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Coherent ideals of 1-distributive lattices

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Abstract. In this paper, we study coherent ideals of pseudocomplemented 1-distributive lattices. We give a set of conditions for an ideal to be a coherent ideal. We also prove some conditions for a pseudocomplemented 1-distributive lattice to be weakly Stone lattice.

AMS Subject Classifications: 06A12, 06A99, 06B10.

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1. Introduction and Background

J. C. Varlet [3] have studied the generalizations of the notion of pseudocomplementedness. W. H. Cornish [6] have studied congruences of pseudocomplemented distributive lattice and ideals of pseudocomplemented semilattices are studied by T. S. Blyth [2]. M. S. Rao [1] studies cohenrent ideals and median prime ideals for pseudocomplemented distributive lattices. In this article we generalize some of these results for pseudocomplemented 1-distributive lattices.

Definition 1.1. *A lattice L with 1 is called 1-distributive if for any $p, q, r \in L$, $p \vee q = 1 = p \vee r$ implies $p \vee (q \wedge r) = 1$.*

The pentagonal lattice P_5 (see the diagram in Figure 1) is 1-distributive but not distributive. Thus, not every 1-distributive lattice is a distributive lattice. The diamond lattice M_3 (see the diagram in Figure 1) is not 1-distributive.

Definition 1.2. *In a 1-distributive lattice L for all $p \in L$*

$$q \leqslant p^* \text{ if and only if } p \wedge q = 0,$$

then the element p^* is called the pseudocomplement of p.

Definition 1.3. *Let L be a 1-distributive lattice. L is called a pseudocomplemented 1-distributive lattice if every element in L has a pseudocomplement.*

Now we discuss about some basic definitions and properties of pseudocomplemented 1-distributive lattices.

Definition 1.4. *Let L be a pseudocomplemented 1-distributive lattice and I be a non-empty subset of L. I is called an ideal if*

(i) $p \in L$, $q \in I$ *with* $p \leqslant q$ *implies* $p \in I$,

(ii) $p, q \in I$ *implies* $p \vee q \in I$.

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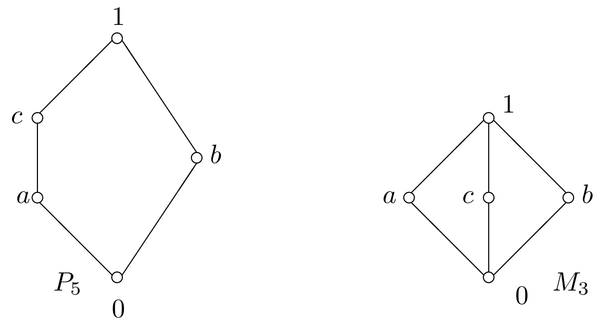

Figure 1: The pentagonal lattice and the diamond lattice

Definition 1.5. *Let L be a pseudocomplemented 1-distributive lattice and I be an ideal of L, I is called a proper ideal if $I \neq L$.*

Definition 1.6. *Let L be a 1-distributive lattice, a proper ideal I of L is called a minimal ideal if I is not belonging to any other proper ideal, that is, if there exists a proper ideal J of L such that $J \subseteq I$, then $I = J$.*

Definition 1.7. *Let L be a pseudocomplemented 1-distributive lattice and P be an ideal. P is called a prime ideal if for any $a, b \in L$ with $a \wedge b \in P$ implies that $a \in P$ or $b \in P$.*

Definition 1.8. *Let L be a pseudocomplemented 1-distributive lattice and I be an ideal of L. Then I is said to be a *-ideal if $x^{**} \in I$ for every $x \in I$.*

Definition 1.9. *Let L be a 1-distributive lattice, an element $a \in L$ is called dense if $a^* = 0$. The set of all dense elements is denoted by $D(L)$.*

The following well known identities (see [2, 4–6]) are used throughout this paper.

(1) $a \leq b$ implies $b^* \leq a^*$.

(2) $a \leq a^{**}$

(3) $a = a^{***}$

(4) $(a \vee b)^* = a^* \wedge b^*$

(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(6) $a \wedge (a \wedge b)^* = a \wedge b^*$.

The identity (6) is used rarely (see [2] for semilattices and see [7] for lattices). For the background of 1-distributive lattices, we refer the reader to [8, 9].

In Section 2, we give the definition of coherent ideal of a 1-distributive lattice. We prove some conditions for a ideal to be a coherent ideal. We discuss about Stone lattices and weakly Stone lattices. We prove some conditions for pseudocomplemented 1-distributive lattice to be weakly Stone lattice.

2. Main Results

Let L be a pseudocomplemented 1-distributive lattice and A be any non-empty subset of L. We define the following set:

$$A^{\tau} = \{x \in L \mid a^* \vee x^* = 1 \text{ for all } a \in A\}.$$

From the definition it can be easily said that, $\{0\}^\tau = L$ and $L^\tau = \{0\}$.

M. Sambasiva Rao (see [1]) proved that A^τ is an ideal for distributive pseudocomplemented lattice. We see that this theorem is also true for P_5 (see Figure 1), which is not distributive. So we have the following theorem.

Theorem 2.1. *Let L be a pseudocomplemented 1-distributive lattice and A be any non-empty subset of L. Then A^τ is an ideal of L.*

Proof. Clearly $0 \in A^\tau$. Let $x, y \in A^\tau$. Then $x^* \vee a^* = 1$ and $y^* \vee a^* = 1$ for all $a \in A$. As L is a 1-distributive lattice, we have $a^* \vee (x^* \wedge y^*) = 1$. Thus $a^* \vee (x \vee y)^* = 1$ and so $x \vee y \in A^\tau$.

Now let $y \in L$ and $x \in A^\tau$ with $y \leq x$. Then $x^* \leq y^*$ and hence $1 = x^* \vee a^* \leq y^* \vee a^*$. So $y \in A^\tau$ and A^τ is an ideal. ■

Remark 2.2. *If $A \cap A^\tau \neq \phi$ then $A \cap A^\tau = \{0\}$. Because if, $t \in A \cap A^\tau$ then $t^* \vee t^* = 1$. This implies $t = 0$ and so $A \cap A^\tau = \{0\}$.*

Now we have the following identities.

Theorem 2.3. *Let A and B be any two non-empty subsets of a pseudocomplemented 1-distributive lattice. Then*

(i) $A \subseteq B$ implies that $B^\tau \subseteq A^\tau$;

(ii) $A \subseteq A^{\tau\tau}$;

(iii) $A^\tau = A^{\tau\tau\tau}$;

(iv) $A^\tau = L$ if and only if $A = \{0\}$.

Proof. (i) Let $A \subseteq B$ and let $x \in B^\tau$. Then $x^* \vee b^* = 1$ for all $b \in B$. Since $A \subseteq B$, this implies $x^* \vee a^* = 1$ for all $a \in A$ and hence $x \in A^\tau$.

(ii) Let $x \in A$. Then if, $a \in A^\tau$ we have $x^* \vee a^* = 1$. So $x \in A^{\tau\tau}$.

(iii) From (ii), we can write $A^\tau \subseteq A^{\tau\tau\tau}$. Let $t \in A^{\tau\tau\tau}$. Then $t^* \vee a^* = 1$ for all $a \in A^{\tau\tau}$ and this implies $t \in A^\tau$.

(iv) Let $A^\tau = L$ and $x \in A^\tau$. This implies $x^* \vee a^* = 1$ for all $a \in A$. This implies $a^* = 1$ for all $a \in A$. So $A = \{0\}$. The reverse inclusion is obvious. ■

Now we have this following theorem.

Theorem 2.4. *Let L be a pseudocomplemented 1-distributive lattice, I and J be any two ideals of L. Then $(I \vee J)^\tau = I^\tau \cap J^\tau$.*

Proof. Clearly $(I \vee J)^\tau \subseteq I^\tau \cap J^\tau$. To prove $I^\tau \cap J^\tau \subseteq (I \vee J)^\tau$, let $x \in I^\tau \cap J^\tau$ and let $t \in I \vee J$. Then $x^* \vee i^* = 1 = x^* \vee j^*$ and $t \leq i \vee j$ for some $i \in I$ and $j \in J$. As L is 1-distributive, we have $x^* \vee (i^* \wedge j^*) = 1$ and this implies $x^* \vee (i \vee j)^* = 1$. Since $t \leq i \vee j$ implies $(i \vee j)^* \leq t^*$, we have $x^* \vee t^* = 1$. Hence $x \in (I \vee J)^\tau$. This completes the proof. ■

Now we have the following corollary.

Corollary 2.5. *Let L be a pseudocomplemented 1-distributive lattice and let $a, b \in L$, then we have*

(i) $a \leq b$ implies that $(b)^\tau \subseteq (a)^\tau$;

(ii) $(a \vee b)^\tau = (a)^\tau \cap (b)^\tau$;

(iii) $(a)^\tau = L$ if and only if $a = 0$;

(iv) $a \in (b)^\tau$ implies $a \wedge b = 0$;

(v) $a^* = b^*$ *implies* $(a)^\tau = (b)^\tau$*;*

(vi) $a \in D(L)$ *implies* $(a)^\tau = \{0\}$.

Definition 2.6. *Let* L *be an pseudocomplemented* 1*-distributive lattice. An element* $a \in L$ *is called closed if* $a = a^{**}$*. The set of all closed elements of* L *is denoted by* $B(L)$*. Thus*

$$B(L) = \{a \in L \mid a = a^{**}\}.$$

Clearly, $0, 1 \in B(L)$.

Definition 2.7. *Let* L *be a pseudocomplemented* 1*-distributive lattice, an element* $a \in L$ *is said to be a Stone element if it satisfies the Stone identity:*

$$a^* \vee a^{**} = 1$$

The set of all Stone elements of L *is denoted by* $S(L)$*. Thus*

$$S(L) = \{a \in L \mid a^* \vee a^{**} = 1\}.$$

.

Definition 2.8. *A pseudocomplemented* 1*-distributive lattice* L *is called Stone lattice if* $a^* \vee a^{**} = 1$ *for all* $a \in L$.

Definition 2.9. *Let* L *be a pseudocomplemented* 1*-distributive lattice and* A *be any non-empty subset of* L*. Define*

$$A^\perp = \{x \in L \mid x \wedge a = 0 \text{ for all } a \in A\}.$$

The set $A^\perp$ *is called the annihilator of* A*. If* $a \in A$ *then the annihilator of* $\{a\}$ *is denoted by* $a^\perp$ *and defined as*

$$a^\perp = \{x \in L \mid x \wedge a = 0\}$$

Now we have the following lemma.

Lemma 2.10. *Let* L *be a pseudocomplemented* 1*-distributive lattice. Then* $A^\perp$ *is an ideal of* L *for any non-empty subset* A *of* L.

Proof. Let L be a pseudocomplemented 1-distributive lattice and $A \subseteq L$. Then $A^\perp = \{x \in L \mid x \wedge a = 0 \text{ for all } a \in A\}$ is the annihilator of A.

Let $p, q \in A^\perp$. So $p \wedge a = 0$ and $q \wedge a = 0$ for all $a \in A$. This implies $p \leq a^*$ and $q \leq a^*$ and thus $p \vee q \leq a^*$. So $(p \vee q) \wedge a = 0$ and thus $p \vee q \in I$. Again let $p \in A^\perp$ and $t \in L$ with $t \leq p$. Thus $t \wedge a \leq p \wedge a = 0$ implies $t \in A^\perp$.

■

Definition 2.11. *Let* L *be a pseudocomplemented* 1*-distributive lattice and* I *is an ideal of* L*. Then* I *is called annihilator ideal if* $I = A^\perp$*, for any nonempty subset* A *of* L.

Now we have the following theorem.

Theorem 2.12. *Let* L *be a pseudocomplemented* 1*-distributive lattice,* I *be an ideal of* L *such that* $I = A^\perp$ *where* $A^\perp$ *is annihilator of* $A \subseteq L$*. Then*

(i) for any ideal J *of* L*,* $I \cap J = \{0\}$ *if and only if* $J \subseteq I^\perp$*;*

(ii) $I \cap I^\perp = \{0\}$*;*

(iii) for any ideal J of L, $J \subseteq I$ implies $I^{\perp} \subseteq J^{\perp}$;

(iv) $I = I^{\perp\perp}$.

Proof. (i) Let $I \cap J = \{0\}$ and $b \in J$. Then $b \wedge a = 0$ for all $a \in I$. This implies $b \in I^{\perp}$. So $J \subseteq I^{\perp}$. Conversely let $J \subseteq I^{\perp}$ and let $t \in I^{\perp}$. Thus $t \wedge i = 0$ for all $i \in I$. So $j \wedge i = 0$ for all $j \in J$. So $I \cap J = \{0\}$.

(ii) Let $t \in I \cap I^{\perp}$. This implies $t \in I$ and $t \wedge i = 0$ for all $i \in I$. Hence $t = 0$.

(iii) Let $J \subseteq I$ and let $t \in I^{\perp}$. This implies $t \wedge i = 0$ for all $i \in I$ and thus $t \wedge j = 0$ for all $j \in J$. Thus $t \in J^{\perp}$.

(iv) Using condition (i) and (ii), $A \subseteq A^{\perp\perp}$. Then by (iii), $A^{\perp\perp\perp} \subseteq A^{\perp}$. Again by (ii), $A^{\perp} \cap A^{\perp\perp} = \{0\}$ and by (i) $A^{\perp} \subseteq A^{\perp\perp\perp}$. So $A^{\perp} = A^{\perp\perp\perp}$. Thus $I = I^{\perp\perp}$. ■

Now we have the following lemma.

Lemma 2.13. *Let L be a pseudocomplemented 1-distributive lattice and I and J are two annihilator ideals of L. Then $I \vee J = (I^{\perp} \cap J^{\perp})^{\perp}$.*

Proof. Obviously $(I^{\perp} \cap J^{\perp}) \subseteq J^{\perp}$ and $(I^{\perp} \cap J^{\perp}) \subseteq J^{\perp}$. Then by Theorem 2.12, $I = I^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$ and $J = J^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$. Hence $I \vee J \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$.

Now let K be another annihilator ideal of L containing I and J. Then we have by Theorem 2.12, $K^{\perp} \subseteq I^{\perp}$ and $K^{\perp} \subseteq J^{\perp}$. So $(I^{\perp} \cap J^{\perp})^{\perp} \subseteq K^{\perp\perp} = K$. Thus $(I^{\perp} \cap J^{\perp})^{\perp}$ is the smallest annihilator ideal of L containing I and J. So $I \vee J = (I^{\perp} \cap J^{\perp})^{\perp}$. ■

Now we have this nice result.

Theorem 2.14. *Let L be a pseudocomplemented 1-distributive lattice and let $a, b \in L$. Then the following conditions are equivalent in L:*

(i) L is Stone lattice;

(ii) for any ideal I of L, $I^{\tau} = I^{\perp}$;

(iii) for $a \in L$, $(a)^{\tau} = a^{\perp}$;

(iv) for any two ideals I, J of L, $I \cap J = \{0\}$ if and only if $I \subseteq J^{\tau}$;

(v) for $a, b \in L$, $a \wedge b = 0$ implies $a^* \vee b^* = 1$;

(vi) for $a \in L$, $(a)^{\tau\tau} = (a^*)^{\tau}$.

Proof. $(i) \Rightarrow (ii)$: Let I be an ideal of a Stone lattice L. Clearly $I^{\tau} \subseteq I^{\perp}$. To prove the converse part, let $x \in I^{\perp}$. Then $x \wedge y = 0$ for all $y \in I$. So we have $x \leq y^*$ and thus $x^{**} \leq y^{***} = y^*$. Since L is Stone lattice, we have $1 = x^* \vee x^{**}$ and as $x^* \vee x^{**} \leq x^* \vee y^*$, we get $1 \leq x^* \vee y^*$. Hence $x \in I^{\tau}$.

$(ii) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (iv)$: It is obvious by Theorem 2.12.

$(iv) \Rightarrow (v)$: Let $a, b \in L$ with $a \wedge b = 0$. This implies $(a] \cap (b] = \{0\}$ and so $(a] \subseteq (b]^{\tau}$. Hence $a \in (b]^{\tau}$ and $a^* \vee b^* = 1$.

$(v) \Rightarrow (vi)$: Let $a \in L$. Since $a \wedge a^* = 0$, we have $a^* \vee a^{**} = 1$. So $a^* \in (a)^{\tau}$. Hence $(a)^{\tau\tau} \subseteq (a^*)^{\tau}$. Conversely let $x \in (a^*)^{\tau}$ and and $t \in (a)^{\tau}$. Now $t \in (a)^{\tau}$ implies $t^* \vee a^* = 1$ and $a^{**} \wedge t^{**} = 0$ and so $t^{**} \leq a^*$.

Therefore we have

$$\begin{aligned} & x \in (a^*)^\tau \\ \Rightarrow & x^* \vee a^{**} = 1 \\ \Rightarrow & x^{**} \wedge a^* = 0 \\ \Rightarrow & x^{**} \wedge t^{**} = 0 \\ \Rightarrow & t \wedge x = 0. \end{aligned}$$

Thus from condition (v), we have $t^* \vee x^* = 1$ for all $t \in (a)^\tau$. Hence $x \in (a)^{\tau\tau}$.

$(vi) \Rightarrow (i)$: Let $a \in L$. Since $(a)^{\tau\tau} = (a^*)^\tau$ and $a \in (a)^{\tau\tau}$, we have $a^* \vee a^{**} = 1$. This completes the proof. ■

Now we give the definition of coherent ideal of a pseudocomplemented 1-distributive lattice.

Definition 2.15. *Let L be a pseudocomplemented 1-distributive lattice and I be an ideal of L. Then I is called a coherent ideal, if for all $x, y \in L$, $(x^\tau) = (y)^\tau$ and $x \in I$ implies that $y \in I$.*

It is quiet easy to prove that $(x)^\tau$ is a coherent ideal for all $x \in L$.

Lemma 2.16. *Let L be a pseudocomplemented 1-distributive lattice and I be an ideal of L, then I is a coherent ideal if for any $x \in I$, $(x)^{\tau\tau} \subseteq I$.*

Proof. From theorem 2.14, $(x)^{\tau\tau} = (x^*)^\tau$. If $x \in I$ and $a \in (x)^{\tau\tau} = (x^*)^\tau$, we have $a \leq a^{**} \leq x^{**} \in I$. Hence $(x)^{\tau\tau} \subseteq I$. ■

Now we have the following result.

Theorem 2.17. *Let L be a pseudocomplemented 1-distributive lattice, then the following conditions are equivalent in L:*

(i) L is Boolean lattice;

(ii) every principle ideal is a coherent ideal;

(iii) every ideal is a coherent ideal;

(iv) every prime ideal is a coherent ideal;

(v) for $a, b \in L$, $(a)^\tau = (b)^\tau$ implies $a = b$;

(vi) for $a, b \in L$, $a^* = b^*$ implies $a = b$.

Proof. $(i) \Rightarrow (ii)$: Suppose L is Boolean. This implies every element $x \in L$ is closed, that is $x = x^{**}$. Let $(x]$ be a principal ideal and let $a, b \in L$ with $(a)^\tau = (b)^\tau$. Let $a \in (x]$. We have to prove that $b \in (x]$. Now $a \vee a^* = 1$ implies that $x \vee a^* = 1$ since $a \in (x]$. This implies $a^* \vee x^{**} = 1$. Hence

$$\begin{aligned} & a^* \vee x^{**} = 1 \\ \Rightarrow & x^* \in (a)^\tau = (b)^\tau \\ \Rightarrow & b^* \vee x^{**} = 1 \\ \Rightarrow & b^{**} \wedge x^* = 0 \\ \Rightarrow & b^{**} \leq x^{**} \\ \Rightarrow & b \leq x^{**}. \end{aligned}$$

So $b \in (x^{**}] = (x]$.

$(ii) \Rightarrow (iii)$: Let $a, b \in L$ and I is an ideal. Also let $a \in I$ and $(a)^\tau = (b)^\tau$. So $(a] \subseteq I$. By (ii), $(a]$ is a coherent ideal for all $a \in L$, so we have $b \in (a]$. So $b \in I$.

$(iii) \Rightarrow (iv)$: It is obviuos.

$(iv) \Rightarrow (v)$: Suppose (iv) holds. Let $a, b \in L$ with $(a)^\tau = (b)^\tau$. If $a \neq b$, there exists a prime ideal P such that $a \in P$ but $b \notin P$. But P is coherent, so we have $b \in I$, which is a contradiction. So $a = b$.

$(v) \Rightarrow (vi)$: By corollary 2.5, this condition holds.

$(vi) \Rightarrow (i)$: Let (vi) holds. So there exists unique complement for all $x \in L$. Hence (i) holds. ■

Definition 2.18. *For any ideal I of a pseudocomplemented 1-distributive lattice L, define $\xi(I)$ as follows*

$$\xi(I) = \{x \in L \mid (x)^\tau \vee I = L\}.$$

Now we have the following lemma which is very useful for studying coherent ideal.

Lemma 2.19. *Let L be a pseudocomplemented 1-distributive lattice and I, J, K be ideals of L. If $I \vee J = L$ and $I \vee K = L$, then $I \vee (J \cap K) = L$.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and let I, J, K be ideals of L. Consider $I \vee J = L$ and $I \vee K = L$. So $i_1 \vee j_1 = 1$ for some $i_1 \in I$ and $j_1 \in J$ and $i_2 \vee k_1 = 1$ for some $i_2 \in I$ and $k_1 \in K$. This implies

$$i_1 \vee i_2 \vee j_1 = 1 \text{ and } i_1 \vee i_2 \vee k_1 = 1.$$

As L is 1-distributive we have $(i_1 \vee i_2) \vee (j_1 \wedge k_1) = 1$ where $i_1 \vee i_2 \in I$ and $j_1 \wedge k_1 \in J \cap K$ (Since $j_1 \wedge k_1 \leq j_1 \in J$ and $j_1 \wedge k_1 \leq k_1 \in K$). This implies $(i_1 \vee i_2) \vee (j_1 \wedge k_1) \in I \vee (J \cap K)$. So $1 \in I \vee (J \cap K)$ and thus $I \vee (J \cap K) = L$. ■

Now we have the following theorem.

Theorem 2.20. *Let L be a pseudocomplemented 1-distributive latticee and I be an ideal of L. Then $\xi(I) = \{x \in L \mid (x)^\tau \vee I = L\}$ is an ideal.*

Proof. Obviously $0 \in \xi(I)$. Let $a, b \in \xi(I)$. This implies $(a)^\tau \vee I = L$ and $(b)^\tau \vee I = L$. Then from Lemma 2.4 and Lemma 2.19, we have $(a \vee b)^\tau \vee I = ((a)^\tau \cap (b)^\tau) \vee I = L$. So $a \vee b \in \xi(I)$.

Now consider, $a \in \xi(I)$ with $b \leq a$ for $a, b \in L$. So $((a)^\tau \vee I) = L$. Then by 2.5, we have $(a)^\tau \subseteq (b)^\tau$ and so $((a)^\tau \vee I) \subseteq ((b)^\tau \vee I) = L$. So $b \in \xi(I)$. This completes the proof. ■

Observe that, if we consider $I = (a]$ in P_5(see Figure 1), we have $\xi(I) = (b]$, which is not a subset of $(a]$. Now we have the following lemma.

Lemma 2.21. *For any *-ideal I of a pseudocomplemented 1-distributive lattice L, $\xi(I) \subseteq I$.*

Proof. Let $x \in \xi(I)$. Then $x \leq a \vee b$ for any $a \in (x)^\tau$ and $b \in I$. Since $a \in (x)^\tau$, we get $x \wedge a = 0$. Also $b \in I$ implies $x \wedge b \in I$. Now $x \leq a \vee b$ implies $x = x \wedge (a \vee b)$. So $x^{**} = x^{**} \wedge (a \vee b)^{**} = x^{**} \wedge (a^* \wedge b^*)^*$. Since $x \wedge a = 0$ implies $a^* \geq x^{**}$, we have $(a^* \wedge b^*)^* \leq (x^{**} \wedge b^*)^*$. Therefore $x^{**} \leq x^{**} \wedge (x^{**} \wedge b^*)^*$. Thus $x^{**} \leq x^{**} \wedge b^{**}$ since $a \wedge (a \wedge b)^* = a \wedge b^*$. Now $x \wedge b \in I$ and I is *-ideal, so this implies $x^{**} \leq (x \wedge b)^{**} \in I$. Hence $x \in I$ and consequently $\xi(I) \subseteq I$. ■

Definition 2.22. *An ideal I of a pseudocomplemented 1-distributive lattice L is called strongly coherent if $I = \xi(I)$ and is called τ-closed if $I = I^{\tau\tau}$.*

Observe that, $(c]$ in P_5(see Figure 1) is a strongly coherent ideal. Also observe that $(0]$ is the smallest τ-closed ideal and L is the largest τ-closed ideal.

It is easy to prove that every strongly coherent ideal is a coherent ideal. Now we have the following lemma.

Lemma 2.23. *Every τ-closed ideal of a pseudocomplemented 1-distributive lattice L is a coherent ideal.*

Proof. Let I be a τ-closed ideal of L. Let $x, y \in L$ and $(x)^\tau = (y)^\tau$. Let $x \in I$ and so $(x)^{\tau\tau} \subseteq I^{\tau\tau}$. Then from 2.3, we have $y \in (y)^{\tau\tau} = (x)^{\tau\tau} \subseteq I^{\tau\tau} = I$. Hence I is coherent. ■

Definition 2.24. *A pseudocomplemented 1-distributive lattice L is called weakly Stone if $(x)^\tau \vee (x)^{\tau\tau} = L$ for all $x \in L$.*

Theorem 2.25. *Let L be a pseudocomplemented 1-distributive Stone lattice, then L is weakly Stone lattice.*

Proof. Let L be a pseudocomplemented 1-distributive lattice and let L is Stone. For all $x \in L$, we have $x^* \vee x^{**} = 1$. Then from Lemma 2.13 and Theorem 2.12, $x^\perp \vee x^{\perp\perp} = (x^{\perp\perp} \cap x^{\perp\perp\perp})^\perp = (x^{\perp\perp} \cap x^\perp)^\perp = (0)^\perp = L$. Now let $a \in L$. Then $a = b \vee c$ for some $b \in x^\perp$ and $c \in x^{\perp\perp}$. Now from theorem 2.14, we have $x^\perp = (x)^\tau$. This implies $a \in (x)^\tau$ and $c \wedge t = 0$ for $t \in x^\perp = (x)^\tau$. Thus from theorem 2.14, we have $c \wedge t = 0$ implies $c^* \vee t^* = 1$ for $t \in (x)^\tau$. So $c \in (x)^{\tau\tau}$. Hence we have $a \in (x)^\tau \vee (x)^{\tau\tau}$ and so $L \subseteq (x)^\tau \vee (x)^{\tau\tau}$. This completes the proof. ■

M. S. Rao (see [1]) showed that the converse part of the above theorem is not true for distributive lattices and so it is not true for 1-distributive lattices also. We conclude this article with the following nice result.

Theorem 2.26. *Let L be a pseudocomplemented 1-distributive lattice. Then the following conditions are equivalent in L:*

(i) L is a weakly Stone lattice;

(ii) every τ-closed ideal of L is strongly coherent;

(iii) for each $x \in L$, $(x)^{\tau\tau}$ is strongly coherent.

Proof. $(i) \Rightarrow (ii)$: Consider L is weakly Stone lattice and I is a τ-closed ideal of L. Then $I^{\tau\tau} = I$. Let $x \in I^\tau$ and as $x^{***} = x^*$, we have $x^{**} \in I^\tau$. So I^τ is p-ideal and by 2.20, $\xi(I) \subseteq I$.

Conversely let $x \in I$. Then using theorem 2.3, we can easily prove that $(x)^{\tau\tau} \subseteq I^{\tau\tau}$. This implies $L = (x)^\tau \vee (x)^{\tau\tau} \subseteq (x)^\tau \vee I^{\tau\tau} = (x)^\tau \vee I$. So $x \in \xi(I)$. Therefore I is strongly coherent ideal.

$(ii) \Rightarrow (iii)$: Since for $x \in L$, $(x)^{\tau\tau\tau} = (x)^\tau$ and so $(x)^{\tau\tau}$ is always τ-closed.

$(iii) \Rightarrow (i)$: Let $x \in L$. Then from condition (iii), $(x)^{\tau\tau}$ is strongly coherent and so $\xi((x)^{\tau\tau}) = (x)^{\tau\tau}$. Since $x \in (x)^{\tau\tau} = \xi((x)^{\tau\tau})$, we have $(x)^\tau \vee (x)^{\tau\tau} = L$. Thus (i) holds. ■

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