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Approximation of time separating stochastic processes by neural networks revisited

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Abstract. Here we study the univariate quantitative approximation of time separating stochastic process over the whole real line by the normalized bell and squashing type neural network operators. Activation functions here are of compact support. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged stochastic function or its high order derivative. The approximations are pointwise and with respect to the Lp norm. The feed-forward neural networks are with one hidden layer. We finish with a great variety of special applications.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

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Contents

1. Introduction

The first author in [2] and [3], was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" activation functions are assumed to be of compact support. The functions under approximation were from the whole $\mathbb{R}$ into $\mathbb{R}$. Here we perform quantitative approximations of time separating stochastic processes by these neural network operators. We follow the above-described pattern

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and produce pointwise and L_p quantitative estimates. This article is a continuation of [4], where the activation functions had been over the whole real line.
We give several interesting applications. Specific motivations came by:

1. Stationary Gaussian processes with an explicit representation such as

$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$

   where ξ_1, ξ_2 are independent random variables with the standard normal distribution, see [6].

2. By the "Fourier model" of a stationary process, see [7].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \ x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network.

2. About Neural Networks Approximation

In this section we follow [3].

Definition 2.1. *(see [5]) A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

Example 2.2. (1) $b(x)$ *can be the characteristic function over* $[-1, 1]$.
(2) $b(x)$ *can be the hat function over* $[-1, 1]$*, i.e.,*

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

In the article we follow we study the pointwise convergence with rates over the real line, to the unit operator, of the "normalized bell type neural network operators",

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{1}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$. The terms in the ratio of sums (1) can be nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \leq T, \ \text{i.e.} \ \left| x - \frac{k}{n} \right| \leq \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{2}$$

In order to have the desired order of numbers

$$-n^2 \le nx - Tn^{\alpha} \le nx + Tn^{\alpha} \le n^2, \tag{3}$$

it is sufficient enough to assume that

$$n \ge T + |x|. \tag{4}$$

When $x \in [-T, T]$ it is enough to assume $n \ge 2T$ which implies (3).

Proposition 2.3. *Let $a \le b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \le card(k) \le (b-a)+1.$$

Note 2.4. *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 2.3 we get that*

$$card(k) \ge \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \ge 1$, if

$$2Tn^{\alpha} - 1 \ge 1 \text{ iff } n \ge T^{-\frac{1}{\alpha}}.$$

So to have the desired order (3) and $card(k) \ge 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \ge \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{5}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling. Here comes the first result we use.

Theorem 2.5. *([3], Ch.1) Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \ge \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(f))(x) - f(x)| \le \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right), \tag{6}$$

where ω_1 is the first modulus of continuity of f.

The second result we use follows.

Theorem 2.6. *([3], Ch.1) Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \ge \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(H_n(f))(x) - f(x)| \le \left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{7}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(7)$\to 0$, therefore L.H.S.(7)$\to 0$, i.e., (7) gives us with rates the pointwise convergence of $(H_n(f))(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Corollary 2.7. *([3], Ch.1) Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $x \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \ge \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \ge 1$. Then*

$$\|H_n(f) - f\|_{p,[-T^*,T^*]} \le \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{8}$$

From (8) we get the L_p convergence of $H_n(f)$ to f with rates.

Corollary 2.8. *([3], Ch.1) Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $x \in [-T^*,T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T+T^*, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p \geq 1$. Then*

$$\|H_n(f) - f\|_{p,[-T^*,T^*]} \leq \tag{9}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|f^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Here from (9) we get again the L_p convergence of $H_n(f)$ to f with rates.

2.1. The "Normalized Squashing Type Operators" and their Convergence to the Unit with Rates

We need

Definition 2.9. *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function" (see also [5]).*

Let $f : \mathbb{R} \to \mathbb{R}$ be either uniformly continuous or continuous and bounded.
For $x \in \mathbb{R}$ we define the "normalized squashing type operator"

$$(K_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}, \tag{10}$$

$0<\alpha<1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$(K_n(f))(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{W(x)}, \tag{11}$$

where

$$W(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).$$

Here we give the pointwise convergence with rates of $(K_n f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Theorem 2.10. *([3], Ch.1) Under the above terms and assumptions we obtain*

$$|K_n(f)(x) - f(x)| \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right). \tag{12}$$

We also give

Theorem 2.11. *([3], Ch.1) Let $x \in \mathbb{R}$, $T>0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(K_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{13}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(f)$ to f with rates.

Note 2.12. *The maps H_n, K_n are positive linear operators reproducing constants, in particular*

$$H_n(1) = K_n(1) = 1. \tag{14}$$

3. Time Seperating Stochastic Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on Ω with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : \mathbb{R} \to \mathbb{R}$, such that $h_i(t)$, $i = 1, 2, \ldots \ldots m$ are all uniformly continuous or $h_i(t)$ $i = 1, 2, \ldots \ldots m$ are all continuous and bounded for every $i = 1, 2, \ldots \ldots m$.
Clearly, then

$$Y(t, \omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega), t \in \mathbb{R}, \tag{15}$$

is a quite common stochastic process separating time.
We can assume that $h_i \in C^r(\mathbb{R}), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation

$$(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C(\mathbb{R}) \text{ or } C^r(\mathbb{R}). \tag{16}$$

A classical example of a stochastic process separating time is

$$(\sin t) Y_1(\omega) + (\cos t) Y_2(\omega), t \in \mathbb{R}.$$

Notice that $|\sin t| \leq 1$ and $|\cos t| \leq 1$.
Another typical example is

$$\sinh(t) Y_1(\omega) + \cosh(t) Y_2(\omega), t \in \mathbb{R}. \tag{17}$$

In this article we will apply the results of section 2, to $f(t) = (EY)(t)$. We will finish with several applications. See the related [6], [7].

4. Main Results

We present the following general approximation of the seperating stochastic processes by neural network operators.

Theorem 4.1. *Let $(EY)(t)$ as in (16), Let also $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(E(Y)))(t) - (E(Y))(t)| \leq \omega_1\left((E(Y)), \frac{T}{n^{1-\alpha}}\right), \tag{18}$$

where ω_1 is the first modulus of continuity of $E(Y)$.

Proof. $E(Y)$ are uniformly continuous or continuous and bounded in $\mathbb{R}$, Thus, the conclusion comes from Theorem 2.5. ∎

Our second main result follows.

Theorem 4.2. *Let $(EY)(t)$ as in (16), Let also $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(E(Y)))(t) - (E(Y))(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(E(Y))^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{19}$$

$$\omega_1\left((E(Y))^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(19)$\to 0$, therefore L.H.S.(19)$\to 0$, i.e., (19) gives us with rates the pointwise convergence of $(H_n(E(Y)))(t) \to (E(Y))(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. Notice that Let $(E(Y)) \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $(E(Y))^{(N)}$ is a uniformly continuous function or $(E(Y))^{(N)}$ is continuous and bounded. Thus, the conclusion comes from Theorem 2.5. ■

We continue with,

Corollary 4.3. *Let $(EY)(t)$ as in (16). Let also $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(E(Y)) - E(Y)\|_{p,[-T^*,T^*]} \leq \omega_1\left(E(Y)\, \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{20}$$

From (20) we get the L_p convergence of $H_n(E(Y))$ to $E(Y)$ with rates.

Corollary 4.4. *Let $(EY)(t)$ as in (16). Let also $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(E(Y)) - E(Y)\|_{p,[-T^*,T^*]} \leq \tag{21}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|(E(Y))^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left((E(Y))^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

We also give the next

Theorem 4.5. *Let $t \in \mathbb{R}$ and $(EY)(t)$ as in (16). Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10). We obtain*

$$|K_n(EY)(t) - (EY)(t)| \leq \omega_1\left(EY, \frac{T}{n^{1-\alpha}}\right). \tag{22}$$

Proof. From Theorem 2.10. ■

We also give

Theorem 4.6. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let also $(EY)(t)$ as in (16). Then*

$$|(K_n(EY))(t) - (EY)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EY)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{23}$$

$$\omega_1\left((EY)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n\,(EY)$ to (EY) with rates.

Proof. $(EY) \in C^N(\mathbb{R})$. Further more $(EY)^{(N)}$ is a uniformly continuous function or $(EY)^{(N)}$ is continuous and bounded. Hence the conclusion comes from Theorem 2.11. ■

5. Applications

For the next applications we consider (Ω, F, P) be a probability space and Y_1, Y_2 be real valued random variables on Ω with finite expectations. We consider the stochastic processes $Z_i(t,\omega)$ for $i = 1,2,3,4$ where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_1(t,\omega) = \sin(\xi t)\, Y_1(\omega) + \cos(\xi t)\, Y_2(\omega), \tag{24}$$

where $\xi > 0$ is fixed;

$$Z_2(t,\omega) = sech\,(\mu t)\, Y_1(\omega) + \tanh(\mu t)\, Y_2(\omega), \tag{25}$$

where $\mu > 0$ is fixed.
Here $sechx := \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, x \in \mathbb{R}$.

$$Z_3(t,\omega) = \frac{1}{1+e^{-\ell_1 t}} Y_1(\omega) + \frac{1}{1+e^{-\ell_2 t}} Y_2(\omega), \tag{26}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_4(t,\omega) = e^{-e^{-\mu_1 t}} Y_1(\omega) + e^{-e^{-\mu_2 t}} Y_2(\omega), \tag{27}$$

where $\mu_1, \mu_2 > 0$ are fixed;

The expectations of $Z_i, i = 1,2,3,4$ are

$$(EZ_1)(t) = \sin(\xi t)\, E(Y_1) + \cos(\xi t)\, E(Y_2), \tag{28}$$

$$(EZ_2)(t) = sech\,(\mu t)\, E(Y_1) + \tanh(\mu t)\, E(Y_2), \tag{29}$$

$$(EZ_3)(t) = \frac{1}{1+e^{-\ell_1 t}} E(Y_1) + \frac{1}{1+e^{-\ell_2 t}} E(Y_2), \tag{30}$$

$$(EZ_4)(t) = e^{-e^{-\mu_1 t}} E(Y_1) + e^{-e^{-\mu_2 t}} E(Y_2). \tag{31}$$

For the next $(EZ_i)(t), i = 1,2,3,4$ are as defined in relations between (28) and (31) respectively.
We present the following result.

Proposition 5.1. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1,2,3,4$*

$$|(H_n(EZ_i))(t) - (EZ_i)(t)| \leq \omega_1\left((EZ_i), \frac{T}{n^{1-\alpha}}\right), \tag{32}$$

where ω_1 is the first modulus of continuity of (EZ_i).

Proof. From Theorem 4.1. ■

We also give

Proposition 5.2. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$*

$$|(H_n(EZ_i))(t) - (EZ_i)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_i)^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{33}$$

$$\omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(33)$\to 0$, therefore L.H.S.(33)$\to 0$, i.e., (33) gives us with rates the pointwise convergence of $(H_n(EZ_i))(t) \to (EZ_i)(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. From Theorem 4.2. ■

We continue with,

Corollary 5.3. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$*

$$\|H_n(EZ_i) - (EZ_i)\|_{p,[-T^*,T^*]} \leq \omega_1\left((EZ_i)\ \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{34}$$

From (34) we get the L_p convergence of $H_n((EZ_i))$ to (EZ_i) with rates.

Corollary 5.4. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$*

$$\|H_n((EZ_i)) - (EZ_i)\|_{p,[-T^*,T^*]} \leq \tag{35}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|(EZ_i)^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Proposition 5.5. *Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10). Then for $i = 1, 2, 3, 4$ we obtain*

$$|K_n(EZ_i)(t) - (EZ_i)(t)| \leq \omega_1\left((EZ_i), \frac{T}{n^{1-\alpha}}\right). \tag{36}$$

Proof. From Theorem 4.5. ■

We also give

Proposition 5.6. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$*

$$|(K_n(EZ_i))(t) - (EZ_i)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_i)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{37}$$

$$\omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(EZ_i)$ to (EZ_i) with rates.

Proof. From Theorem 4.6. ■

6. Specific Applications

Let $(\Omega, \mathcal{F}, P)$, where Ω is the set of non-negative integers, be a probability space, $Y_{1,1}, Y_{2,1}$ be real-valued random variables on Ω following Poisson distributions with parameters $\lambda_1, \lambda_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,1}(t, \omega)$ for $i = 1, 2, 3, 4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,1}(t, \omega) = \sin(\xi t) Y_{1,1}(\omega) + \cos(\xi t) Y_{2,1}(\omega), \tag{38}$$

where $\xi > 0$ is fixed;

$$Z_{2,1}(t, \omega) = sech(\mu t) Y_{1,1}(\omega) + \tanh(\mu t) Y_{2,1}(\omega), \tag{39}$$

where $\mu > 0$ is fixed.

$$Z_{3,1}(t, \omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_{1,1}(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_{2,1}(\omega), \tag{40}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,1}(t, \omega) = e^{-e^{-\mu_1 t}} Y_{1,1}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,1}(\omega), \tag{41}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,1}) = \lambda_1$ and $E(Y_{2,1}) = \lambda_2$, the expectations of $Z_{i,1}, i = 1, 2, 3, 4,$ are

$$(EZ_{1,1})(t) = \lambda_1 \sin(\xi t) + \lambda_2 cos(\xi t), \tag{42}$$

$$(EZ_{2,1})(t) = \lambda_1 sech(\mu t) + \lambda_2 \tanh(\mu t), \tag{43}$$

$$(EZ_{3,1})(t) = \frac{\lambda_1}{1 + e^{-\ell_1 t}} + \frac{\lambda_2}{1 + e^{-\ell_2 t}}, \tag{44}$$

$$(EZ_{4,1})(t) = \lambda_1 e^{-e^{-\mu_1 t}} + \lambda_2 e^{-e^{-\mu_2 t}}. \tag{45}$$

For the next we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{R}$, be a probability space, $Y_{1,2}, Y_{2,2}$ be real-valued random variables on Ω following Gaussian distributions with expectations $\hat{\mu}_1, \hat{\mu}_2 \in \mathbb{R}$ respectively.
We consider the stochastic processes $Z_{i,2}(t, \omega)$ for $i = 1, 2, 3, 4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,2}(t, \omega) = \sin(\xi t) Y_{1,2}(\omega) + \cos(\xi t) Y_{2,2}(\omega), \tag{46}$$

where $\xi > 0$ is fixed;

$$Z_{2,2}(t, \omega) = sech(\mu t) Y_{1,2}(\omega) + \tanh(\mu t) Y_{2,2}(\omega), \tag{47}$$

where $\mu > 0$ is fixed.

$$Z_{3,2}(t, \omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_{1,2}(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_{2,2}(\omega), \tag{48}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,2}(t, \omega) = e^{-e^{-\mu_1 t}} Y_{1,2}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,2}(\omega), \tag{49}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,2}) = \hat{\mu}_1$ and $E(Y_{2,2}) = \hat{\mu}_2$, The expectations of $Z_{i,2}, i = 1, 2, 3, 5$ are

$$(EZ_{1,2})(t) = \hat{\mu}_1 \sin(\xi t) + \hat{\mu}_2 cos(\xi t), \tag{50}$$

$$(EZ_{2,2})(t) = \hat{\mu}_1 sech(\mu t) + \hat{\mu}_2 \tanh(\mu t), \tag{51}$$

$$(EZ_{3,2})(t) = \frac{\hat{\mu}_1}{1 + e^{-\ell_1 t}} + \frac{\hat{\mu}_2}{1 + e^{-\ell_2 t}}, \tag{52}$$

$$(EZ_{4,2})(t) = \hat{\mu}_1 e^{-e^{-\mu_1 t}} + \hat{\mu}_2 e^{-e^{-\mu_2 t}}. \tag{53}$$

Furthermore, we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on Ω following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,3}(t, \omega)$ for $i = 1, 2, 3, 4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,3}(t, \omega) = \sin(\xi t) Y_{1,3}(\omega) + \cos(\xi t) Y_{2,3}(\omega), \tag{54}$$

where $\xi > 0$ is fixed;

$$Z_{2,3}(t, \omega) = sech(\mu t) Y_{1,3}(\omega) + \tanh(\mu t) Y_{2,3}(\omega), \tag{55}$$

where $\mu > 0$ is fixed.

$$Z_{3,3}(t, \omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_{1,3}(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_{2,3}(\omega), \tag{56}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,3}(t, \omega) = e^{-e^{-\mu_1 t}} Y_{1,3}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,3}(\omega), \tag{57}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,3}) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)$ and $E(Y_{2,3}) = \Gamma\left(1 + \frac{1}{\gamma_2}\right)$, where $\Gamma(\cdot)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1, 2, 3, 4$, are

$$(EZ_{1,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \sin(\xi t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right) cos(\xi t), \tag{58}$$

$$(EZ_{2,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) sech(\mu t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \tanh(\mu t), \tag{59}$$

$$(EZ_{3,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \frac{1}{1 + e^{-\ell_1 t}} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \frac{1}{1 + e^{-\ell_2 t}}, \tag{60}$$

$$(EZ_{4,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) e^{-e^{-\mu_1 t}} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) e^{-e^{-\mu_2 t}}. \tag{61}$$

We present the following result.

Proposition 6.1. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$|(H_n(EZ_{i,k}))(t) - (EZ_{i,k})(t)| \leq \omega_1\left((EZ_{i,k}), \frac{T}{n^{1-\alpha}}\right), \tag{62}$$

where ω_1 is the first modulus of continuity of $(EZ_{i,k})$.

Proof. From Proposition 5.1. ∎

We also give

Proposition 6.2. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$|(H_n(EZ_{i,k}))(t) - (EZ_{i,k})(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_{i,k})^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{63}$$

$$\omega_1\left((EZ_{i,k})^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(63)$\to 0$, therefore L.H.S.(63)$\to 0$, i.e., (63) gives us with rates the pointwise convergence of $(H_n(EZ_{i,k}))(t) \to (EZ_{i,k})(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. From Proposition 5.2. ■

We continue with,

Corollary 6.3. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $t \in [-T^*,T^*]$, $T^*>0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T+T^*, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p \geq 1$. Then for $i=1,2,3,4$ and $k=1,2,3$*

$$\left\| H_n\left(EZ_{i,k}\right) - \left(EZ_{i,k}\right)\right\|_{p,[-T^*,T^*]} \leq \omega_1\left(\left(EZ_{i,k}\right)\ \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{64}$$

From (64) we get the L_p convergence of $H_n\left(\left(EZ_i\right)\right)$ to $\left(EZ_{i,k}\right)$ with rates.

Corollary 6.4. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $t \in [-T^*,T^*]$, $T^*>0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T+T^*, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p \geq 1$. Then for $i=1,2,3,4$ and $k=1,2,3$*

$$\left\| H_n\left(\left(EZ_{i,k}\right)\right) - \left(EZ_{i,k}\right)\right\|_{p,[-T^*,T^*]} \leq \tag{65}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|\left(EZ_{i,k}\right)^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left(\left(EZ_{i,k}\right)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Proposition 6.5. *Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10), for $i=1,2,3,4$ and $k=1,2,3$ we obtain*

$$\left|K_n\left(EZ_{i,k}\right)(t) - \left(EZ_{i,k}\right)(t)\right| \leq \omega_1\left(\left(EZ_{i,k}\right), \frac{T}{n^{1-\alpha}}\right). \tag{66}$$

Proof. From Proposition 5.5. ■

We also give

Proposition 6.6. *Let $t \in \mathbb{R}$, $T>0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i=1,2,3,4$ and $k=1,2,3$*

$$\left|\left(K_n\left(EZ_{i,k}\right)\right)(t) - \left(EZ_{i,k}\right)(t)\right| \leq \left(\sum_{j=1}^{N} \frac{\left|\left(EZ_{i,k}\right)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{67}$$

$$\omega_1\left(\left(EZ_{i,k}\right)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n\left(EZ_{i,k}\right)$ to $\left(EZ_{i,k}\right)$ with rates.

Proof. From Proposition 5.6. ■

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