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Characterizations for pseudoparalel submanifolds of Lorentz-Sasakian space forms

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Abstract. In this study, invariant total geodesic submanifolds, which are important submanifolds of Lorentz-Sasakian space forms, have been investigated. An important class of the considered invariant submanifolds, called pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel invariant submanifolds, has been defined and the characterizations of Lorentz-Sasakian space forms for these types of invariant submanifolds have been revealed. Then, conditions are given for these obtained invariant submanifolds to be total geodesic by means of concircular and projective curvature tensors.

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Contents

1. Introduction

$\phi-$sectional curvature plays the an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [1]. P. Alegre and D. Blair described generalized Sasakian space forms [2]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [3]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity. Sasakian space forms, generalized Sasakian space forms and Lorentz-Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained ([4]-[8]).

Many mathematicians have considered the submanifolds of manifolds such as $K-$paracontak, Lorentzian para-Kenmotsu, almost Kenmotsu and studied their various characterizations ([9],[10],[11]).

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In this study, invariant total geodesic submanifolds, which are important submanifolds of Lorentz-Sasakian space forms, have been investigated. An important class of the considered invariant submanifolds, called pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel invariant submanifolds, has been defined and the characterizations of Lorentz-Sasakian space forms for these types of invariant submanifolds have been revealed. Then, conditions are given for these obtained invariant submanifolds to be total geodesic by means of concircular and projective curvature tensors.

Starting from this part of the article, for the sake of brevity, Lorentz Sasakian space form with $\mathcal{LSS}$-form, pseudoparallel submanifold with $\mathcal{P}$-submanifolds, 2-pseudoparallel submanifold with 2-$\mathcal{P}$ submanifold, Ricci generalized pseudoparallel submanifold with $\mathcal{RGP}$-submanifold and 2-Ricci generalized pseudoparallel submanifold with 2- $\mathcal{RGP}$ submanifold will be shown.

2. Preliminary

Let $\tilde{\Psi}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{\Psi}$ Lorentz manifold with (ϕ, ξ, η, g) structure tensors satisfies the following conditions, this manifold is called a Lorentz-Sasakian manifold

$$\phi^2\Lambda_1 = -\Lambda_1 + \eta(\Lambda_1)\xi, \eta(\xi) = 1, \eta(\phi\Lambda_1) = 0,$$

$$g(\phi\Lambda_1, \phi\Lambda_2) = g(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2), \eta(\Lambda_1) = -g(\Lambda_1, \xi),$$

$$\left(\tilde{\nabla}_{\Lambda_1}\phi\right)\Lambda_2 = -g(\Lambda_1, \Lambda_2)\xi - \eta(\Lambda_2)\Lambda_1, \tilde{\nabla}_{\Lambda_1}\xi = -\phi\Lambda_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemann metric g.

The plane section Π in $T_x\tilde{\Psi}$. If the Π plane is spanned by Λ_1 and $\phi\Lambda_1$, this plane is called the ϕ-section. The curvature of the ϕ-section is called the ϕ-sectional curvature. If the Lorentz-Sasakian manifold has a constant ϕ-sectional curvature, this manifold is called the $\mathcal{LSS}-$form and is denoted by $\tilde{\Psi}(c)$. The curvature tensor of the $\mathcal{LSS}-$form $\tilde{\Psi}(c)$ is defined as

$$\begin{aligned}
\tilde{R}(\Lambda_1, \Lambda_2)\Lambda_3 &= \left(\tfrac{c-3}{4}\right)\{g(\Lambda_2, \Lambda_3)\Lambda_1 - g(\Lambda_1, \Lambda_3)\Lambda_2\} \\
&+ \left(\tfrac{c+1}{4}\right)\{g(\Lambda_1, \phi\Lambda_3)\phi\Lambda_2 - g(\Lambda_2, \phi\Lambda_3)\phi\Lambda_1 \\
&+2g(\Lambda_1, \phi\Lambda_2)\phi\Lambda_3 + \eta(\Lambda_2)\eta(\Lambda_3)\Lambda_1 - \eta(\Lambda_1)\eta(\Lambda_3)\Lambda_2 \\
&+g(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\xi - g(\Lambda_2, \Lambda_3)\eta(\Lambda_1)\xi\},
\end{aligned} \tag{1}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \chi\left(\tilde{\Psi}\right)$.

Lemma 2.1. *Let $\tilde{\Psi}(c)$ be the $(2m+1)-$dimensional $\mathcal{LSS}-$form. The following relations are provided for the $\mathcal{LSS}-$forms.*

$$\tilde{\nabla}_{\Lambda_1}\xi = -\phi\Lambda_1, \tag{2}$$

$$\left(\tilde{\nabla}_{\Lambda_1}\phi\right)\Lambda_2 = -g(\Lambda_1, \Lambda_2)\xi - \eta(\Lambda_2)\Lambda_1, \tag{3}$$

$$\left(\tilde{\nabla}_{\Lambda_1}\eta\right)\Lambda_2 = g(\phi\Lambda_1, \Lambda_2), \tag{4}$$

$$\tilde{R}(\xi, \Lambda_2)\Lambda_3 = -g(\Lambda_2, \Lambda_3)\xi - \eta(\Lambda_3)\Lambda_2, \tag{5}$$

$$\tilde{R}(\xi, \Lambda_2)\xi = \eta(\Lambda_2)\xi - \Lambda_2, \tag{6}$$

$$\tilde{R}(\Lambda_1,\Lambda_2)\xi=\eta(\Lambda_2)\Lambda_1-\eta(\Lambda_1)\Lambda_2, \tag{7}$$

$$S(\Lambda_1,\xi)=-\left[\frac{(c+1)-4m}{2}\right]\eta(\Lambda_1), \tag{8}$$

where $\tilde{R}, S$ and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $\tilde{\Psi}(c)$, respectively.

Let Ψ be the immersed submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. Let the tangent and normal subspaces of Ψ in $\tilde{\Psi}(c)$ be $\Gamma(T\Psi)$ and $\Gamma\left(T^{\perp}\Psi\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(T\Psi)$ and $\Gamma\left(T^{\perp}\Psi\right)$ are

$$\tilde{\triangledown}_{\Lambda_1}\Lambda_2=\triangledown_{\Lambda_1}\Lambda_2+h(\Lambda_1,\Lambda_2), \tag{9}$$

$$\tilde{\triangledown}_{\Lambda_1}\Lambda_5=-A_{\Lambda_5}\Lambda_1+\triangledown^{\perp}_{\Lambda_1}\Lambda_5, \tag{10}$$

respectively, for all $\Lambda_1,\Lambda_2\in\Gamma(T\Psi)$ and $\Lambda_5\in\Gamma\left(T^{\perp}\Psi\right)$, where $\triangledown$ and $\triangledown^{\perp}$ are the connections on Ψ and $\Gamma\left(T^{\perp}\Psi\right)$, respectively, h and A are the second fundamental form and the shape operator of Ψ. There is a relation

$$g(A_{\Lambda_5}\Lambda_1,\Lambda_2)=g(h(\Lambda_1,\Lambda_2),\Lambda_5) \tag{11}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form h is defined as

$$\left(\tilde{\triangledown}_{\Lambda_1}h\right)(\Lambda_2,\Lambda_3)=\triangledown^{\perp}_{\Lambda_1}h(\Lambda_2,\Lambda_3)-h(\triangledown_{\Lambda_1}\Lambda_2,\Lambda_3)-h(\Lambda_2,\triangledown_{\Lambda_1}\Lambda_3). \tag{12}$$

Specifically, if $\tilde{\triangledown}h=0$, Ψ is said to be in the parallel second fundamental form or $1-$parallel.

Let R be the Riemann curvature tensor of Ψ. In this case, the Gauss equation can be expressed as

$$\begin{aligned}\tilde{R}(\Lambda_1,\Lambda_2)\Lambda_3&=R(\Lambda_1,\Lambda_2)\Lambda_3+A_{h(\Lambda_1,\Lambda_3)}\Lambda_2-A_{h(\Lambda_2,\Lambda_3)}\Lambda_1\\&+\left(\tilde{\triangledown}_{\Lambda_1}h\right)(\Lambda_2,\Lambda_3)-\left(\tilde{\triangledown}_{\Lambda_1}h\right)(\Lambda_1,\Lambda_3).\end{aligned} \tag{13}$$

Let Ψ be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, the tensor field $Q(A,T)$ is defined as

$$\begin{aligned}Q(A,T)(\Lambda_1,...,X_k;X,Y)&=-T((X\wedge_A Y)\Lambda_1,...,X_k)\\&-...-T(\Lambda_1,...,X_{k-1},(X\wedge_A Y)X_k),\end{aligned} \tag{14}$$

where

$$(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y,$$

$k\geq 1,\Lambda_1,\Lambda_2,...,X_k,X,Y\in\Gamma(T\Psi)$

3. Invariant Pseudoparalel submanifolds of Lorentz-Sasakian space forms

Let Ψ be the immersed submanifold of a $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\phi(T_{x_1}\Psi)\subset T_{x_1}\Psi$ in every x_1 point, the Ψ manifold is called invariant submanifold. From this section of the article, we will assume that the manifold Ψ is the invariant submanifold of the $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. So it is clear from (3) and (9) that

$$h(\Lambda_1,\xi)=0, h(\phi\Lambda_1,\Lambda_2)=h(\Lambda_1,\phi\Lambda_2)=\phi h(\Lambda_1,\Lambda_2) \tag{15}$$

for all $\Lambda_1,\Lambda_2\in\Gamma(T\Psi)$.

Lemma 3.1. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. The second fundamental form h of the submanifold Ψ is parallel if and only if Ψ is the total geodesic submanifold.*

Proof. The proof of the theorem is easily obtained if we choose $\Lambda_3 = \xi$ in (12) and make the necessary adjustments. ■

Definition 3.2. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.h$ and $Q(g,h)$ are linearly dependent, M is called $\mathcal{P}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_1 on the set $M_1 = \{\Lambda_1 \in \Psi | \, h(\Lambda_1) \neq g(\Lambda_1)\}$ such that

$$\tilde{R}.h = L_1 Q(g,h).$$

If $L_1 = 0$ specifically, Ψ is called a semiparallel submanifold.

Theorem 3.3. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is $\mathcal{P}$-submanifold, then Ψ is either a total geodesic or $L_1 = -1$.*

Proof. Let's assume that Ψ is a $\mathcal{P}$-submanifold. So, we can write

$$\left(\tilde{R}(\Lambda_1,\Lambda_2)h\right)(\Lambda_4,\Lambda_5) = L_1 Q(g,h)(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2),$$

that is

$$\begin{aligned} &\tilde{R}^{\perp}(\Lambda_1,\Lambda_2)h(\Lambda_4,\Lambda_5) - h(R(\Lambda_1,\Lambda_2)\Lambda_4,\Lambda_5) \\ &-h(\Lambda_4,R(\Lambda_1,\Lambda_2)\Lambda_5) = -\lambda_1\{h((\Lambda_1 \wedge_g \Lambda_2)\Lambda_4,\Lambda_5) \\ &+h(\Lambda_4,(\Lambda_1 \wedge_g \Lambda_2)\Lambda_5)\}, \end{aligned} \tag{16}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5 \in \Gamma(T\Psi)$. If we choose $\Lambda_5 = \xi$ in (16) and make use of (7), (15), we get

$$(1+L_1)\{\eta(\Lambda_2)h(\Lambda_4,\Lambda_1) - \eta(\Lambda_1)h(\Lambda_4,\Lambda_2)\} = 0. \tag{17}$$

IIf we choose $\Lambda_2 = \xi$ in (17), we obtain

$$(1+L_1)h(\Lambda_4,\Lambda_1) = 0.$$

This completes the proof. ■

Corollary 3.4. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. Ψ is semiparallel if and only if Ψ is total geodesic submanifold.*

Definition 3.5. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.\tilde{\bigtriangledown}h$ and $Q\left(g,\tilde{\bigtriangledown}h\right)$ are linearly dependent, M is called 2-$\mathcal{P}$ submanifold.*

Equivalent to this definition, it can be said that there is a function L_2 on the set $M_2 = \left\{\Lambda_1 \in \Psi | \, \tilde{\bigtriangledown}h(\Lambda_1) \neq g(\Lambda_1)\right\}$ such that

$$\tilde{R}.\tilde{\bigtriangledown}h = L_2 Q\left(g,\tilde{\bigtriangledown}h\right).$$

If $L_2 = 0$ specifically, Ψ is called a 2-semiparallel submanifold.

Theorem 3.6. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is 2-$\mathcal{P}$ submanifold, then Ψ is a total geodesic submanifold.*

Proof. Let's assume that Ψ is a 2-$\mathcal{P}$ submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1,\Lambda_2\right)\tilde{\bigtriangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3\right)=L_2Q\left(g,\tilde{\bigtriangledown}h\right)\left(\Lambda_4,\Lambda_5,\Lambda_3;\Lambda_1,\Lambda_2\right), \tag{18}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5,\Lambda_3\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_3=\xi$ in (18), we can write

$$\begin{aligned}
&R^{\perp}\left(\xi,\Lambda_2\right)\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\xi\right)-\left(\tilde{\bigtriangledown}_{R(\xi,\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)\\
&-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,R\left(\xi,\Lambda_2\right)\xi\right)\\
&=-L_2\left\{\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)+\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)\right.\\
&\left.+\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\left(\xi\wedge_g\Lambda_2\right)\xi\right)\right\}.
\end{aligned} \tag{19}$$

Let's calculate all the expressions in (19). So, we can write

$$\begin{aligned}
&R^{\perp}\left(\xi,\Lambda_2\right)\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=R^{\perp}\left(\xi,\Lambda_2\right)\left\{\bigtriangledown^{\perp}_{\Lambda_4}h\left(\Lambda_5,\xi\right)\right.\\
&\left.-h\left(\bigtriangledown_{\Lambda_4}\Lambda_5,\xi\right)-h\left(\Lambda_5,\bigtriangledown_{\Lambda_4}\xi\right)\right\}\\
&=R^{\perp}\left(\xi,\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{R(\xi,\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{R(\xi,\Lambda_2)\Lambda_4}h\left(\Lambda_5,\xi\right)-h\left(\bigtriangledown_{R(\xi,\Lambda_2)\Lambda_4}\Lambda_5,\xi\right)\\
&-h\left(\Lambda_5,\bigtriangledown_{R(\xi,\Lambda_2)\Lambda_4}\xi\right)=-\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{\Lambda_4}h\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)-h\left(\bigtriangledown_{\Lambda_4}R\left(\xi,\Lambda_2\right)\Lambda_5,\xi\right)\\
&-h\left(R\left(\xi,\Lambda_2\right)\Lambda_5,\bigtriangledown_{\Lambda_4}\xi\right)=-\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,R\left(\xi,\Lambda_2\right)\xi\right)=\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\eta\left(\Lambda_2\right)\xi-\Lambda_2\right)\\
&=\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\eta\left(\Lambda_2\right)\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\\
&=-h\left(\Lambda_5,\Lambda_4\eta\left(\Lambda_2\right)\xi+\eta\left(\Lambda_2\right)\bigtriangledown_{\Lambda_4}\xi\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\\
&=\eta\left(\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\bigtriangledown}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\left(\tilde{\bigtriangledown}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\right)\left(\Lambda_5,\xi\right)=\bigtriangledown^{\perp}_{(\xi\wedge_g\Lambda_2)\Lambda_4}h\left(\Lambda_5,\xi\right)-h\left(\bigtriangledown_{(\xi\wedge_g\Lambda_2)\Lambda_4}\Lambda_5,\xi\right)\\
&-h\left(\Lambda_5,\bigtriangledown_{(\xi\wedge_g\Lambda_2)\Lambda_4}\xi\right)=\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right),
\end{aligned} \tag{24}$$

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)=\nabla^{\perp}_{\Lambda_4}h\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)-h\left(\nabla_{\Lambda_4}\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\xi\right)$$
$$-h\left(\left(\xi\wedge_g\Lambda_2\right)\Lambda_5,\nabla_{\Lambda_4}\xi\right)=\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right), \tag{25}$$

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\left(\xi\wedge_g\Lambda_2\right)\xi\right)=\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,-\eta\left(\Lambda_2\right)\xi+\Lambda_2\right)$$
$$=\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,-\eta\left(\Lambda_2\right)\xi\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right) \tag{26}$$
$$=-\phi\eta\left(\Lambda_2\right)h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right).$$

If we substitute $(20), (21), (22), (23), (24), (25), (26)$ for (19), we obtain

$$R^{\perp}\left(\xi,\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)+\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right)+\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right)$$
$$-\eta\left(\Lambda_2\right)\phi h\left(\Lambda_5,\Lambda_4\right)+\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)=-L_2\left\{\phi\eta\left(\Lambda_4\right)h\left(\Lambda_5,\Lambda_2\right)\right. \tag{27}$$
$$\left.+\phi\eta\left(\Lambda_5\right)h\left(\Lambda_2,\Lambda_4\right)-\phi\eta\left(\Lambda_2\right)h\left(\Lambda_5,\Lambda_4\right)-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\Lambda_5,\Lambda_2\right)\right\}$$

If we choose $\Lambda_5=\xi$ and use (15), we get

$$\phi h\left(\Lambda_2,\Lambda_4\right)+\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)=-L_2\left\{\phi h\left(\Lambda_2,\Lambda_4\right)\right.$$
$$\left.-\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)\right\}. \tag{28}$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{\Lambda_4}h\right)\left(\xi,\Lambda_2\right)=\phi h\left(\Lambda_2,\Lambda_4\right). \tag{29}$$

If (29) is written instead of (28), we obtain

$$h\left(\Lambda_2,\Lambda_4\right)=0.$$

This completes the proof. ■

Corollary 3.7. *The total geodesic of the invariant 2-pseudoparalell submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form is independent of the choice of L_2.*

Definition 3.8. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.h$ and $Q(S,h)$ are linearly dependent, M is called $\mathcal{RGP}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_3 on the set $M_3=\{\Lambda_1\in\Psi|\, h(\Lambda_1)\neq S(\Lambda_1)\}$ such that

$$\tilde{R}.h=L_3Q\left(S,h\right).$$

Theorem 3.9. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is $\mathcal{RGP}$-submanifold, then Ψ is either a total geodesic or $L_3=\dfrac{-2}{(c+1)-4m}$ provided $4m\neq(c+1)$.*

Proof. Let's assume that Ψ is a $\mathcal{RGP}$-submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1, \Lambda_2\right) h\right)\left(\Lambda_4, \Lambda_5\right) = L_3 Q\left(S, h\right)\left(\Lambda_4, \Lambda_5; \Lambda_1, \Lambda_2\right),$$

that is

$$\begin{aligned}&\tilde{R}^{\perp}\left(\Lambda_1, \Lambda_2\right) h\left(\Lambda_4, \Lambda_5\right) - h\left(R\left(\Lambda_1, \Lambda_2\right) \Lambda_4, \Lambda_5\right)\\ &-h\left(\Lambda_4, R\left(\Lambda_1, \Lambda_2\right) \Lambda_5\right) = -\lambda_3\left\{h\left(\left(\Lambda_1 \wedge_g \Lambda_2\right) \Lambda_4, \Lambda_5\right)\right.\\ &\left.+h\left(\Lambda_4, \left(\Lambda_1 \wedge_g \Lambda_2\right) \Lambda_5\right)\right\},\end{aligned} \tag{30}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5 \in \Gamma\left(T\Psi\right)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (30) and make use of $(8), (15)$, we get

$$\left[1 + \frac{(c+1) - 4m}{2} L_3\right] h\left(\Lambda_4, \Lambda_2\right) = 0.$$

It is clear from the last equation that either

$$h\left(\Lambda_4, \Lambda_2\right) = 0,$$

or

$$L_3 = \frac{-2}{(c+1) - 4m}.$$

This completes the proof. ■

Definition 3.10. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If $\tilde{R}.\tilde{\bigtriangledown} h$ and $Q\left(S, \tilde{\bigtriangledown} h\right)$ are linearly dependent, M is called 2-$\mathcal{RGP}$-submanifold.*

Equivalent to this definition, it can be said that there is a function L_4 on the set $M_4 = \left\{\Lambda_1 \in \Psi | \tilde{\bigtriangledown} h\left(\Lambda_1\right) \neq S\left(\Lambda_1\right)\right\}$ such that

$$\tilde{R}.\tilde{\bigtriangledown} h = L_4 Q\left(S, \tilde{\bigtriangledown} h\right).$$

Theorem 3.11. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ is 2-$\mathcal{RGP}$-submanifold, then Ψ is either a total geodesic or $L_4 = \dfrac{2}{4m - (c+1)}$ provided $4m \neq (c+1)$.*

Proof. Let's assume that Ψ is a 2-$\mathcal{RGP}$-submanifold. So, we can write

$$\left(\tilde{R}\left(\Lambda_1, \Lambda_2\right) \tilde{\bigtriangledown} h\right)\left(\Lambda_4, \Lambda_5, \Lambda_3\right) = L_4 Q\left(S, \tilde{\bigtriangledown} h\right)\left(\Lambda_4, \Lambda_5, \Lambda_3; \Lambda_1, \Lambda_2\right), \tag{31}$$

for all $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5, \Lambda_3 \in \Gamma\left(T\Psi\right)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (31), we can write

$$\begin{aligned}&R^{\perp}\left(\xi, \Lambda_2\right)\left(\tilde{\bigtriangledown}_{\Lambda_4} h\right)\left(\xi, \Lambda_3\right) - \left(\tilde{\bigtriangledown}_{R\left(\xi, \Lambda_2\right)\Lambda_4} h\right)\left(\xi, \Lambda_3\right)\\ &-\left(\tilde{\bigtriangledown}_{\Lambda_4} h\right)\left(R\left(\xi, \Lambda_2\right)\xi, \Lambda_3\right) - \left(\tilde{\bigtriangledown}_{\Lambda_4} h\right)\left(\xi, R\left(\xi, \Lambda_2\right)\Lambda_3\right)\\ &= -L_4\left\{\left(\tilde{\bigtriangledown}_{\left(\xi \wedge_S \Lambda_2\right)\Lambda_4} h\right)\left(\xi, \Lambda_3\right) + \left(\tilde{\bigtriangledown}_{\Lambda_4} h\right)\left(\left(\xi \wedge_S \Lambda_2\right)\xi, \Lambda_3\right)\right.\\ &\left.+\left(\tilde{\bigtriangledown}_{\Lambda_4} h\right)\left(\xi, \left(\xi \wedge_S \Lambda_2\right)\Lambda_3\right)\right\}.\end{aligned} \tag{32}$$

Let's calculate all the expressions in (32). So, we can write

$$\begin{aligned} &R^{\perp}(\xi,\Lambda_2)\left(\tilde{\nabla}_{\Lambda_4}h\right)(\xi,\Lambda_3)=R^{\perp}(\xi,\Lambda_2)\left\{\nabla^{\perp}_{\Lambda_4}h(\xi,\Lambda_3)\right.\\ &\left.-h(\nabla_{\Lambda_4}\Lambda_3,\xi)-h(\Lambda_3,\nabla_{\Lambda_4}\xi)\right\}\\ &=R^{\perp}(\xi,\Lambda_2)\phi h(\Lambda_3,\Lambda_4), \end{aligned} \tag{33}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{R(\xi,\Lambda_2)\Lambda_4}h\right)(\xi,\Lambda_3)=\nabla^{\perp}_{R(\xi,\Lambda_2)\Lambda_4}h(\xi,\Lambda_3)-h\left(\nabla_{R(\xi,\Lambda_2)\Lambda_4}\xi,\Lambda_3\right)\\ &-h\left(\xi,\nabla_{R(\xi,\Lambda_2)\Lambda_4}\Lambda_3\right)=-\phi\eta(\Lambda_4)h(\Lambda_2,\Lambda_3), \end{aligned} \tag{34}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)(R(\xi,\Lambda_2)\xi,\Lambda_3)=\left(\tilde{\nabla}_{\Lambda_4}h\right)(\eta(\Lambda_2)\xi-\Lambda_2,\Lambda_3)\\ &-\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3)=\nabla^{\perp}_{\Lambda_4}h(\eta(\Lambda_2)\xi,\Lambda_3)-h(\nabla_{\Lambda_4}\eta(\Lambda_2)\xi,\Lambda_3)\\ &-h(\eta(\Lambda_2)\xi,\nabla_{\Lambda_4}\Lambda_3)-\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3)\\ &=\phi\eta(\Lambda_2)h(\Lambda_4,\Lambda_3)-\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3), \end{aligned} \tag{35}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)(\xi,R(\xi,\Lambda_2)\Lambda_3)=\nabla^{\perp}_{\Lambda_4}h(\xi,R(\xi,\Lambda_2)\Lambda_3)-h(\nabla_{\Lambda_4}\xi,R(\xi,\Lambda_2)\Lambda_3)\\ &-h(\xi,\nabla_{\Lambda_4}R(\xi,\Lambda_2)\Lambda_3)=-\phi\eta(\Lambda_3)h(\Lambda_4,\Lambda_2) \end{aligned} \tag{36}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{(\xi\wedge_S\Lambda_2)\Lambda_4}h\right)(\xi,\Lambda_3)=\nabla^{\perp}_{(\xi\wedge_S\Lambda_2)\Lambda_4}h(\xi,\Lambda_3)-h\left(\nabla_{(\xi\wedge_S\Lambda_2)\Lambda_4}\xi,\Lambda_3\right)\\ &-h\left(\xi,\nabla_{(\xi\wedge_S\Lambda_2)\Lambda_4}\Lambda_3\right)=\tfrac{(c+1)-4m}{2}\phi\eta(\Lambda_4)h(\Lambda_2,\Lambda_3), \end{aligned} \tag{37}$$

$$\begin{aligned} &\left(\tilde{\nabla}_{\Lambda_4}h\right)((\xi\wedge_S\Lambda_2)\xi,\Lambda_3)=\left(\tilde{\nabla}_{\Lambda_4}h\right)(S(\Lambda_2,\xi)\xi-S(\xi,\xi)\Lambda_2,\Lambda_3)\\ &=\tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{\Lambda_4}h\right)(-\eta(\Lambda_2)\xi+\Lambda_2,\Lambda_3)\right\}\\ &=\tfrac{(c+1)-4m}{2}\left\{-\nabla^{\perp}_{\Lambda_4}h(\eta(\Lambda_2)\xi,\Lambda_3)+h(\nabla_{\Lambda_4}\eta(\Lambda_2)\xi,\Lambda_3)\right.\\ &\left.h(\eta(\Lambda_2)\xi,\nabla_{\Lambda_4}\Lambda_3)+\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3)\right\}\\ &=\tfrac{(c+1)-4m}{2}\left\{\left(\tilde{\nabla}_{\Lambda_4}h\right)(\Lambda_2,\Lambda_3)-\phi\eta(\Lambda_2)h(\Lambda_4,\Lambda_3)\right\}, \end{aligned} \tag{38}$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi,\left(\xi \wedge_S \Lambda_2\right) \Lambda_3\right)=\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, S\left(\Lambda_2, \Lambda_3\right) \xi-S\left(\xi, \Lambda_3\right) \Lambda_2\right) \\
&=\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, S\left(\Lambda_2, \Lambda_3\right) \xi\right)+\tfrac{(c+1)-4 m}{2}\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, \eta\left(\Lambda_3\right), \Lambda_2\right) \\
&=\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_3\right) h\left(\Lambda_4, \Lambda_2\right) .
\end{aligned} \tag{39}
$$

If we substitute $(33),(34),(35),(36),(37),(38),(39)$ for (32), we obtain

$$
\begin{aligned}
&R^{\perp}\left(\xi, \Lambda_2\right) \phi h\left(\Lambda_3, \Lambda_4\right)+\phi \eta\left(\Lambda_4\right) h\left(\Lambda_2, \Lambda_3\right)-\phi \eta\left(\Lambda_2\right) h\left(\Lambda_4, \Lambda_3\right) \\
&+\eta\left(\Lambda_3\right) \phi h\left(\Lambda_4, \Lambda_2\right)+\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \Lambda_3\right)=-L_4\left\{\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_4\right) h\left(\Lambda_2, \Lambda_3\right)\right. \\
&\left.-\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_2\right) h\left(\Lambda_4, \Lambda_3\right)+\tfrac{(c+1)-4 m}{2} \phi \eta\left(\Lambda_3\right) h\left(\Lambda_4, \Lambda_2\right)+\tfrac{(c+1)-4 m}{2}\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \Lambda_3\right)\right\}
\end{aligned} \tag{40}
$$

If we choose $\Lambda_3=\xi$ in (40) and use (15), we get

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \xi\right)+\phi h\left(\Lambda_4, \Lambda_2\right)=-\tfrac{(c+1)-4 m}{2} L_4\left\{\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\Lambda_2, \xi\right)\right. \\
&\left.+\phi h\left(\Lambda_4, \Lambda_2\right)\right\} .
\end{aligned} \tag{41}
$$

On the other hand, it is clear that

$$
\left(\tilde{\nabla}_{\Lambda_4} h\right)\left(\xi, \Lambda_2\right)=\phi h\left(\Lambda_2, \Lambda_4\right) . \tag{42}
$$

If (42) is written instead of (41), we obtain

$$
2 \phi h\left(\Lambda_2, \Lambda_4\right)=[4 m-(c+1)] L_4 \phi h\left(\Lambda_2, \Lambda_4\right) .
$$

It is clear from the last equality

$$
h\left(\Lambda_2, \Lambda_4\right)=0 \text { or } L_4=\frac{2}{4 m-(c+1)} .
$$

This completes the proof. ∎

4. Total geodesic submanifolds on concircular and projective curvature tensor

In this section, the invariant submanifold Ψ of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$ will be considered with the concircular and projective curvature tensor. The concircular curvature tensor is defined as

$$
\tilde{Z}\left(\Lambda_1, \Lambda_2\right) \Lambda_3=R\left(\Lambda_1, \Lambda_2\right) \Lambda_3-\frac{r}{2 m(2 m+1)}\left[g\left(\Lambda_2, \Lambda_3\right) \Lambda_1-g\left(\Lambda_1, \Lambda_3\right) \Lambda_2\right], \tag{43}
$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \chi\left(\tilde{\Psi}\right)$. If we choose $\Lambda_1=\Lambda_3=\xi$ in (43) and use (6), we get

$$
\tilde{Z}\left(\xi, \Lambda_2\right) \xi=-\left[1+\frac{r}{2 m(2 m+1)}\right]\left[-\eta\left(\Lambda_2\right) \xi+\Lambda_2\right] . \tag{44}
$$

Theorem 4.1. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ satisfies the condition $\tilde{Z}\left(\Lambda_1, \Lambda_2\right) h=L_5 Q(g, h)$,then Ψ is either total geodesic or $L_5=-\left(1+\frac{r}{2 m(2 m+1)}\right)$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(\tilde{Z}\left(\Lambda_1,\Lambda_2\right)h\right)\left(\Lambda_4,\Lambda_5\right)=L_5Q\left(g,h\right)\left(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2\right), \tag{45}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_5=\xi$ in (45) and use (15), we get

$$-h\left(\Lambda_4,\tilde{Z}\left(\xi,\Lambda_2\right)\xi\right)=-L_5h\left(\Lambda_4,\Lambda_2\right). \tag{46}$$

If we use (44) out of (46), we obtain

$$\left[\left(1+\frac{r}{2m\left(2m+1\right)}\right)+L_5\right]h\left(\Lambda_4,\Lambda_2\right)=0.$$

This completes the proof. ■

Theorem 4.2. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If Ψ satisfies the condition $\tilde{Z}\left(\Lambda_1,\Lambda_2\right)h=L_6Q\left(S,h\right)$, then Ψ is total geodesic or $L_6=\frac{2[r+2m(2m+1)]}{2m(2m+1)[(c+1)-4m]}$ and $(c+1)\neq 4m$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(\tilde{Z}\left(\Lambda_1,\Lambda_2\right)h\right)\left(\Lambda_4,\Lambda_5\right)=L_6Q\left(S,h\right)\left(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2\right), \tag{47}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_5=\xi$ in (47) and use (15), we get

$$-h\left(\Lambda_4,\tilde{Z}\left(\xi,\Lambda_2\right)\xi\right)=L_6S\left(\xi,\xi\right)h\left(\Lambda_4,\Lambda_2\right). \tag{48}$$

If we use (44) and (8) out of (48), we obtain

$$\left[\left(1+\frac{r}{2m\left(2m+1\right)}\right)+\left(\frac{(c+1)-4m}{2}\right)L_6\right]h\left(\Lambda_4,\Lambda_2\right)=0.$$

This completes the proof. ■

The projective curvature tensor is defined as

$$P\left(\Lambda_1,\Lambda_2\right)\Lambda_3=R\left(\Lambda_1,\Lambda_2\right)\Lambda_3-\frac{1}{2m}\left[S\left(\Lambda_2,\Lambda_3\right)\Lambda_1-S\left(\Lambda_1,\Lambda_3\right)\Lambda_2\right], \tag{49}$$

for all $\Lambda_1,\Lambda_2,\Lambda_3\in\chi\left(\tilde{\Psi}\right)$. If we choose $\Lambda_1=\Lambda_3=\xi$ in (49) and use (6), (8), we get

$$P\left(\xi,\Lambda_2\right)\xi=\frac{c+1}{4m}\left[\eta\left(\Lambda_2\right)\xi-\Lambda_2\right]. \tag{50}$$

Theorem 4.3. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}\left(c\right)$. If Ψ satisfies the condition $P\left(\Lambda_1,\Lambda_2\right)h=L_7Q\left(g,h\right)$, then Ψ is either total geodesic or $L_7=-\frac{c+1}{4m}$.*

Proof. Let's assume that Ψ satisfies the condition

$$\left(P\left(\Lambda_1,\Lambda_2\right)h\right)\left(\Lambda_4,\Lambda_5\right)=L_7Q\left(g,h\right)\left(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2\right), \tag{51}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5\in\Gamma\left(T\Psi\right)$. If we choose $\Lambda_1=\Lambda_5=\xi$ in (51) and use (15), we get

$$-h\left(\Lambda_4,P\left(\xi,\Lambda_2\right)\xi\right)=-L_7h\left(\Lambda_4,\Lambda_2\right). \tag{52}$$

If we use (50) out of (52), we obtain

$$\left[\frac{c+1}{4m}+L_7\right]h\left(\Lambda_4,\Lambda_2\right)=0.$$

This completes the proof. ■

Theorem 4.4. *Let Ψ be the invariant submanifold of the $(2m+1)-$dimensional $\mathcal{LSS}-$form $\tilde{\Psi}(c)$. If Ψ satisfies the condition $P(\Lambda_1,\Lambda_2)h = L_8 Q(S,h)$, then Ψ is either total geodesic or $L_8 = \frac{2(c+1)}{4m[4m-(c+1)]}$ and $(c+1) \neq 4m$.*

Proof. Let's assume that Ψ satisfies the condition

$$(P(\Lambda_1,\Lambda_2)h)(\Lambda_4,\Lambda_5) = L_8 Q(S,h)(\Lambda_4,\Lambda_5;\Lambda_1,\Lambda_2), \tag{53}$$

for all $\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5 \in \Gamma(T\Psi)$. If we choose $\Lambda_1 = \Lambda_5 = \xi$ in (53) and use (15), we get

$$-h(\Lambda_4, P(\xi,\Lambda_2)\xi) = L_8 S(\xi,\xi) h(\Lambda_4,\Lambda_2). \tag{54}$$

If we use (50) and (8) out of (54), we obtain

$$\left[\frac{c+1}{4m} + \frac{[(c+1)-4m]}{2} L_8\right] h(\Lambda_4,\Lambda_2) = 0.$$

This completes the proof. ■

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