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Polynomial stability of a Rayleigh system with distributed delay

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Abstract. In this paper we study the polynomial stability of a Rayleigh system with distributed delay in dynamic control. After studying the existence and uniqueness of the solution, we showed polynomial stability and finally proved that this polynomial stability is the best that can be had by establishing that there is no exponential stability. Our contribution is the introduction of the distributed delay term in the control.

AMS Subject Classifications: 34D20, 35B40, 35L70.

Keywords: Rayleigh beam equation, dynamic boundary control, distributed delay, spectral analysis, Rational stabilization.

Contents

1 Introduction **388**

2 Existence and uniqueness of solution **390**

3 Strong stability **396**

4 Polynomial stability **398**

5 Exponential unstability **406**

6 Acknowledgement **410**

1. Introduction

In this paper we focus on the Rayleigh problem subject to a single dynamic control with a distributed delay as follows

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$$\begin{cases} u_{tt}(x,t) - \gamma u_{xxtt}(x,t) + u_{xxxx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) + \eta(t) = 0, \\ u_{xxx}(1,t) - \gamma u_{xtt}(1,t) = 0, \ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_{xt}(1,t) + \beta_1\eta(t) + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)\eta(t-s)ds = 0, \ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \ \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{C} \\ \eta(-t) = f_0(.,-t), \ \forall\, t \in (0,\tau_2), \end{cases} \tag{1.1}$$

where η denotes the dynamical control, $\int_{\tau_1}^{\tau_2} \beta_2(s)\eta(t-s)ds$ is the time delay, β_1 is a positive constants and the initial data (u_0, u_1, f_0) belong to a suitable space. The damping of the system is made via the indirect damping mechanism.
Throughout this paper, we assume that $\beta_2 : [\tau_1;\tau_2] \to \mathbb{R}$, β_2 is in $L^{+\infty}$ and is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} \beta_2(s)ds < \beta_1. \tag{1.2}$$

It should be that D. Mercier and al. studied in [9] the problem

$$\begin{cases} u_{tt}(x,t) - \gamma u_{xxtt}(x,t) + u_{xxxx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) + \eta(t) = 0, \\ u_{xxx}(1,t) - \gamma u_{xtt}(1,t) = 0 \ \ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_{xt}(1,t) + \beta\eta(t) = 0 \ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \ \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{C} \end{cases} \tag{1.3}$$

where β is a positive constant and η the dynamical control.
A study in which they showed the polynomial decay of the solution of the system (1.3).

Then, the important and interesting case when the Rayleigh beam equation is damped by only one dynamical boundary with distributed delay remaine open. The aim of this paper is to fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary feedback whith distributed delay (1.1).

The paper is organized as follows: In the second part we will establish the well posedness of problems (1.1) using semi-group theory. In the sections 3 and 4 respectively we will establish the strong and polynomial stability and finally in section 5 the absence of an exponential decay.

2. Existence and uniqueness of solution

Here we study the well posedness for the problem (1.1) using the semigroup theory.
As we did in [11, 12] and [13] let's

$$z(\rho,t,s)=\eta(t-s\rho),\quad \rho\in(0,1), s\in(\tau_1,\tau_2),\ t>0. \tag{2.1}$$

Now the problem (1.1) is equivalent to

$$\begin{cases} u_{tt}(x,t)-\gamma u_{xxtt}(x,t)+u_{xxxx}(x,t)=0 \text{ in }]0,1[\times(0,+\infty)\\ sz_t(\rho,t)+z_\rho(\rho,t)=0 \text{ in } (0,1)\times(0,+\infty)\\ u(0,t)=u_x(0,t)=0\\ u_{xx}(1,t)+\eta(t)=0,\\ u_{xxx}(1,t)-\gamma u_{xtt}(1,t)=0\ \forall\, t\in(0,+\infty)\\ \eta_t(t)-u_{xt}(1,t)+\beta_1\eta(t)+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds=0\ \ \forall\, t\in(0,+\infty)\\ u(\cdot,0)=u_0,\quad u_t(\cdot,0)=u_1 \text{ in }]0,1[,\quad \eta(0)=\eta_0\in\mathbb{C}\\ z(\rho,0,s)=f_0(.,-\rho\tau)\ \ \forall\,\rho\in(0,1), s\in(\tau_1,\tau_2),\\ z(0,t,s)=\eta(t)\ \ \forall\, t\in(0,+\infty) \end{cases} \tag{2.2}$$

The well posedness of problem (1.1) follows from standard semigroup theory.

Now let

$$V=\left\{u\in H^1(0,1), u(0)=0\right\},\ \ \|u\|_V^2=\int_0^1(|u|^2+\gamma\,|u_x|^2)dx$$

$$W=\left\{u\in H^2(0,1), u(0)=0, u_x(0)=0\right\},\ \ \|u\|_W^2=\int_0^1|u_{xx}|^2\,dx$$

and the energy space

$$\mathcal{H}=W\times V\times\mathbb{C}\times L^2\Big((0,1)\times(\tau_1,\tau_2)\Big)$$

with the inner product

$$\left\langle\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\z^*\end{pmatrix}\right\rangle_{\mathcal{H}}=\int_0^1 u_{xx}\overline{u^*_{xx}}\,dx+\int_0^1(v\overline{v^*}+\gamma v_x\overline{v^*_x})\,dx+\eta\overline{\eta^*}+\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)|z|^2dsd\rho.$$

Let u, η and z be smooth solutions of the system. Then multiplying the first equation of the system by $\overline{\Phi}\in W$ and integrating by part on $(0,1)$, we get

$$\int_0^1 u_{tt}\overline{\Phi}-\gamma u_{xxtt}\overline{\Phi}\,dx+\int_0^1 u_{xxxx}\overline{\Phi}\,dx=0 \tag{2.3}$$

Setting

$$I=-\int_0^1 \gamma u_{xxtt}\overline{\Phi}\,dx+\int_0^1 u_{xxxx}\overline{\Phi}\,dx$$

We obtain that

$$\begin{aligned}I&=\gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx-\gamma u_{ttx}(1)\overline{\Phi(1)}+\gamma u_{ttx}(0)\overline{\Phi(0)}+\int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx\\&+u_{xxx}(1)\overline{\Phi(1)}-u_{xxx}(0)\overline{\Phi(0)}-u_{xx}(1)\overline{\Phi_x(1)}+u_{xx}(0)\overline{\Phi_x(0)}\\&=\gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx+\int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx+\eta\overline{\Phi_x(1)}\end{aligned}$$

Now the relation 2.3 becomes

$$\int_0^1 u_{tt}\overline{\Phi}dx+\gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx+\int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx+\eta\overline{\Phi_x(1)}=0 \tag{2.4}$$

Now we define the linear operators $A\in\mathcal{L}(W,W')$, $B\in\mathcal{L}(\mathbb{R},V')$, $C\in\mathcal{L}(V,V')$, by the following way

$$< Au,\Phi >_{W'\times W}=\int_0^1 u_{xx}\overline{\Phi_{xx}}dx,\ \ \forall\, u,\Phi\in W$$

$$< B\eta,\Phi >_{W'\times W}=\eta\overline{\Phi_x(1)},\ \ \forall\,\eta\in\mathbb{R},\forall\Phi\in W$$

$$< Cu,\Phi >_{V'\times V}=\int_0^1 (u\overline{\Phi}+\gamma u_x\overline{\Phi_x})dx,\ \ \forall\, u,\Phi\in W$$

Then by means of the Lax-Milgram theorem, the operator A (resp. C) is the canonical isomorphism of W (resp. V) onto W' (resp. V'). Then we can formulate the variational equation 2.4 as :

$$Cu_{tt}+Au+B\eta=0,\ \ in\ W'.$$

If we assume that $Ay+B\eta\in\ V'$, then we obtain that :

$$u_{tt}+C^{-1}(Au+B\eta)=0,\ \ in\ V$$

If we denote by

$$\mathcal{U}=\Big(u,u_t,\eta,z\Big)^{\mathsf{T}},$$

one has

$$\mathcal{U}_t=(u_t,u_{tt},\eta_t,z_t)^{\mathsf{T}}=\Big(u_t,-C^{-1}(Au+B\eta),u_{xt}(1)-\beta_1\eta-\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds,-s^{-1}z_\rho\Big)^{\mathsf{T}}.$$

Therefore problem (2.2) can be rewritten as:

$$\begin{cases}\mathcal{U}_t=\mathcal{A}\mathcal{U}\\ \mathcal{U}(0)=(u_0,u_1,\eta_0,f_0(.,-\rho s))^{\mathsf{T}},\end{cases} \tag{2.5}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\,(u,v,\eta,z)^{\mathsf{T}}=\Big(u_t,-C^{-1}(Au+B\eta),u_{xt}(1)-\beta_1\eta-\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds,-s^{-1}z_\rho\Big)^{\mathsf{T}},$$

with domain

$$\mathcal{D}(\mathcal{A})=\Big\{\,(u,v,\eta,z)^{\mathsf{T}}\in\mathcal{H},v\in W,Au+B\eta\in V'\text{ and }z\in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)\,\big|\,z(0)=\eta\ \Big\},$$

As in [19] let's prove the following lemma.

Lemma 2.1. *Let* $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{H}$. *Then* $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ *if and only if* $u \in W \cap H^3(0,1)$, $v \in W$, $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ *and* $z(0)=\eta$ *such as*
$u_{xxx}(1)+\gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1)=0$;
$u_{xx}(1)+\eta=0$.

Proof. The sufficiency is obvious. Indeed let $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{H}$.
Assume $u \in W \cap H^3(0,1)$, $v \in W$, $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ and $z(0)=\eta$ such as
$u_{xxx}(1)+\gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1)=0$ and $u_{xx}(1)+\eta=0$.

We know

$z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ and $z(0)=\eta$;
$u \in W \cap H^3(0,1) \Rightarrow u \in W$;
As $W \subset V$, $v \in W \Rightarrow v \in V$.
Moreover, if $u_{xxx}(1)+\gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1)=0$, this implies that the equation is well posed and this necessarily leads to
$Au+B\eta \in V'$.
So $(u,v,\eta,z)^{\mathsf{T}} \in D(\mathcal{A})$

To prove the necessity, let $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ and
$\mathcal{A}(u,v,\eta,z)^{\mathsf{T}}=(g,k,h,q)^{\mathsf{T}}$. Then we obtain

$$\begin{cases} v=g\in W \\ -C^{-1}(Au+B\eta)=k \\ v_x(1)-\beta_1\eta-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds=h \\ -s^{-1}z_\rho=q\in L^2\Big((0,1)\times(\tau_1,\tau_2)\Big). \end{cases} \tag{2.6}$$

If the relation $z(0)=\eta$ is obvious, we obtain from the first and last equations of the system (2.6) that $v \in W$, and then $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$.
Then since $k \in V$ and $C: V \longrightarrow V'$ is an isomorphism, so the equation $(2.6)_2$ can be rewritten as

$Au+B\eta=-Ck$ in $V' \subset W'$

So for all $\psi \in W$ we have

$$\int_0^1 u_{xx}\overline{\psi_{xx}}\,dx+\eta\overline{\psi_x}(1)=-\int_0^1(k\overline{\psi}+\gamma k_x\overline{\psi_x})dx \tag{2.7}$$

This means

$$\int_0^1 u_{xx}\overline{\psi_{xx}}\,dx+\eta\overline{\psi_x}(1)+\int_0^1(k\overline{\psi}+\gamma k_x\overline{\psi_x})dx=0 \tag{2.8}$$

On the one hand, let's take $\phi \in C_0^\infty(0,1)$ and take $\psi=\displaystyle\int_0^x\phi(s)ds$.
We know $\psi_x=\phi$ and $\psi_{xx}=\phi_x$
By replacing in (2.8) we obtain

$$\int_0^1 u_{xx}\overline{\phi_x}\,dx+\eta\overline{\phi}(1)+\int_0^1\Big[k\Big(\int_0^x\overline{\phi(s)}ds\Big)\Big]dx+\int_0^1\gamma k_x\overline{\phi}dx=0 \tag{2.9}$$

Since $\phi \in C_0^\infty(0,1)$ then $\overline{\phi}(1)=0$, so we get

$$\int_0^1 u_{xx}\overline{\phi_x}\,dx + \int_0^1 \Big[k\Big(\int_0^x \overline{\phi(s)}ds\Big)\Big]dx + \int_0^1 \gamma k_x\overline{\phi}dx = 0 \tag{2.10}$$

In integration by parts we have

$$\Big[u_{xx}\overline{\phi}\Big]_0^1 - \int_0^1 u_{xxx}\overline{\phi}\,dx + \Big[\Big(\int_1^x k(s)ds\Big).\Big(\int_0^x \overline{\phi(s)}ds\Big)\Big]_0^1 - \int_0^1 \Big(\int_1^x k(s)ds\Big)\overline{\phi(x)}dx + \int_0^1 \gamma k_x\overline{\phi}dx = 0 \tag{2.11}$$

But $\Big[\Big(\int_1^x k(s)ds\Big).\Big(\int_0^x \overline{\phi(s)}ds\Big)\Big]_0^1 = \Big[u_{xx}\overline{\phi}\Big]_0^1 = 0$

Consequently, the (2.11) equation can be rewritten

$$\int_0^1 u_{xxx}\overline{\phi(x)}\,dx - \int_0^1 \Big(\int_1^x k(s)ds\Big)\overline{\phi(x)}dx + \int_0^1 \gamma k_x\overline{\phi(x)}dx = 0$$

By inverting the 1 and x terminals in $\displaystyle\int_1^x k(s)ds$ we have

$$\int_0^1 u_{xxx}\overline{\phi(x)}\,dx = -\int_0^1 \Big[\Big(\int_x^1 k(s)ds\Big)dx + \gamma k_x\Big]\overline{\phi(x)}dx, \forall \phi \in W$$

However

$$u_{xxx} = \int_x^1 k(s)ds + \gamma k_x \ pp\ in\ L^2(0,1) \tag{2.12}$$

This leads to $u \in H^3(0,1)\cap W$.
In particular, (2.12) allows us to write

$$u_{xxx}(1) - \gamma k_x(1) = 0 \tag{2.13}$$

while $k_x(1) = -\Big[C^{-1}(Au+B\eta)\Big]_x(1)$
From which we finally obtain

$$u_{xxx}(1) + \gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1) = 0 \tag{2.14}$$

On the other hand, for any $\phi \in V$ such that $\phi(1)=1$, let's pose $\psi = \displaystyle\int_0^x \phi(s)ds$.
Based on the previous calculations, we have

$$\int_0^1 u_{xx}\overline{\phi_x(x)}\,dx + \eta + \int_0^1 \Big[\int_x^1 k(s)ds + \gamma k_x\Big]\overline{\phi(x)}dx = 0 \tag{2.15}$$

From (2.12) we have $\displaystyle\int_x^1 k(s)ds + \gamma k_x = u_{xxx}$
By replacing in (2.15) we obtain

$$\int_0^1 u_{xx}\overline{\phi_x(x)}\,dx + \eta + \int_0^1 u_{xxx}\overline{\phi(x)}dx = 0$$

By integration by parts we have

$$u_{xx}(1)\overline{\phi(1)} - u_{xx}(0)\overline{\phi(0)} - \int_0^1 u_{xxx}\overline{\phi}\,dx + \eta + \int_0^1 u_{xxx}\overline{\phi(x)}dx = 0$$

This implies that

$$u_{xx}(1)\overline{\phi(1)} - u_{xx}(0)\overline{\phi(0)} + \eta = 0$$

Since $\overline{\phi(1)} = 1$ and $\overline{\phi(0)} = 0$, we finally obtain

$$u_{xx}(1) + \eta = 0 \tag{2.16}$$

The neccessity is also proved.

■

We can now state the following existence results.

Theorem 2.2.
Assume that (1.2) *holds. Then for any datum* $U_0 = (u_0, u_1, \eta_0, f_0)$ *belongs to* $\mathcal{H}$*, the problem (1.1) has one and only one weak solution* $U = (u, u_t, \eta, z)$ *verifying:*

$$\begin{cases} u \in C\left([0,\infty), V\right) \cap C^1\left([0,\infty), L^2(0,1)\right) \\ \eta \in C\left([0,\infty)\right) \end{cases} \tag{2.17}$$

Moreover, if $U_0 = (u_0, u_1, \eta_0, f_0)$ *belongs to* $\mathcal{D}(\mathcal{A})$*, then problem (1.1) has one and only one strong solution* $U = (u, u_t, \eta, z)$ *which satisfies*

$$\begin{cases} u \in C\left([0,\infty), H^2(0,1) \cap V\right) \cap C^1\left([0,\infty), V\right) \cap C^2\left([0,\infty), L^2(0,1)\right) \\ \eta \in C^1\left([0,\infty)\right). \end{cases} \tag{2.18}$$

Proof. We have

$$\begin{aligned}
\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ -C^{-1}(Au + B\eta) \\ v_x(1) - \beta_1\eta - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)z(1,t,s)ds \\ -s^{-1}z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= (v,u)_{W\times W} + (-C^{-1}(Au + B\eta), v)_{V\times V} \\
&+ \left(v_x(1) - \beta_1\eta - \int_{\tau_1}^{\tau_2} \beta_2(s)z(1,t,s)ds\right).\overline{\eta} \\
&- \int_0^1 \int_{\tau_1}^{\tau_2} \beta_2(s)z(\rho)\overline{z_\rho(\rho)}\,ds\,d\rho. \\
&= < Av, u >_{W'\times W} + < -(Au + B\eta), v >_{V'\times V} + v_x(1)\overline{\eta} \\
&- \int_{\tau_1}^{\tau_2} \beta_2(s)z(1,t,s)ds\overline{\eta} - \beta_1|\eta|^2 - \int_0^1 \int_{\tau_1}^{\tau_2} \beta_2(s)z(\rho)\overline{z_\rho(\rho)}\,ds\,d\rho.
\end{aligned}$$

Since $(u,v,\eta,z)^{\mathsf{T}}\in\mathcal{D}(\mathcal{A})$, then $Au+B\eta\in V'$ and $v\in W$ then we have

$$\begin{aligned}<-(Au+B\eta),v>_{V'\times V} &= <-(Au+B\eta),v>_{W'\times W}\\ &= -<Au,v>_{W'\times W} - <B\eta,v>_{W'\times W}\\ &= -<Au,v>_{W'\times W} - \eta\overline{v_x(1)}.\end{aligned}$$

We can deduce

$$\begin{aligned}\Re\left\langle\mathcal{A}\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_{\mathcal{H}} &= \Re\Big(<Av,u>_{W'\times W} - <Au,v>_{W'\times W} + v_x(1)\overline{\eta}-\eta\overline{v_x(1)}\Big)\\ &\quad -\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds)\overline{\eta}\right)-\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds\\ &\quad +\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds-\beta_1|\eta|^2\\ &= -\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds)\overline{\eta}\right)-\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds\\ &\quad +\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds-\beta_1|\eta|^2\\ &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds+\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds-\beta_1|\eta|^2\\ &\quad -\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds+\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds\\ &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds-\beta_1|\eta|^2+\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds\\ &\leq \Big(-\beta_1+\int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2\end{aligned}$$

and

$$\Re\left\langle\mathcal{A}\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_{\mathcal{H}} \leq \Big(-\beta_1+\int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2$$

Now the relation (1.2) allows to conclude that

$$\Re\left\langle\mathcal{A}\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix}\right\rangle_{\mathcal{H}} \leq 0$$

which implies that the operator $\mathcal{A}$ is dissipative.
Let us prove that the operator $\lambda I-\mathcal{A}$ is surjective for at least one $\lambda>0$.
For $(f,g,h,k)^{\mathsf{T}}\in\mathcal{H}$, we look for $(u,v,\eta,z)^{\mathsf{T}}\in\mathcal{D}(\mathcal{A})$ solution of

$$\begin{cases}\lambda u-v=f & \text{in }]0,1[\\ \lambda v+C^{-1}(Au+B\eta)=g & \text{in } V'\\ \lambda\eta-v_x(1)+\beta_1\eta+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds=h & \\ \lambda z+s^{-1}z_\rho=k & \text{in }]0,1[.\end{cases}\tag{2.19}$$

Suppose that we have found u with the appropriate regularity. It means that we have also found η. Then $v = \lambda u - f$ and we can determine z by solving the system

$$\begin{cases} s^{-1} z_\rho + \lambda z = k \ \text{ in }]0,1[\\ z(0) = \eta. \end{cases} \tag{2.20}$$

We obtain

$$z(\rho) = \eta e^{-\lambda s \rho} + s e^{-\lambda s \rho} \int_0^\rho k(\sigma) e^{\lambda s \sigma} \, d\sigma.$$

In particular

$$z(1) = \eta e^{-\lambda s} + \tau e^{-\lambda s} \int_0^1 k(\sigma) e^{\lambda s \sigma} \, d\sigma.$$

The function u verifies now

$$\begin{cases} \lambda^2 C u + A u = C(g + \lambda f) - B\eta \ \text{ in } V' \\ u(0) = 0 \\ u_x(0) = 0 \end{cases} \tag{2.21}$$

By using Lax-Milgram's Lemma, the problem (2.21) admits a unique weak solution. Indeed multiplying the first equation by $v \in V$ and by integrating formally by parts we get

$$a(u, v) = F(v), \forall\, v \in V, \tag{2.22}$$

where the bilinear and continuous form a is given by

$$a(u, v) = \int_0^1 \left(\lambda^2 \gamma u_x v_x + \lambda^2 u v + u_{xx} v_{xx} \right) dx \quad \forall\, u, v \in V,$$

while the linear form F is

$$F(v) = \int_0^1 (g + \lambda f) v + \gamma (g + \lambda f)_x v_x \, dx - \eta v_x(1), \quad \forall\, v \in V.$$

Since a is clearly strongly coercive on V and F is continuous on V, by Lax-Milgram's Lemma, problem (2.21) admits a unique solution $u \in V$. By taking test functions $v \in \mathcal{D}(0;1)$, we recover the first identity of (2.21). This guarantees that u belongs to $H^2(0,1)$. Using now Green's formula, we see that u satisfies the third identity of (2.21).

Finally, we define η and v by setting

$$v = \lambda u - f \text{ and } \eta = \frac{v_x(1) - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1, t, s) ds + h}{\beta_1 + \lambda}$$

This shows that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and it generates a $\mathcal{C}_0$-semigroup of contractions in $\mathcal{H}$, under Lumer-Phillips theorem. So, we have found $(u, v, \eta, z)^T \in \mathcal{D}(\mathcal{A})$ which verifies (2.21). The proof ends by using the Hille-Yosida theorem. ∎

3. Strong stability

The main results of this section reads as follows.

Theorem 3.1.
The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t \geq 0}$ is strongly stable on the energy space $\mathcal{H}$, that is for any $U_0 \in \mathcal{H}$,

$$\lim_{t \longrightarrow 0} \left\| e^{t\mathcal{A}} U_0 \right\|_{\mathcal{H}} = 0.$$

Proof. We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [3, 6, 18]. According this theory, since the resolvent of $\mathcal{A}$ is compact, it suffices to establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis. For our purpose, it is easy to prove that the resolvent of the operator $\mathcal{A}$ defined in (2.5) is compact. We are ready now to achieve the proof of theorem 3.1 with the following lemma.

Lemma 3.2.
There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. By contradiction argument, we assume that there exists at least one $i\lambda \in \sigma(\mathcal{A})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ on the imaginary axis. Let $U = (u, v, \eta, z)^{\mathsf{T}} \in D(\mathcal{A})$ be the corresponding normalized eigenvector, that is verifying $\|U\| = 1$ and

$$\mathcal{A}(u, v, \eta, z)^{\mathsf{T}} = i\lambda(u, v, \eta, z)^{\mathsf{T}}, \tag{3.1}$$

which is equivalent to

$$\begin{cases} v - i\lambda u = 0 & \text{in }]0,1[\\ -C^{-1}(Au + B\eta) - i\lambda v = 0 & \text{in } V' \\ v_x(1) - \beta_1\eta - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds - i\lambda\eta = 0 & \\ s^{-1}z_\rho + i\lambda z = 0 & \text{in }]0,1[. \end{cases} \tag{3.2}$$

Recalling the dissipativity of $\mathcal{A}$ and setting

$$\Lambda_1 = \beta - \int_{\tau_1}^{\tau_2} \beta_2(s)ds \tag{3.3}$$

in the proof of theorem 2.2, it follows that

$$0 = \Re e \left\langle \mathcal{A}(u, v, \eta, z)^{\mathsf{T}}, (u, v, \eta, z)^{\mathsf{T}} \right\rangle_{\mathcal{H}} \leq -\Lambda \left|\eta\right|^2 \tag{3.4}$$

So we deduce that $\eta = z = 0$.

Now (3.2) becomes

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0,1) \\ C^{-1}Au + i\lambda v = 0 \quad \text{in } (0,1) \\ v_x(1,.) = 0. \end{cases} \tag{3.5}$$

From the first equation of (3.5) we deduce that

$$u(1) = 0$$

Setting $v = i\lambda u$, it remains to find $u \in V$ which verifies

$$\begin{cases} Au - \lambda^2 Cu = 0 \quad \text{in } (0,1) \\ u_x(1) = 0 \\ u(1) = 0. \end{cases} \tag{3.6}$$

By Cauchy-Kowalevski theorem, there exists a nonempty neighbourhood $\mathcal{O}$ of 1 such that $u = 0$ in $\mathcal{O} \cap (0,1)$. Then the unicity theorem of Holmgren (see [7]) allows to conclude that

$$u = 0, \quad \text{on } (0,1). \tag{3.7}$$

We deduce that $(u, v, \eta, z)^{\mathsf{T}} = (0,0,0,0)^{\mathsf{T}}$ which contradicts the fact that $\|U\| = 1$. We conclude that $\mathcal{A}$ has no eigenvalue on the imaginary axis. ■

As the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are full satisfied, the proof of theorem 3.1 is thus completed. ■

4. Polynomial stability

In this section, we shall analyze the rational decays rate in the form t^{-1} of the energy of system. For that purpose we recall first the following result due to Borichev and Tomilov [4].

Lemma 4.1.
Let $\mathbf{A}$ be the generator of a C_0-semigroup of bounded operators on a Hilbert space $\mathbf{X}$ such that $i\mathbb{R} \subset \rho(\mathbf{A})$. Then we have the polynomial decay

$$\left\|e^{t\mathbf{A}}U_0\right\| \leq \frac{C}{t^{1/\theta}}\left\|U_0\right\|,\ t > 0,$$

if and only if

$$\limsup_{|\lambda|\to+\infty} \frac{1}{|\lambda|^{\theta}}\left\|(i\lambda - \mathbf{A})^{-1}\right\| < \infty.$$

The main result of this section is the following theorem

Theorem 4.2.
The semigroup of system (1.1) *decays polynomially as*

$$\left\|e^{t\mathcal{A}}U_0\right\| \leq \frac{C}{t}\left\|U_0\right\|,\ \forall\, U_0 \in \mathcal{D}(\mathcal{A}),\ \forall\, t > 0. \tag{4.1}$$

Proof. It suffices to show following the results in [10, 20] and the above theorem, that for any $U = (u, v, \eta, z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ and
$F = (f, g, h, k)^{\mathsf{T}} \in \mathcal{H}$, the solution of

$$(i\lambda I - \mathcal{A})\,U = F \tag{4.2}$$

verifies

$$\|U\|_{\mathcal{H}} \leq C\lambda\|F\|_{\mathcal{H}}; \tag{4.3}$$

where $\lambda > 0$ and $C > 0$.

Problem (1.1) without delay is the following one

$$\begin{cases} u_{tt}(x,t) - \gamma u_{xxtt}(x,t) + u_{xxxx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) + \eta(t) = 0, \\ u_{xxx}(1,t) - \gamma u_{xtt}(1,t) = 0\ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_{xt}(1,t) + \beta_1\eta(t) = 0\ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{C} \\ \eta(t-\tau) = f_0(t-\tau)\ \forall\, t \in (0,\tau), \end{cases}$$

which is well-posed in

$$\mathcal{H}_0 := W \times V \times \mathbb{C} \tag{4.4}$$

endowed with the norm

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 := \|u_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \gamma\,\|v_x\|_{L^2(0,1)}^2 + |\eta|^2. \tag{4.5}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0\,(u,v,\eta)^{\mathsf{T}} := \left(v, -C^{-1}(Au+B\eta), v_x(1) - \beta_1\eta\right)^{\mathsf{T}} \tag{4.6}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{\,(u,v,\eta)^{\mathsf{T}} \in \mathcal{H}, v\in W, Au+B\eta \in V'\,\right\}, \tag{4.7}$$

Thanks to [9], the operator $\mathcal{A}_0$ generates a polynomial stable semigroup with optimal decay rate t^{-1}. Therefore the solution $(u^*,v^*,\eta^*)^{\mathsf{T}}$ of

$$(i\lambda I - \mathcal{A}_0)\begin{pmatrix} u^*\\ v^*\\ \eta^*\end{pmatrix} = \begin{pmatrix} u\\ v\\ \eta\end{pmatrix} \tag{4.8}$$

verifies

$$\left\|(u^*,v^*,\eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \leq C_0\lambda \left\|(u,v,\eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \tag{4.9}$$

where C_0 is a positive constant.

On the other hand the system (4.8) can be rewritten as

$$\begin{cases} i\lambda u^* - v^* = u\\ i\lambda v^* + C^{-1}(Au^*+B\eta^*) = v\\ i\lambda\eta^* - v_x^*(1) + \beta_1\eta^* = \eta. \end{cases} \tag{4.10}$$

Let $\alpha \in \mathbb{R}$, with the help of integrations by parts and using (4.10) we have

$$
\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\ v\\ \eta\\ z\end{pmatrix}, \begin{pmatrix}u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} i\lambda u - v\\ i\lambda v + C^{-1}(Au+B\eta)\\ i\lambda\eta - v_x(1)+\beta_1\eta + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\\ i\lambda z + s^{-1}z_\rho\end{pmatrix}, \begin{pmatrix}u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}}\\
&= \Big(i\lambda u - v, u^*\Big)_{W\times W} + \Big(i\lambda v + C^{-1}(Au+B\eta), v^*\Big)_{V\times V}\\
&\quad +\Big(i\lambda\eta - v_x(1)+\beta_1\eta+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\Big)\overline{\eta^*}\\
&\quad +\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)(i\lambda z + s^{-1}z_\rho)\overline{z}\,ds\,d\rho\\
&= \int_0^1 (i\lambda u - v)_{xx}\,\overline{u^*_{xx}}dx + \int_0^1 \big(i\lambda v + C^{-1}(Au+B\eta)\big)\,\overline{v^*}dx\\
&\quad +\gamma\int_0^1 \big(i\lambda v + C^{-1}(Au+B\eta)\big)_x\,\overline{v^*}_x dx\\
&\quad +\left(i\lambda\eta - v_x(1)+\beta_1\eta+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\right)\overline{\eta^*} + i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho\\
&\quad +\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad +\int_0^1 C^{-1}(Au+B\eta)\overline{v^*} + \gamma C^{-1}(Au+B\eta)\overline{v^*}dx\\
&\quad -\overline{\Big[i\lambda\eta^* - v^*_x(1)+\beta_1\eta^*\Big]}\eta + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds.\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2 + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad + < CC^{-1}(Au+B\eta), v^* >_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2 + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad + < Au+B\eta, v^* >_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho
\end{aligned}
$$

$$
\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\ v\\ \eta\\ z\end{pmatrix},\begin{pmatrix}u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx-\int_0^1 v_{xx}\overline{u^*_{xx}}dx+i\lambda\int_0^1 v\overline{v^*}dx+i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{v^*_x(1)}\eta-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +<Au,v^*>_{V'\times V}+<B\eta,v^*>_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx-\int_0^1 v_{xx}\overline{u^*_{xx}}dx+i\lambda\int_0^1 v\overline{v^*}dx+i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{v^*_x(1)}\eta-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +\int_0^1 u_{xx}\overline{v^*_{xx}}dx+\eta\overline{v^*_x(1)}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\int_0^1 u_{xx}\overline{(i\lambda u^*-v^*)_{xx}}dx-\int_0^1 v\overline{(i\lambda v^*)}+\gamma v_x\overline{(i\lambda v^*_x)}dx\\
&\quad -\int_0^1 v_{xx}\overline{u^*_{xx}}dx-|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\int_0^1 u_{xx}\overline{u_{xx}}dx-<Cv,i\lambda v^*>_{V'\times V}-\overline{<Au^*,v>_{V'\times V}}\\
&\quad -|\eta|^2+\beta_2 z(1)\overline{\eta^*}-\overline{<B\eta^*,v>_{V'\times V}}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-(v,i\lambda v^*)_{V\times V}-\overline{(C^{-1}Au^*,v)_{V\times V}}\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{(C^{-1}B\eta^*,v)_{V\times V}}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-(v,i\lambda v^*)_{V\times V}-(v,C^{-1}Au^*)_{V\times V}\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-(v,C^{-1}B\eta^*)_{V\times V}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-\Big(v,C^{-1}(Au^*+B\eta^*)+i\lambda v^*\Big)_{V\times V}-|\eta|^2\\
&\quad +\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho
\end{aligned}
$$

$$\begin{aligned}\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\\alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= -\|u_{xx}\|^2_{L^2(0,1)}-(v,v)_{V\times V}-|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-\|v\|^2_{L^2(0,1)}-\gamma\|v_x\|^2_{L^2(0,1)}-|\eta|^2\\
&\quad +2\beta_1\eta\overline{\eta^*}+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|(u,v,\eta)\|^2_{\mathcal{H}_0}+2\beta_1\eta\overline{\eta^*}+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds.\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\end{aligned}$$

So

$$\begin{aligned}\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} = \Re\left\langle F,\begin{pmatrix}u^*\\v^*\\\eta^*\\\alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} + \Re\Big(2\beta_1\eta\overline{\eta^*}\Big)+\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big)\\
+\Re\Big(\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big)\end{aligned}\tag{4.11}$$

Take $\alpha=\dfrac{-1}{\varepsilon}$ with $\varepsilon>0$.Then (4.11) becomes

$$\begin{aligned}\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} = \Re\left\langle F,\begin{pmatrix}u^*\\v^*\\\eta^*\\\frac{-1}{\varepsilon}z\end{pmatrix}\right\rangle_{\mathcal{H}} + \Re\Big(2\beta_1\eta\overline{\eta^*}\Big)+\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big)\\
-\Re\Big(\frac{1}{\varepsilon}\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big)\end{aligned}\tag{4.12}$$

We have by Young's inegality

$$\begin{aligned}\Re\Big(2\beta_1\eta\overline{\eta^*}\Big) &\leq 2\beta_1|\eta|.|\eta^*|\\
&\leq \frac{\beta_1^2}{\varepsilon}|\eta|^2+\varepsilon|\eta^*|^2\end{aligned}\tag{4.13}$$

Then by Fubbini

$$-\Re\Big(\frac{1}{\varepsilon}\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big) = -\Re\Big(\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\Big[|z|^2\Big]_0^1 ds\Big)$$
$$= -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0)|^2\,ds$$
$$= -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds.|\eta|^2 \tag{4.14}$$

Moreover, by the Cauchy-Schwarz inequality

$$\Re\left\langle F, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ \alpha z \end{pmatrix}\right\rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}.\|(u^*,v^*,\eta^*)^\mathsf{T}\|_{\mathcal{H}_0} + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|0,0,0,z)^\mathsf{T}\|_{\mathcal{H}}$$
$$\leq \|F\|_{\mathcal{H}}.\|(u^*,v^*,\eta^*)^\mathsf{T}\|_{\mathcal{H}_0} + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|u,v,\eta,z)^\mathsf{T}\|_{\mathcal{H}}$$
$$\leq C_0\lambda\|F\|_{\mathcal{H}}.\|(u,v,\eta)^\mathsf{T}\|_{\mathcal{H}_0} + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.15}$$

Finally, Young's inequality gives us

$$\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big) \leq \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2ds + \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds.|\eta^*|^2 \tag{4.16}$$

Summing (4.13),(4.14),(4.15) and (4.16) we have

$$\Big\|(u,v,\eta)^\mathsf{T}\Big\|^2_{\mathcal{H}_0} \leq \frac{\beta_1^2}{\varepsilon}|\eta|^2 + \varepsilon|\eta^*|^2 - \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds.|\eta|^2$$
$$+ C_0\lambda\|F\|_{\mathcal{H}}.\|(u,v,\eta)^\mathsf{T}\|_{\mathcal{H}_0} + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2ds$$
$$+\frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds|\eta^*|^2$$
$$\leq \Big(\frac{\beta_1^2}{\varepsilon} + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds\Big)|\eta|^2 + \varepsilon\Big(1+\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta^*|^2$$
$$+ \Big(C_0\lambda + \frac{1}{\varepsilon}\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.17}$$

Using the fact that $\mathcal{A}$ is dissipative and Cauchy-Schwarz inequality we have

$$\Big(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2 \leq \Re\left\langle (i\lambda I - \mathcal{A})\,U, U\right\rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.18}$$

This leads to

$$|\eta|^2 \leq \frac{1}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.19}$$

Note also that (4.9) and the dissipativity of $\mathcal{A}_0$ give

$$\beta_1|\eta^*|^2 \leq \Re\left\langle (i\lambda I - \mathcal{A}_0)\,(u^*, v^*, \eta^*)^{\mathsf{T}}, (u^*, v^*, \eta^*)^{\mathsf{T}} \right\rangle_{\mathcal{H}_0} \tag{4.20}$$

$$\leq \|(u, v, \eta)^{\mathsf{T}}\|_{\mathcal{H}_0}.\|(u^*, v^*, \eta^*)^{\mathsf{T}}\|_{\mathcal{H}_0} \tag{4.21}$$

$$\leq C_0\lambda\|(u, v, \eta)^{\mathsf{T}}\|^2_{\mathcal{H}_0} \tag{4.22}$$

This means that

$$|\eta^*|^2 \leq \frac{C_0\lambda}{\beta_1}\|(u, v, \eta)^{\mathsf{T}}\|^2_{\mathcal{H}_0} \tag{4.23}$$

In other words

$$|\eta^*|^2 \leq \frac{C_0\lambda}{\beta_1}\|U\|^2_{\mathcal{H}} \tag{4.24}$$

Using (4.19) and (4.24) in (4.17) we get

$$\left\|(u, v, \eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} \leq C_1\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \varepsilon\lambda C_2\|U\|^2_{\mathcal{H}} + \Big(C_0\lambda + \frac{1}{\varepsilon}\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.25}$$

where C_1 and C_2 are constants that do not depend on λ defined by

$$C_1 = \frac{\dfrac{\beta_1^2}{\varepsilon} + \dfrac{1}{2\varepsilon}\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)\,ds}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)ds}$$

and

$$C_2 = \frac{C_0\Big(1 + \dfrac{1}{2}\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)ds\Big)}{\beta_1}$$

Let $\varepsilon = \dfrac{1}{2C_2\lambda}$, so $C_2\lambda\varepsilon = \dfrac{1}{2}$.Hence (4.25) becomes

$$\left\|(u, v, \eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} \leq \Big(C_1 + C_3\lambda\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \frac{1}{2}\|U\|^2_{\mathcal{H}} \tag{4.26}$$

with $C_3 = C_0 + 2C_2$.
If we add $\displaystyle\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho$ member by member we have

$$\frac{1}{2}\,\|U\|^2_{\mathcal{H}} \leq \Big(C_1 + C_3\lambda\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho \tag{4.27}$$

Now we need a better estimate for

$$\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho$$

From (4.2) we get

$$\begin{cases} s^{-1}z_\rho + i\lambda z = k \;\text{ in }]0,1[\\ z(0) = \eta. \end{cases} \tag{4.28}$$

We obtain

$$z(\rho) = \eta e^{-i\lambda s\rho} + s\int_0^\rho k(\sigma)e^{i\lambda(\sigma-\rho)}\,d\sigma.$$

By the triangular inequality we have

$$|z(\rho)| \le |\eta| + s\int_0^\rho |k(\sigma)|\,d\sigma.$$

This implies that

$$|z(\rho)|^2 \le |\eta|^2 + 2|\eta|s\int_0^\rho |k(\sigma)|\,d\sigma + s^2\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2. \tag{4.29}$$

On the one hand, using Cauchy-Schwarz inequality, we have

$$\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2 \le \Big(\int_0^\rho |k(\sigma)|^2\,d\sigma\Big)\Big(\int_0^\rho d\sigma\Big) \le \int_0^\rho |k(\sigma)|^2\,d\sigma. \tag{4.30}$$

On the other hand, Young's inequality gives us

$$2s|\eta|s\int_0^\rho |k(\sigma)|\,d\sigma \le |\eta|^2 + s^2\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2 \le |\eta|^2 + s^2\int_0^\rho |k(\sigma)|^2\,d\sigma \tag{4.31}$$

Using (4.30) and (4.31) in (4.29) we get

$$|z(\rho)|^2 \le 2|\eta|^2 + 2s^2\int_0^\rho |k(\sigma)|^2\,d\sigma. \tag{4.32}$$

Let's now integrate (4.32) on $(0,1)\times(\tau_1,\tau_2)$. We have

$$\begin{aligned}
\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z(\rho)|^2 ds d\rho &\le 2\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|\eta|^2 ds d\rho + 2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s^3\int_0^\rho |k(\sigma)|^2\,d\sigma ds d\rho\\
&\le 2\int_0^1 d\rho.\int_{\tau_1}^{\tau_2} s\beta_2(s)ds|\eta|^2 + 2\int_{\tau_1}^{\tau_2}\beta_2(s)s^3\int_0^1 |k(\sigma)|^2 ds d\rho\\
&\le 2\tau_2\int_{\tau_1}^{\tau_2}\beta_2(s)ds|\eta|^2 + 2\tau_2^2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s|k(\rho,s)|^2 ds d\rho\\
&\le 2\tau_2\beta_1|\eta|^2 + 2\tau_2^2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s|k(\rho,s)|^2 ds d\rho.
\end{aligned} \tag{4.33}$$

Using (4.19) and the definition of the norm in $\mathcal{H}$ we deduce from (4.33) that

$$\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z(\rho)|^2 ds d\rho \le C_4\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 2\tau_2^2\|F\|_{\mathcal{H}}^2. \tag{4.34}$$

with

$$C_4 = \frac{2\tau_2\beta_1}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}$$

Combining (4.27) and (4.34) we get

$$\|U\|_{\mathcal{H}}^2 \le 2\Big(C_1 + C_3\lambda + C_4\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 4\tau_2^2\|F\|_{\mathcal{H}}^2 \tag{4.35}$$

Taking $ambda$ to be sufficiently large, we obtain

$$\|U\|_{\mathcal{H}}^2 \le C_3\lambda\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 4\tau_2^2\|F\|_{\mathcal{H}}^2 \tag{4.36}$$

$$\le C\Big(\lambda\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\Big) \tag{4.37}$$

where $C \ge max\left\{C_3, 4\tau_2^2\right\}$
Hence the result

$$\|U\|_{\mathcal{H}} \le C\lambda\|F\|_{\mathcal{H}}. \tag{4.38}$$

Therefore $\limsup_{\lambda\to+\infty} \dfrac{1}{\lambda}\left\|(i\lambda - \mathbf{A})^{-1}\right\| < \infty$, whence the semi-group decreases polynomially according to the rate t^{-1}.

■

5. Exponential unstability

In this section, we show that the semigroup generated by the operator $\mathcal{A}$ is not exponentially stable. For that we use the frequency domain approach (see Huang [8] and Pruss [5]), namely the below result.

Lemma 5.1. *A contraction semigroup on a Hilbert space is exponentially stable if and only if*

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}) \tag{5.1}$$

and

$$\sup_{|\lambda|\to\infty} \|\,(i\lambda I - \mathcal{A})^{-1}\,\| < +\infty. \tag{5.2}$$

$\rho(\mathcal{A})$ *denotes the resolvent set of the operator* $\mathcal{A}$.

We state on the following result that constitutes the main of this section

Theorem 5.2. *The system* (2.2) *is not exponentially stable on the* $\mathcal{H}$ *energy space.*

Proof. Following the lemma (5.1), we prove that the condition (5.2) is not satisfied satisfied in the sense that there are sequences (λ_n), (U_n) and (F_n) such that

$$(i\lambda_n - \mathcal{A})U_n = F_n; \tag{5.3}$$

$$\|F_n\|_{\mathcal{H}} = O(1); \tag{5.4}$$

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty. \tag{5.5}$$

Note that this technique was used in [15], [2], [16], [17] and in several other articles
Let $U_n = (u^n, v^n, \eta^n, z^n)^T$ et $F_n = (f^{1n}, f^{2n}, f^{3n}, f^{4n})^T$

Assuming that (5.3) is verified, we have

$$\begin{cases} i\lambda_n u^n - v^n = f^{1n} \\ i\lambda_n v^n + C^{-1}(Au^n + B\eta^n) = f^{2n} \\ i\lambda_n \eta^n - v_x^n(1) + \beta_1\eta^n + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z^n(1,t,s)ds = f^{3n} \\ i\lambda_n z^n + s^{-1} z_\rho^n = f^{4n}. \end{cases} \tag{5.6}$$

We are looking for a particular solution defined for $f^{1n} = f^{3n} = f^{4n} = 0$ and $f^{2n}(x) = e^{\frac{1}{\sqrt{\gamma}}x} - e^{\frac{-1}{\sqrt{\gamma}}x}$ solution of the differential equation $-\gamma f_{xx} + f = 0$.
The system becomes

$$\begin{cases} v^n = i\lambda_n u^n \\ -\lambda_n^2 C u^n + Au^n + B\eta^n = Cf_{2n} \\ i\lambda_n \eta^n - i\lambda_n u_x^n(1) + \beta_1 \eta^n + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z^n(1,t,s)ds = 0 \\ i\lambda_n z^n + s^{-1} z_\rho^n = 0. \end{cases} \tag{5.7}$$

Using the definition of the operators A, B and C, we obtain for any $\Phi \in W$ the following variational formulation

$$\int_0^1 u_{xx}^n \overline{\Phi_{xx}}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} + \gamma u_x^n \overline{\Phi_x}\, dx + \eta^n \overline{\Phi_x(1)} = \int_0^1 f^{2n}\overline{\Phi} + \gamma f_x^{2n} \overline{\Phi_x} dx \tag{5.8}$$

Integration by parts gives

$$\begin{aligned} \Big[u_{xx}^n \overline{\Phi_x}\Big]_0^1 - \Big[u_{xxx}^n \overline{\Phi}\Big]_0^1 + \int_0^1 u_{xxxx}^n \overline{\Phi}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} dx - \lambda_n^2 \gamma \Big[u_x^n \overline{\Phi}\Big]_0^1 \\ + \lambda_n^2 \gamma \int_0^1 u_{xx}^n \overline{\Phi} dx + \eta^n \overline{\Phi_x(1)} = \int_0^1 f^{2n} \overline{\Phi} dx + \gamma \Big[f_x^{2n} \overline{\Phi}\Big]_0^1 - \gamma \int_0^1 f_{xx}^{2n} \overline{\Phi} dx \end{aligned} \tag{5.9}$$

This leads to

$$\begin{aligned} u_{xx}^n(1)\overline{\Phi_x}(1) - u_{xx}^n(0)\overline{\Phi_x}(0) - u_{xxx}^n(1)\overline{\Phi}(1) + u_{xxx}^n(0)\overline{\Phi}(0) + \int_0^1 u_{xxxx}^n \overline{\Phi}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} dx \\ - \lambda_n^2 \gamma u_x^n(1)\overline{\Phi}(1) + \lambda_n^2 \gamma u_x^n(0)\overline{\Phi}(0) + \lambda_n^2 \gamma \int_0^1 u_{xx}^n \overline{\Phi} dx + \eta^n \overline{\Phi_x(1)} \\ = \int_0^1 \Big[-\gamma f_{xx}^{2n} + f^{2n}\Big] \overline{\Phi} dx + \gamma f_x^{2n}(1)\overline{\Phi}(1) - \gamma f_x^{2n}(0)\overline{\Phi}(0) \end{aligned} \tag{5.10}$$

Since $\Phi(0) = \Phi_x(0) = 0$ and $-\gamma f_{xx}^{2n} + f^{2n} = 0$, (5.10) can be written as

$$\int_0^1 \Big[u_{xxxx}^n + \lambda_n^2 u_{xx}^n - \lambda_n^2 \gamma u^n\Big] \overline{\Phi}\, dx + \Big[u_{xx}^n(1) + \eta^n\Big] \overline{\Phi_x}(1) - \Big[u_{xxx}^n(1) + \lambda_n^2 \gamma u_x^n(1) + \gamma f_x^{2n}(1)\Big] \overline{\Phi}(1) = 0 \tag{5.11}$$

This is equivalent to the system

$$\begin{cases} u_{xxxx}^n + \lambda_n^2 u_{xx}^n - \lambda_n^2 \gamma u^n = 0; \\ u_{xx}^n(1) + \eta^n = 0; \\ u_{xxx}^n(1) + \lambda_n^2 \gamma u_x^n(1) + \gamma f_x^{2n}(1) = 0; \\ u^n(0) = u_x^n(0) = 0. \end{cases} \tag{5.12}$$

Let's now try to express η^n as a function of u^n. To do this, we'll solve the equation of the (5.7) system, which is

$$i\lambda_n z^n + s^{-1} z_\rho^n = 0 \tag{5.13}$$

The solution of (5.13) is of the form

$$z^n(\rho, s) = Ce^{-i\lambda_n s\rho} \tag{5.14}$$

Now $z^n(0) = \eta^n(t)$ so $C := \eta^n(t)$.
Thus (5.14) is written as

$$z^n(\rho, s) = \eta^n(t)e^{-i\lambda_n s\rho} \tag{5.15}$$

When we derive this solution with respect to t and with respect to ρ we obtain the equation

$$\eta^n_t - i\lambda_n\eta^n = 0 \tag{5.16}$$

After integration, we also obtain that (5.16) has the solution
$\eta^n = ke^{i\lambda_n t}$, with $k \in \mathbb{C}$.
Since $\eta^n(0) = \eta^n_0$ we obtain $k = \eta^n_0$, from which $\eta^n = \eta^n_0 e^{i\lambda_n t}$.
Replacing η^n by $\eta^n_0 e^{i\lambda_n t}$ in (5.15) gives us

$$z(\rho) = \eta^n_0 e^{i\lambda_n(t-s\rho)}.$$

In particular

$$z^n(1) = \eta^n_0 e^{i\lambda_n(t-s)} = \eta^n e^{-i\lambda_n s}.$$

From the third equation of (5.7) we finally obtain by replacing $z^n(1)$ by $\eta^n e^{-i\lambda_n s}$

$$\eta^n = \frac{i\lambda_n u^n_x(1)}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} \tag{5.17}$$

The (5.12) system thus becomes

$$\begin{cases} u^n_{xxxx} + \lambda_n^2 u^n_{xx} - \lambda_n^2\gamma u^n = 0; \\ u^n_{xx}(1) + \dfrac{i\lambda_n}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} u^n_x(1) = 0; \\ u^n_{xxx}(1) + \lambda_n^2\gamma u^n_x(1) + \gamma f^{2n}_x(1) = 0; \\ u^n(0) = u^n_x(0) = 0. \end{cases} \tag{5.18}$$

For the rest of the proof, let's assume, as in article [9]

$$\lambda_n = \frac{n\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1) \tag{5.19}$$

In other words

$$\lambda_n = \frac{n\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + l(n) \; with \; \lim_{n\to+\infty} l(n) = 0 \tag{5.20}$$

It is clear that from a certain rank $n \geq n_0, n_0$ very large

$$\frac{i\lambda_n}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} \approx 1$$

and

$$\begin{aligned} u^n_{xxx}(1) + \lambda_n^2\gamma u^n_x(1) + \gamma f^{2n}_x(1) &= u^n_{xxx}(1) + \lambda_n^2\gamma u^n_x(1) + 2\sqrt{\gamma}ch(\frac{1}{\sqrt{\gamma}}) \\ &\approx u^n_{xxx}(1) + \lambda_n^2\gamma u^n_x(1). \end{aligned}$$

We therefore conclude that when $\lambda_n \longrightarrow +\infty$ the system (5.18) is equivalent to the system

$$\begin{cases} u^n_{xxxx} + \lambda_n^2 u^n_{xx} - \lambda_n^2 \gamma u^n = 0; \\ u^n_{xx}(1) + u^n_x(1) = 0; \\ u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1) = 0; \\ u^n(0) = u^n_x(0) = 0. \end{cases} \tag{5.21}$$

On the one hand, Serge Nicaise and associates have shown in [9] that (5.21) admits a solution verifying

$$\|u^n\|_W \sim n^2 \; et \; \|u^n\|_V \sim n \; when \; n \longrightarrow +\infty$$

This gives us (5.5).

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty.$$

On the other hand, according to the choice of F_n we have

$$\begin{aligned}
\|F_n\|^2_{\mathcal{H}} &= \int_0^1 \Big[f^{2n}(x)\Big]^2 + \gamma\Big[f^{2n}_x(x)\Big]^2 dx \\
&= \int_0^1 \Big[e^{\frac{1}{\sqrt{\gamma}}x} - e^{\frac{-1}{\sqrt{\gamma}}x}\Big]^2 + \Big[e^{\frac{1}{\sqrt{\gamma}}x} + e^{\frac{-1}{\sqrt{\gamma}}x}\Big]^2 dx \\
&= \int_0^1 \Big[e^{\frac{2}{\sqrt{\gamma}}x} - 2 + e^{\frac{-2}{\sqrt{\gamma}}x}\Big] + \Big[e^{\frac{2}{\sqrt{\gamma}}x} + 2 + e^{\frac{-2}{\sqrt{\gamma}}x}\Big]^2 dx \\
&= \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}x} - 2x - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}x}\Big]_0^1 + \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}x} + 2x - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}x}\Big]_0^1 \\
&= \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}} - 2 - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}}\Big] + \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}} + 2 - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}}\Big] \\
&\quad - \Big[\frac{\sqrt{\gamma}}{2} - \frac{\sqrt{\gamma}}{2}\Big] - \Big[\frac{\sqrt{\gamma}}{2} - \frac{\sqrt{\gamma}}{2}\Big] \\
&= \sqrt{\gamma}\Big(\frac{e^{\frac{2}{\sqrt{\gamma}}} - e^{\frac{-2}{\sqrt{\gamma}}}}{2}\Big) - 2 + \sqrt{\gamma}\Big(\frac{e^{\frac{2}{\sqrt{\gamma}}} - e^{\frac{-2}{\sqrt{\gamma}}}}{2}\Big) + 2 \\
&= 2.sh\Big(\frac{2}{\sqrt{\gamma}}\Big)
\end{aligned}$$

This means that

$$\|F_n\|_{\mathcal{H}} = O(1) \tag{5.22}$$

Finally, we've found sequences (λ_n), (U_n) and (F_n) satisfying $(5.3)-(5.5)$. Consequently, the proof of Theorem (5.2) is complete. ■

Conclusion

In this paper we have studied a Rayleigh-type problem with a distributed delay. We used the tools of functional analysis and semi-group theory to obtain the existence, uniqueness and polynomial decay. However, we have established that this polynomial decay is the best in the sense that it is impossible to have an exponential decay. In the future, we'd like to continue our study by replacing the distributed delay with a variable delay.

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