}(w_{1,i})(x_0+te)-N_{p_i}(w_{1,i})(x_0)\end{cases}$$

Using standard argument as in [4], we divide these inequalities by $t>0$ and $t<0$ and let t tend to $0^{\pm}$ to get

$$\begin{cases}\dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0)\leq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0)\leq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\end{cases}\quad\text{and}\quad\begin{cases}\dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0)\geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0)\geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\end{cases}$$

for all $j\in\{1,\cdots,N\}$. Thus, $\begin{cases}\dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0)= \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0)= \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0)\end{cases}$ for all $j\in\{1,\cdots,N\}$. Hence,

$$\nabla N_{p_i}(\varphi_{2,i})(x_0)=\nabla N_{p_i}(w_{1,i})(x_0)=\nabla N_{p_i}(k\varphi_{1,i})(x_0)=k^{p_i-1}\nabla N_{p_i}(\varphi_{1,i})(x_0)$$

and one easily checks that $\nabla\left(\dfrac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}\right)(x_0)=0$ which yields

$$\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x)=\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x_0)=\left(\frac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}\right)^{p_i-1}=k_i^{p_i-1},$$

for all $x\in\Omega$. Consequently $\varphi_{2,i}=k_i\varphi_{1,i}$ and one can write $u_{2,i}=k_iu_{1,i}$ for all $i\in\{1,\cdots,n\}$. ■

3.3. Proof of Theorem 2.3

Proof of Theorem 2.3. Assume that **(A)** holds. From Theorem 2.2, there is a unique $\lambda_1(V,.,m_1,\cdots,m_n)$ solution of $\mu_1(.,\lambda)\equiv 0$ so that $\lambda_1(V,.,m_1,\cdots,m_n)$ is an eigensurface of (1.1) and

$$\begin{aligned}\mu_1'(\beta,\lambda_1(V,\beta,m_1,\cdots,m_n))&=-M_\beta(u_{1,0},\cdots,u_{n,0})<0=\mu_1(\beta,\lambda_1(V,\beta,m_1,\cdots,m_n))\\&=E_{\beta,V}(u_{1,0},\cdots,u_{n,0})-\lambda_1(V,\beta,m_1,\cdots,m_n)M_\beta(u_{1,0},\cdots,u_{n,0})\end{aligned}$$

for all $\beta\in\mathbb{R}^N$, with $(u_{1,0},\cdots,u_{n,0})\in\mathcal{M}_\beta$. Then, the positivity can be obtained $E_{\beta,V}(u_{1,0},\cdots,u_{n,0})=\lambda_1(V,\beta,m_1,\cdots,m_n)M_\beta(u_{1,0},\cdots,u_{n,0})>0$ and using $(\overline{u}_{1,0},\cdots,\overline{u}_{n,0})$ with $\overline{u}_{i,0}=\dfrac{u_{i,0}}{[M_\beta(u_{i,0},\cdots,u_{i,0})]^{\frac{1}{p_i}}}$ as an admissible function for $\lambda_1(V,\beta,m_1,\cdots,m_n)$, we reach $E_{\beta,V}(\overline{u}_{1,0},\cdots,\overline{u}_{n,0})=\lambda_1(V,\beta,m_1,\cdots,m_n)$. Next, we can normalize any $(u_1,\cdots,u_n)\in\mathcal{S}_\beta$ and get $E_{\beta,V}(u_1,\cdots,u_n)\geq\lambda_1(V,\beta,m_1,\cdots,m_n)$ which shows that $\lambda_1(V,\beta,m_1,\cdots,m_n)$ is the lowest positive eigensurface of (1.1).

(i)-(ii) follow from Theorem 2.2-(iii).

(iii) Let $i\in\{1,\cdots,n\}$ and pose $\varphi_{p_i}=(\frac{p_i}{\alpha_i+1})^{\frac{1}{p_i}}\varphi_{p_i,m_i}$. Then, a straightforward computation yields $\dfrac{\alpha_i+1}{p_i}M_{i,\beta}(\varphi_{p_i})+\displaystyle\sum_{j=1,j\neq i}^{n}\dfrac{\alpha_j+1}{p_j}M_{j,\beta}(0)=1$ for all $\beta\in\mathbb{R}^N$ and it follows that $\lambda_1(V,\beta,m_1,\cdots,m_n)\leq E_{\beta,V}(0,\cdots,0,\varphi_{p_i},0,\cdots,0)=\Gamma_i^{p_i}(\beta,m_i)$ i.e. $\lambda_1(V,.,m_1,\cdots,m_n)\leq\min\{\Gamma_i^{p_i}(.,m_i),i\in\{1,\cdots,n\}\}$. ■

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