MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 339–366.
http://doi.org/10.26637/mjm1204/002

Mathematical modeling and optimal control of the dynamics of terrorist ideologies

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***R*eceived 20 July 2024; *A*ccepted 11 September 2024**

Abstract. We describe the dynamics of the spread of terrorist ideologies within a population, described as an epidemic. The equations of the model are obtained using a contact process which gives us first-order autonomous non-linear differential equations. Next, the stability of the equilibrium point is established using the basic reproduction number technique; numerical simulations allow us to verify the mathematical results. Finally, optimal control analysis highlight the importance of synergy of action (numbers, equipment, strategy and training) within the defense and security forces, and the importance of patriotism in a nation. In addition, ongoing awareness-raising campaigns are helping to speed up the eradication process.

AMS Subject Classifications: 49K15, 93B05, 93C15, 93D23.

Keywords: Terrorism, modeling, basic reproduction number, optimal control.

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1. Introduction and Background

There is no an international accepted definition of terrorism. According to [1] terrorism is defined by Title 22 of the U.S. Code as politically motivated violence perpetrated in a clandestine manner against noncombatants. Experts on terrorism also include another aspect in the definition: the act is committed in order to create a fearful state of mind in an audience different from the victims. In [2] we have more than 260 other definitions of terrorism compiled by Joseph J. Easson and Alex P. Schmid. This means that terrorism is not easy concept to define because of its many manifestations: kidnappings of diplomats, sequestration of individuals not concerned by the defended cause, acts of sabotage, assassinations, hijackings of planes etc. [3]. Whether or not an act is considered as terrorism also depends on whether a legal, moral, or behavioral perspective is used to interpret the act, see [1] and [4]. Given definition by the Economists T. Sandler and W. Enders in [5] and [6] is very close: terrorism is "the premeditated use, or threat of use, of extra-normal violence to achieve a political objective, through intimidation or the fear of a large audience." The authors point out that an act without specific political motivation must be considered as a criminal offence rather than terrorist. They also consider violence to be targeted at vulnerable target populations not directly involved in political decision-making processes such as terrorists seek to influence. For [7], If a regime constrains the executive branch, then terrorism may be more prevalent. If, however, a regime allows all viewpoints to be represented, then grievances may be held in check, resulting in less terrorism. Regimes that value constituents' lives and property will also act to limit attacks.

Several models have been written in order to provide a good understanding of the problem, see [8], [9] and [10]. In [11] terrorism is described as a new challenge to Nigeria stability. In [12] C.G. Ngari purpose a mathematical model of Kenya domestic radicalization like a desease. Ngari incorporated rehabilitation centers in his model like A. Gambo and M.O. Ibrahim in [13]. M.R. Pooda and al in [14] study the dynamics of narcoterrorism int the Sahel and in [15] they state a multi-objective optimal control of counter-terrorism in the Sahel Region in Africa. All of theses models ignore that defense and security forces can evolve into terrorist. Our model has three major differences from existing models. Firstly, the death rates resulting from fighting are not constant coefficients. They depend on the balance of power between the defense and security forces and the terrorists. Secondly, terrorists are classified according to the roles they play on the chessboard, not in any hierarchical order. Finally, we incorporate into our model the fact that defense and security forces can also become terrorists. We propose in this paper a mathematical model of dynamics behavior of terrorism ideologies using contacts process. Without loss of generality, this model can be applied to the G5 Sahel countries and to any others similarity countries.

2. Model formulation

We divide the population in eight (08) compartments.

$$\begin{aligned}
\mathbf{S}(t) &: \text{Susceptible },\\
\mathbf{D}(t) &: \text{Defense and Security Forces (DSF)},\\
\mathbf{H}(t) &: \text{Homeland Defense Volunteers (HDV)},\\
\mathbf{I}(t) &: \text{Internally Displaced Persons (IDP)}\\
\mathbf{P}(t) &: \text{Prisoners or People in Detention Centers}\\
\mathbf{T}(t) &: \text{Terrorist},\\
\mathbf{T_S}(t) &: \text{Terrorist soldiers},\\
\mathbf{T_L}(t) &: \text{Terrorist leaders.}
\end{aligned}$$

We set

$$A = D + H + T + T_S + T_L \tag{2.1}$$
$$N = S + D + H + I + P + T + T_S + T_L \tag{2.2}$$

and Taking for initial conditions

$$S(0) > 0,\ D(0) > 0,\ H(0) > 0,\ I(0) \geq 0,\ P(0) \geq 0,\ T(0) \geq 0,\ T_S(0) \geq 0,\ T_L(0) \geq 0,\ N(0) \leqslant \frac{\Lambda}{\mu}. \tag{2.3}$$

We understand by susceptible any person capable of adhering to the terrorist ideology. This definition assumes that the person may or may not be aware of this ideology but has not adopted or accepted it. A susceptible is not a supporter of terrorist ideology and therefore she cannot propagate it.

A terrorist is a person who is a supporter of terrorist ideology. He can only propagate it by means which exclude the taking up of arms. As soon as a weapon is taken or violence is used, we have to deal with a terrorist soldier. We include in the class of terrorists all unarmed persons who provide assistance for the success of the terrorist activity. These include intelligence officers and civilians who supply them. Terrorists and terrorist soldiers are not only ideologically convinced people; some act out of coercion, or within certain limits to defend themselves. The terrorist leaders are the masters of the terrorist chessboard: they set the course. They are the ones who organize, decide on the areas to attack and instruct the actions to be carried out.

Internally Displaced Persons are, according to the United Nations High Commissioner for Refugees (U.N.H.C.R.) in [16], people forced to flee within their own country because of the attacks perpetuated by armed terrorist groups. In the practical dictionary of humanitarian law of Doctors Without Borders [17], we can read that they do not constitute a particular legal category and therefore do not benefit from specific protection under international law.

The regular army is designated by the term DSF. The Volunteers for the Defense of the Homeland (HDV) is a groups of armed combatants created by the government in order to better respond to the demands imposed on it by the terrorist hydra. We include in this class any self-defense groups and any other organization whose objective is to fight alongside the DSF for the defense of the homeland.

The term prison or detention center includes areas regularly set up to accommodate persons deprived of their freedom in connection with terrorism as well as probable detention areas which have been set up by the army for its needs and which meet the criteria of prison. The following assumptions complete the model formulation.

First of all, we assume that the compartments are homogeneous and contained within the same territory. Thus, the spatial distribution of terrorist ideology can be omit and everybody in the population has same average natural death rate μ.

As Castillo Chavez and Bao Song in [18], for $i = \overline{1,6}$ ε_i, q and e measure the strengh of the recruitement force and assumed to be proportionnal to the number of contacts per unit time as well as to the likelihood of success. We also denote Λ_5 and Λ_8 as the per-capita recovery rate. Hence, $1/\Lambda_5$ and $1/\Lambda_8$ are the average residence time respectively for terrorists and terrorist soldiers. This assumed that the residence times are exponentially distributed.

The model equations follow a contact process. In other words, the transition from a class A to a class B is obtained after contact with an individual of class B or an individual of another class who shares the convictions that emanate from class B. For example, an individual can only become a terrorist following contact with a terrorist, a terrorist soldier or a terrorist leader. Contact notion is any means by which individuals can stay in touch such as family ties, telephone calls, radio and television broadcasts, sending letters, coded or explicit messages, internet, etc.

Parameters definitions	
Parameters	**Definitions**
η	Death rate due to detention conditions
μ	Natural mortality rate
Λ	Susceptible recruitment rate
Λ_1	DSF recruitment rate from S
Λ_2	DSF out-going rate
Λ_3	HDV recruitment rate from S
Λ_4	HDV drop-out rate
Λ_5	Terrorist soldiers repentance rate
Λ_6	Prisoners out-going rate
Λ_7	Force of radicalization
Λ_8	Terrorist repentance rate
Λ_9	Force of the determination in defense of the homeland
β_1	Terrorist-to-terrorist-soldiers conversion rate
β_2	Terrorist-to-terrorist-leaders conversion rate
β_3	Terrorist-soldiers-to-terrorist-leaders conversion rate
δ_1	DSF death rate due to violent extrmism
δ_2	HDV death rate due to violent extrmism
δ_3	Terrorists death rate due to counter-terrorist activities
δ_4	Terrorist soldiers death rate due to counter-terrorist activities
δ_5	Terrorist leaders death rate due to counter-terrorist activities
ε_1	Strength of the recruitment force from D into T
ε_2	Strength of the recruitment force from D into T_S
ε_3	Strength of the recruitment force from D into T_L
ε_4	Strength of the recruitment force from H into T
ε_4	Strength of the recruitment force from H into T_S
ε_6	Strength of the recruitment force from H into T_L
a	Undergoing judicial process rate from T_S
b	Strength of the recruitment force from P into T_S
h	Undergoing judicial process rate from T
k	Strength of the recruitment force from P into T
l_1	Undergoing judicial process rate from T_L
l_2	Strength of the recruitment force from P into T_L
n	HDV recruitment rate from IDP
m	DSF recruitment rate from IDP
π	DSF recruitment rate from HDV
e	Strength of the recruitment force from I into T_S
q	Strength of the recruitment force from I into T

the model equations are given by:

$$\frac{dS}{dt} = \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T - \left[\mu + \Lambda_9 \frac{T_S}{A+S} + \Lambda_1 \frac{D+H}{A+S} + \Lambda_3 \frac{T_S}{A+S} + \Lambda_7 \frac{T+T_S+T_L}{A+S}\right] S \tag{2.4}$$

$$\frac{dD}{dt} = \left(\frac{\Lambda_1 S}{A+S} + \frac{mI}{A+I}\right)(D+H) + \pi H - \left[\Lambda_2 + \mu + \delta_1 \frac{T_S}{A} + \varepsilon_1 \frac{T+T_S+T_L}{A} + \varepsilon_2 \frac{T_S+T_L}{A} + \varepsilon_3 \frac{T_L}{A}\right] D \tag{2.5}$$

$$\frac{dH}{dt} = \Lambda_3 \frac{T_S}{A+S} S + n \frac{D+H}{A+I} I - \left[\pi + \mu + \Lambda_4 + \delta_2 \frac{T_S}{A} + \varepsilon_4 \frac{T+T_S+T_L}{A} + \varepsilon_5 \frac{T_S+T_L}{A} + \varepsilon_6 \frac{T_L}{A}\right] H \tag{2.6}$$

$$\frac{dI}{dt} = \Lambda_9 \frac{T_S}{A+S} S - \left[\mu + (n+m)\frac{D+H}{A+I} + e \frac{T_S+T_L}{A+I} + q \frac{T+T_S+T_L}{A+I}\right] I \tag{2.7}$$

$$\frac{dP}{dt} = \left[hT + aT_S + l_1 T_L\right] \frac{D+H}{A} - \left[\mu + \eta + \Lambda_6 + l_2 \frac{T_L}{A+P} + b \frac{T_S+T_L}{A+P} + k \frac{T+T_S+T_L}{A+P}\right] P \tag{2.8}$$

$$\frac{dT}{dt} = \left[\Lambda_7 \frac{S}{A+S} + q \frac{I}{A+I} + k \frac{P}{A+P} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A}\right](T + T_S + T_L) - \left[\Lambda_8 + \mu + (D+H)\left(\frac{h}{A} + \frac{\delta_3}{A}\right) + \beta_1 \frac{T_S+T_L}{A} + \beta_2 \frac{T_L}{A}\right] T \tag{2.9}$$

$$\frac{dT_S}{dt} = \left[\beta_1 \frac{T}{A} + \frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e \frac{I}{A+I} + b \frac{P}{A+P}\right](T_S + T_L) - \left[\mu + \Lambda_5 + (D+H)\left(\frac{a}{A} + \frac{\delta_4}{A}\right) + \beta_3 \frac{T_L}{A}\right] T_S \tag{2.10}$$

$$\frac{dT_L}{dt} = \left[\beta_2 \frac{T}{A} + \beta_3 \frac{T_S}{A} + \frac{\varepsilon_3 D + \varepsilon_6 H}{A} + l_2 \frac{P}{A+P}\right] T_L - \left[\mu + (D+H)\left(\frac{l_1}{A} + \frac{\delta_5}{A}\right)\right] T_L \tag{2.11}$$

We get the terrorism network diagram.

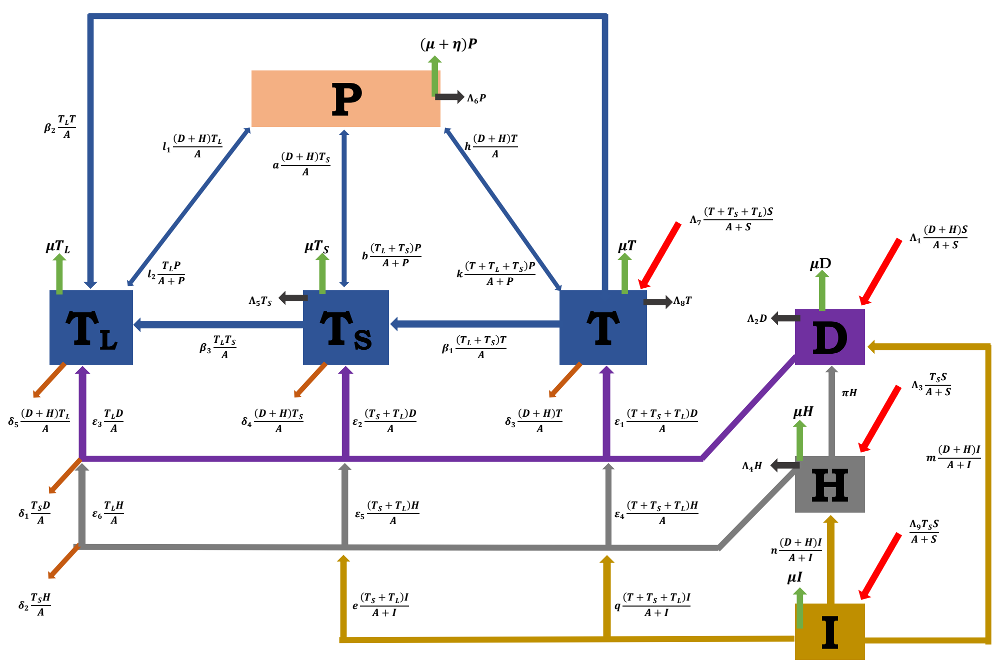

Figure 1: Flow diagram

3. Model Analysis

It's worth mentioning that the parameters of the formulated model are non-negative since the model describes the dynamics of an ideology in an human population. Consequently, it suffices to state that the solutions of the model are non-negative. We denote by $\mathbb{R}_+^8$ the set $[0;+\infty[$ and by Ω the set

$$\Omega = \left\{ (S(t), D(t), H(t), I(t), P(t), T(t), T_S(t), T_L(t)) \in \mathbb{R}_+^8;\ and\ N \leq \frac{\Lambda}{\mu} \right\}. \tag{3.1}$$

Lemma 3.1. *The system* (2.4) - (2.11) *with initial contitions* (2.3) *has a unique solution in* Ω.

Proof. We follow [19] and apply Cauchy-Lypschitz theorem about the existence and the uniqueness of solutions for first-order autonomous systems with initial conditions (2.3). ■

Theorem 3.1. *The feasible region* Ω *is positively invariant and attracting with respect the system* (2.3) - (2.11).

Proof. The vector field associated to the system (2.4) - (2.11) is denoted by

$$\vec{V} = \begin{pmatrix} \dfrac{dS}{dt} \\ \dfrac{dD}{dt} \\ \dfrac{dH}{dt} \\ \dfrac{dI}{dt} \\ \dfrac{dP}{dt} \\ \dfrac{dT}{dt} \\ \dfrac{dT_S}{dt} \\ \dfrac{dT_L}{dt} \end{pmatrix} \tag{3.2}$$

For this demonstration we follow [20], [21] and [22] using the barrier theorem by checking that the vector field is always tangent or pointing inside the boundary $\partial\mathbb{R}_+^8$ of $\mathbb{R}_+^8$.
$\partial\mathbb{R}_+^8 = \{S = 0\} \cup \{D = 0\} \cup \{H = 0\} \cup \{I = 0\} \cup \{P = 0\} \cup \{T = 0\} \cup \{T_S = 0\} \cup \{T_L = 0\}$.

On $\{S = 0\}$, the associated vector field is

$$\vec{V_1} = \begin{pmatrix} \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T \\ \left(\frac{dD}{dt}\right)_{S=0} \\ \left(\frac{dH}{dt}\right)_{S=0} \\ \left(\frac{dI}{dt}\right)_{S=0} \\ \left(\frac{dP}{dt}\right)_{S=0} \\ \left(\frac{dT}{dt}\right)_{S=0} \\ \left(\frac{dT_S}{dt}\right)_{S=0} \\ \left(\frac{dT_L}{dt}\right)_{S=0} \end{pmatrix}$$

We have $\vec{e_1} = (1,0,0,0,0,0,0,0)$ and

$$\vec{V_1} \cdot \vec{e_1} = \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_6 P + \Lambda_8 T \geq 0$$

So, the vector field $\vec{V_1}$ is pointing inside the positive orthan.
The same reasoning can be done for $\{D = 0\}$, $\{H = 0\}$, $\{I = 0\}$, $\{P = 0\}$, $\{T = 0\}$, $\{T_S = 0\}$ and $\{T_L = 0\}$.
We deduce that the set Ω is positively invariant with respect the model.

Moreover,

$$\frac{dN}{dt} = \frac{d\left(S + D + H + I + P + T + T_S + T_L\right)}{dt}$$

$$\frac{dN}{dt} + \mu N \leqslant \Lambda \tag{3.3}$$

According to [23], the solution of (3.3) is given by

$$N(t) \leqslant N(0)\exp(-\mu t) + \frac{\Lambda}{\mu}\left[1 - \exp(-\mu t)\right] \tag{3.4}$$

$$N(0) \leqslant \frac{\Lambda}{\mu} \tag{3.5}$$

Furthermore, for $t \to +\infty$ in (3.4) the total population N approaches the caring capacity constant $\frac{\Lambda}{\mu}$. It means that $\limsup N(t)_{t\to+\infty} \leq \frac{\Lambda}{\mu}$, demonstrating that Ω is attractive within $\mathbb{R}^8_+$; see [13], [21] and [24]. ■

3.1. Basic reproduction number

The basic reproductive number, $\mathcal{R}_0$, is the average number of secondary infections produced by one infected individual during the entire course of infection in a completely susceptible population. I serves as a threshold parameter that predicts whether an infection dies out or keeps persistence in a population. We determine the basic reproduction number by using Watmough and Van den Driessche method in [25]. The population is divided in eight compartments in this order S, D, H, I, P, T, T_S and T_L. For our model, infected compartments are P, T, T_S and T_L and we can discard the compartment P because it doesn't change the basic reproduction number. The next generation matrice is obtained by calculated $G = FV^{-1}$.

Firstly we determine the Terrorist-free equilibrium (TFE) by solving the model equations for $T^* = T_S^* = T_L^* = 0$. It yields

$$E_0 = \left(\frac{\Lambda(\mu+\Lambda_2)}{\mu\Lambda_1},\ \frac{\Lambda\,(\Lambda_1-\Lambda_2-\mu)}{\mu\Lambda_1},\ 0,\ 0,\ 0,\ 0,\ 0 \right) \tag{3.6}$$

Now, we state the basic reproduction number.
Considering $\mathcal{F}_T$, $\mathcal{F}_{T_S}$ and $\mathcal{F}_{T_L}$ as the rates of appearance of newly radicalized individuals respectively in compartments T, T_S and T_L and for $i \in \{T, T_S, T_L\}$ $\nu_i = \nu_i^- - \nu_i^+$ with ν_i^- the rate of transfers of individuals out the class i and ν_i^+ the rate of transfers of individuals into class i we get:

$$F = J_{\mathcal{F}}(E_0) = \begin{bmatrix} \frac{\partial \mathcal{F}_T}{\partial T} & \frac{\partial \mathcal{F}_T}{\partial T_S} & \frac{\partial \mathcal{F}_T}{\partial T_L} \\ \frac{\partial \mathcal{F}_{T_S}}{\partial T} & \frac{\partial \mathcal{F}_{T_S}}{\partial T_S} & \frac{\partial \mathcal{F}_{T_S}}{\partial T_L} \\ \frac{\partial \mathcal{F}_{T_L}}{\partial T} & \frac{\partial \mathcal{F}_{T_L}}{\partial T_S} & \frac{\partial \mathcal{F}_{T_L}}{\partial T_L} \end{bmatrix}(E_0) \quad and \quad V = J_{\nu}(E_0) = \begin{bmatrix} \frac{\partial \nu_T}{\partial T} & \frac{\partial \nu_T}{\partial T_S} & \frac{\partial \nu_T}{\partial T_L} \\ \frac{\partial \nu_{T_S}}{\partial T} & \frac{\partial \nu_{T_S}}{\partial T_S} & \frac{\partial \nu_{T_S}}{\partial T_L} \\ \frac{\partial \nu_{T_L}}{\partial T} & \frac{\partial \nu_{T_L}}{\partial T_S} & \frac{\partial \nu_{T_L}}{\partial T_L} \end{bmatrix}(E_0)$$

$$\text{where} \quad \mathcal{F} = \begin{bmatrix} \mathcal{F}_T \\ \mathcal{F}_{T_S} \\ \mathcal{F}_{T_L} \end{bmatrix} = \begin{bmatrix} \left[\Lambda_7 \frac{S}{A+S} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A} + q\frac{I}{A+I}\right](T+T_S+T_L) \\ \left[\frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e\frac{I}{A+I}\right](T_S+T_L) \\ \left[\frac{\varepsilon_3 D + \varepsilon_6 H}{A}\right] T_L \end{bmatrix}$$

This give us

$$F = \begin{bmatrix} \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 & \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 & \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 \\ 0 & \varepsilon_2 & \varepsilon_2 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{3.7}$$

with

$$\frac{S^*}{D^*+S^*} = \frac{\mu+\Lambda_2}{\Lambda_1} \tag{3.8}$$

$$\frac{D^*}{D^*+S^*} = \frac{\Lambda_1-\Lambda_2-\mu}{\Lambda_1} \tag{3.9}$$

Now, we are locking for V and V^{-1}.

$$\nu = \begin{bmatrix} \nu_T \\ \\ \nu_{T_S} \\ \\ \nu_{T_L} \end{bmatrix} = \begin{bmatrix} \left[\Lambda_8+\mu+(D+H)\left(\frac{h}{A}+\frac{\delta_3}{A}\right)+\beta_1\frac{T_S+T_L}{A}+\beta_2\frac{T_L}{A}\right]T \\ \\ \left[\mu+\Lambda_5+(D+H)\left(\frac{a}{A}+\frac{\delta_4}{A}\right)+\beta_3\frac{T_L}{A}\right]T_S-\left[\beta_1\frac{T}{A}\right](T_S+T_L) \\ \\ \left[\mu+(D+H)\left(\frac{l_1}{A}+\frac{\delta_5}{A}\right)\right]T_L-\left[\beta_2\frac{T}{A}+\beta_3\frac{T_S}{A}\right]T_L \end{bmatrix}$$

It comes that

$$V = \begin{bmatrix} \Lambda_8+\mu+h+\delta_3 & 0 & 0 \\ \\ 0 & \Lambda_5+\mu+a+\delta_4 & 0 \\ \\ 0 & 0 & \mu+l_1+\delta_5 \end{bmatrix} \tag{3.10}$$

and

$$V^{-1} = \begin{bmatrix} \frac{1}{\Lambda_8+\mu+h+\delta_3} & 0 & 0 \\ \\ 0 & \frac{1}{\Lambda_5+\mu+a+\delta_4} & 0 \\ \\ 0 & 0 & \frac{1}{\mu+l_1+\delta_5} \end{bmatrix} \tag{3.11}$$

The next generation matrix denotes $G = FV^{-1}$.
From (3.7), (3.11) and (3.6) we obtain the next generation matrix that is

$$G = \begin{bmatrix} \dfrac{\Lambda_7(\mu+\Lambda_2)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_8+\mu+h+\delta_3)} & \dfrac{\Lambda_7(\mu+\Lambda_1)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_5+\mu+a+\delta_4)} & \dfrac{\Lambda_7(\mu+\Lambda_1)+\Lambda_1\varepsilon_1}{\Lambda_1(\mu+l_1+\delta_5)} \\ 0 & \dfrac{\varepsilon_2}{\Lambda_5+\mu+a+\delta_4} & \dfrac{\varepsilon_2}{\mu+l_1+\delta_5} \\ 0 & 0 & \dfrac{\varepsilon_3}{\mu+l_1+\delta_5} \end{bmatrix} \tag{3.12}$$

The basic reproduction number is the spectral radius of the next generation matrix. Then

$$\mathcal{R}_0 = \max\left\{ \frac{\Lambda_7(\mu+\Lambda_2)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_8+\mu+h+\delta_3)} ; \frac{\varepsilon_2}{\Lambda_5+\mu+a+\delta_4} ; \frac{\varepsilon_3}{\mu+l_1+\delta_5} \right\}. \tag{3.13}$$

Applying theorem of Varga in [26], theorem 2 in [25] or theorem 6 in [27] and as [21] we claim the following local stability result.

Theorem 3.2. *The terrorist free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.*

According to the theorem 3.2, as long as the value of $\mathcal{R}_0$ is less than one, terrorism can never take on the scale of an epidemic. Note that this interpretation depends on the initial conditions, in particular the number of terrorists, terrorist soldiers and leaders in the initial population. To get rid of this dependency, a global stability result is needed.

Theorem 3.3. *The terrorist free equilibrium E_0 is globally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. According to theorem 3.2 the TFE is locally asymptotically stable and according to theorem 3.1 the domain Ω of the feasible solution is attractive. Thus, we follow [20], [21] and [24] to get the global asymptotic stability. ■

3.2. Endemic equilibrium

As soon as the basic reproduction number is greater than one, a single terrorist has a large recruitment capacity and can put the whole nation at risk. We're going to see an explosion in the number of terrorists, terrorist soldiers and terrorist leaders. With soldiers as the armed wing, the result will be more violence and an increase in the number of internally displaced people. There will be more deaths on the DSF and HDV sides. In the long term, the whole nation will be at risk, and in the worst case scenario, we'll have an occupation of the entire territory by armed terrorist groups.

Theorem 3.4. *if $\mathcal{R}_0 > 1$, the terrorist free equilibrium E_0 is unstable.*

Proof. We apply theorem 2 in [25]. ■

4. Numerical Analysis

In this section, we use numerical simulations to verified mathematical analysis results. This mean that for $\mathcal{R}_0 < 1$ we have to see that the populations of terrorists, terrorist soldiers and terrorist leaders are coming to disappear. In the verse, for $\mathcal{R}_0 > 1$ these populations are growing and terrorism ideology is spreading.

The Table 1 gives parameters settings for extinction and the Table 2 gives parameters settings for persistence. The populations of susceptible, DSF, HDV, IDP, prisoners, terrorists, terrorist soldiers and terrorist leaders at initials conditions ($t = 0$) are given by :

$S = 15\,089\,674 \quad D = 23\,000 \quad H = 50\,000 \quad I = 1\,882\,391$

$P = 7\,041 \qquad T = 470\,500 \qquad T_S = 15\,089\,674 \qquad T_L = 25$

Table 1 : Parameters settings for extinction

Parameters and values	Parameters and values
$\eta = 0.0000025$	$\varepsilon_1 = 0.0001$
$\mu = 0.0034247$	$\varepsilon_2 = 0.0001$
$\Lambda = 600$	$\varepsilon_3 = 0.0001$
$\Lambda_1 = 0.05$	$\varepsilon_4 = 0.0000001$
$\Lambda_2 = 0.001$	$\varepsilon_4 = 0.0000001$
$\Lambda_3 = 0.005$	$\varepsilon_6 = 0.0000001$
$\Lambda_4 = 0.0001$	$a = 0.0008$
$\Lambda_5 = 0.0001$	$b = 0.001$
$\Lambda_6 = 0.001$	$h = 0.20635$
$\Lambda_7 = 0.056$	$k = 0.0016$
$\Lambda_8 = 0.0001$	$l_1 = 0.0001$
$\Lambda_9 = 0.01$	$l_2 = 0.0001$
$\beta_1 = 0.9 * 0.12$	$n = 0.01$
$\beta_2 = 0.0792$	$m = 0.001$
$\beta_3 = 0.00005$	$\pi = 0.005$
$\delta_1 = 0.00125$	$\delta_2 = 0.00125$
$\delta_3 = 0.0001$	$\delta_4 = 0.0005$
$\delta_5 = 0.0002$	$q = 0.005$
$e = 0.0025$	

Table 2 : Parameters settings for persistence

Parameters and values	Parameters and values
$\eta = 0.00000025$	$\varepsilon_1 = 0.01$
$\mu = 0.00034247$	$\varepsilon_2 = 0.01$
$\Lambda = 600$	$\varepsilon_3 = 0.01$
$\Lambda_1 = 0.05$	$\varepsilon_4 = 0.001$
$\Lambda_2 = 0.001$	$\varepsilon_4 = 0.001$
$\Lambda_3 = 0.005$	$\varepsilon_6 = 0.001$
$\Lambda_4 = 0.0001$	$a = 0.00008$
$\Lambda_5 = 0.00001$	$b = 0.001$
$\Lambda_6 = 0.001$	$h = 0.0000010635$
$\Lambda_7 = 0.156$	$k = 0.0016$
$\Lambda_8 = 0.0000001$	$l_1 = 0.00001$
$\Lambda_9 = 0.01$	$l_2 = 0.01$
$\beta_1 = 0.1 * 0.12$	$n = 0.01$
$\beta_2 = 0.00792$	$m = 0.001$
$\beta_3 = 0.00005$	$\pi = 0.005$
$\delta_1 = 0.00125$	$\delta_2 = 0.00125$
$\delta_3 = 0.00001$	$\delta_4 = 0.00005$
$\delta_5 = 0.02$	$q = 0.05$
$e = 0.065$	

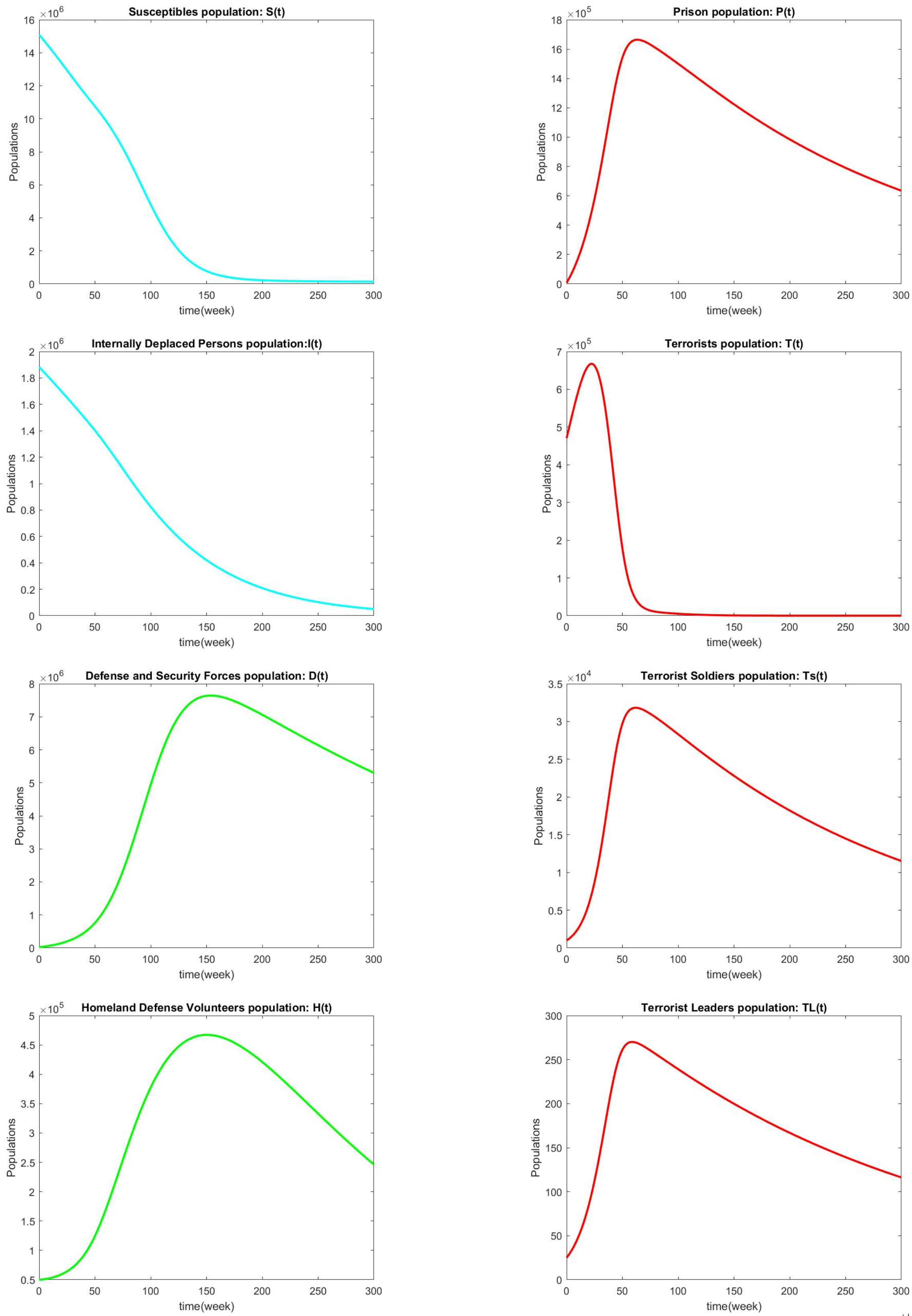

Figure 2: **Evolution of the different populations with extinction values : $\mathcal{R}_0 = 0,0268 < 1$.**

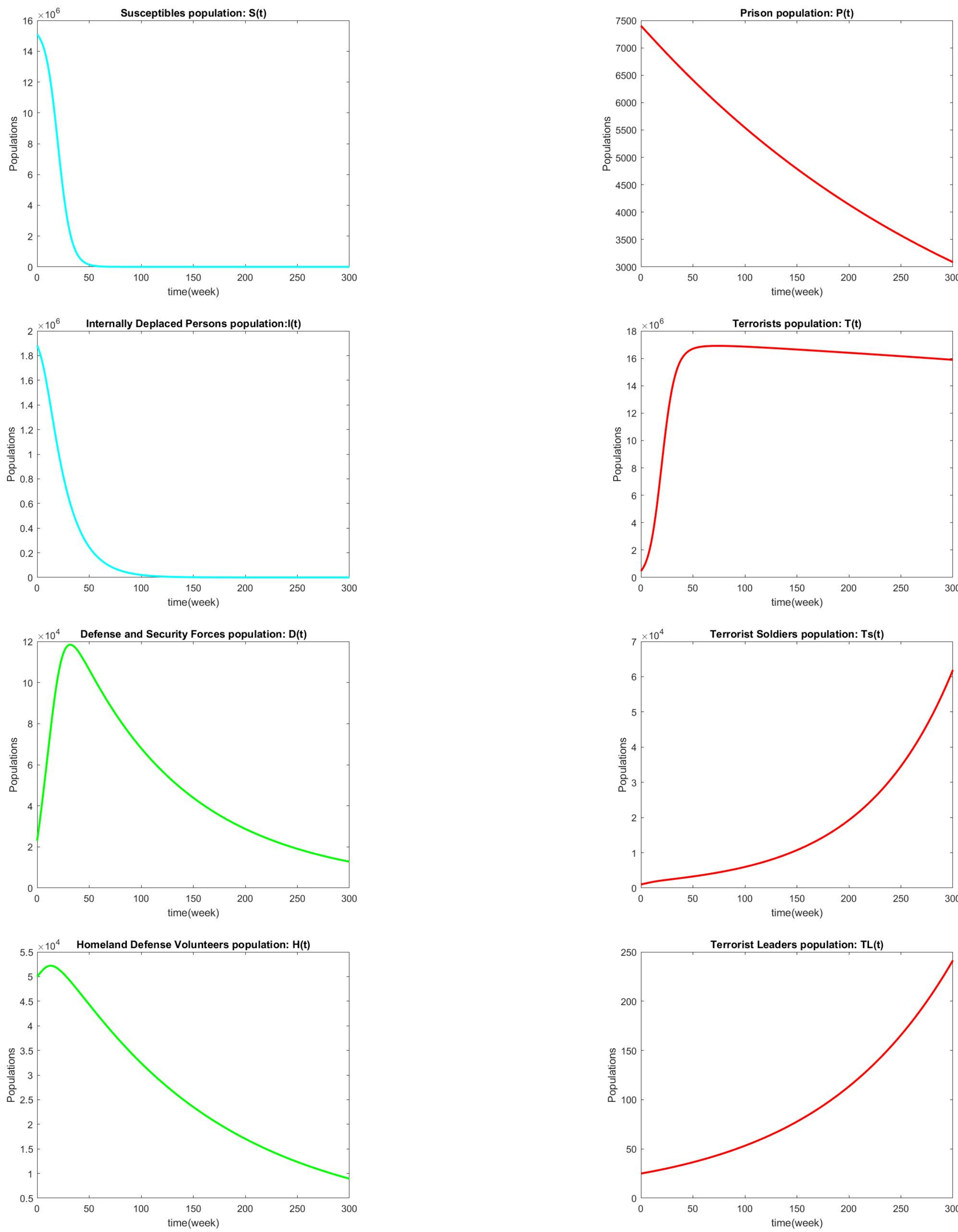

Figure 3: **Evolution of the different populations with persistence values :** $\mathcal{R}_0 = 40.1221 > 1$

Figures Comments:

Figure 2: It shows that terrorist, terrorist soldiers and leaders populations decrease until they stabilize at zero, meaning the extinction of the radicalization and the spread of terrorist ideologies. As HDV were created to help DSF, the decreasing of HDV compartment population is explained by the extinction of the spreading of terrorist ideologies. As a result, the influence that terrorists had within the general population, justifying the existence of the susceptible, no longer exists, hence the number of susceptible naturally stabilizes at zero. IDP who had fled their areas will be able to return and lead a peaceful life again. It therefore goes without saying that the IDV compartment is switched off. However, DSF population is growing. Indeed, the extinction of classes T, T_S and T_L induces the cancellation of the transfer coefficients from DSF class to classes T, T_S and T_L. The only coefficient which ensures the reduction in the number of individuals in the DSF class is the natural mortality rate which is relatively small. Finally, there is no one left to imprison because the terrorists, soldiers and leaders have all disappeared.

Figure 3 : The populations of terrorist, terrorist soldiers and leaders are continuously growing; showing that the terrorist ideology is spreading. There would be more violent and more deaths on the DSF and HDV sides explaining the decreasing of these populations. The slight growth that we are seeing in the first few weeks in the DSF and HDV compartments can be explained by the fact that the government, in response, will increase the recruitment of DSF and HDV to try to contain the growing hydra of terrorism. Since terrorist ideology will be predominant, susceptible and people in the IDP and prisoner compartments will spend less time in their respective compartments. They will be absorbed very quickly in compartments T, Ts and TL; thus contributing to the growth of the number of individuals in these compartments. Thus, in the long term, the whole nation will be in danger. This situation may result in the stabilization of the number of individuals in compartments T, T_s and T_L in the sense that the need for recruitment will decrease to a certain threshold which will be maintained in order to keep the territory under control. At this stage in the evolution of terrorism, the susceptible will no longer be susceptible but terrorists, which explains the stabilization of the number of susceptible at zero.

5. Optimal control model and analysis

5.1. Optimal control model formulation

We introduce three (03) time-dependent control $u_1(t)$, $u_2(t)$ and $u_3(t)$ which are described as follows.

(i) $u_1(t)$ covers all the actions undertaken by government, civil organizations, traditional authorities and political parties to raise awareness through public conferences, preaching and socio-religious seminars. This include television, radio and interactive broadcasts, as well as newspapers articles and pages used in the fight against terrorism.

(ii) $u_2(t)$ represents the ability of DSF and HDV to respond to attacks and carry out preventive operations. This capacity is expressed through military equipment, the quality of that equipment, military training, knowledge and control of the territory, the commitment of the players and their numbers.

(iii) $u_3(t)$ is any action that allows to identify and to neutralise terrorist leaders.

Adding the tree aforementioned time-dependent control we get the control system.

$$\frac{dS}{dt} = \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T - \left[\mu + \Lambda_9 \frac{T_S}{A+S} + \Lambda_1 \frac{D+H}{A+S} + \Lambda_3 \frac{T_S}{A+S} + (1-u_1)\Lambda_7 \frac{T+T_S+T_L}{A+S}\right] S \tag{5.1}$$

$$\frac{dD}{dt} = \left(\frac{\Lambda_1 S}{A+S} + \frac{mI}{A+I}\right)(D+H) + \pi H - \left[\Lambda_2 + \mu + \delta_1 \frac{T_S}{A} + (1-u_1)\varepsilon_1 \frac{T+T_S+T_L}{A} + (1-u_2)\varepsilon_2 \frac{T_S+T_L}{A} + (1-u_3)\varepsilon_3 \frac{T_L}{A}\right] D \tag{5.2}$$

$$\frac{dH}{dt} = \Lambda_3 \frac{T_S}{A+S} S + n \frac{D+H}{A+I} I - \left[\pi + \mu + \Lambda_4 + \delta_2 \frac{T_S}{A} + (1-u_1)\varepsilon_4 \frac{T+T_S+T_L}{A} + (1-u_2)\varepsilon_5 \frac{T_S+T_L}{A} + (1-u_3)\varepsilon_6 \frac{T_L}{A}\right] H \tag{5.3}$$

$$\frac{dI}{dt} = \Lambda_9 \frac{T_S}{A+S} S - \left[\mu + (n+m)\frac{D+H}{A+I} + (1-u_2) e \frac{T_S+T_L}{A+I} + (1-u_1) q \frac{T+T_S+T_L}{A+I}\right] I \tag{5.4}$$

$$\frac{dP}{dt} = [hT + aT_S + l_1 T_L] \frac{D+H}{A} - \left[\mu + \eta + \Lambda_6 + (1-u_3) l_2 \frac{T_L}{A+P} + (1-u_2) b \frac{T_S+T_L}{A+P} + (1-u_1) k \frac{T+T_S+T_L}{A+P}\right] P \tag{5.5}$$

$$\frac{dT}{dt} = (1-u_1)\left[\Lambda_7 \frac{S}{A+S} + q\frac{I}{A+I} + k\frac{P}{A+P} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A}\right](T+T_S+T_L) - \left[\Lambda_8 + \mu + (D+H)\left(\frac{h}{A} + \frac{\delta_3}{A}\right) + (1-u_2)\beta_1 \frac{T_S+T_L}{A} + (1-u_3)\beta_2 \frac{T_L}{A}\right] T \tag{5.6}$$

$$\frac{dT_S}{dt} = (1-u_2)\left[\beta_1 \frac{T}{A} + \frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e\frac{I}{A+I} + b\frac{P}{A+P}\right](T_S+T_L) - \left[\mu + \Lambda_5 + (D+H)\left(\frac{a}{A} + \frac{\delta_4}{A}\right) + (1-u_3)\beta_3 \frac{T_L}{A}\right] T_S \tag{5.7}$$

$$\frac{dT_L}{dt} = (1-u_3)\left[\beta_2 \frac{T}{A} + \beta_3 \frac{T_S}{A} + \frac{\varepsilon_3 D + \varepsilon_6 H}{A} + l_2 \frac{P}{A+P}\right] T_L - \left[\mu + (D+H)\left(\frac{l_1}{A} + \frac{\delta_5}{A}\right)\right] T_L \tag{5.8}$$

5.2. Optimal control model analysis

Our goal is to seek the optimal solution required to minimize the number of terrorists, terrorist soldiers and terrorist leaders responsible for spreading the terrorist ideology int the population at minimum cost. Hence, the objective functional for this control problem is given by

$$\mathcal{J}(u_1,u_2,u_3) = \min_{0\le u_1,u_2,u_3\le 1} \int_0^T \left(\omega_1 T(t) + \omega_2 T_s(t) + \omega_3 T_L(t) + \omega_4 u_1^2 + \omega_5 u_2^2 + \omega_6 u_3^2\right) dt \tag{5.9}$$

where, constants $\omega_i, i = 1,2,...,6$ are positive weights required to balance the corresponding terms in the objective functional. The optimal controls u_1^*, u_2^* and u_3^* we are looking for are the solutions of the problem

$$\mathcal{J}\left(u_1^*, u_2^*, u_3^*\right) = \min\{\mathcal{J}(u_1,u_2,u_3) : u_1,u_2,u_3 \in \mathcal{U}\}. \tag{5.10}$$

$$\mathcal{U} = \{(u_1,u_2,u_3) : (u_1(t),\, u_2(t),\, u_3(t)) \text{ are measurable for } t \in [0;T]\} \tag{5.11}$$

Theorem 5.1. *The problem of optimal control (5.1)-(5.11) has a unique solution in $\mathcal{U}$.*

Proof. Luke's results [28] assure us of the existence of solutions for system (5.1)-(5.8). Since the state variables are bounded, the set containing the system's solutions is bounded. Consequently, we obtain the result by applying Flemming-Rishel's theorem; [29] and [30]. ■

Pontryagin's maximum principle [31] gives the necessary conditions that the control u_1^*, u_2^* and u_3^* must satifsy. These conditions allow us to determine the optimal values of the control u_1^*, u_2^* and u_3^*, using the

Hamiltonian of the system. This Hamilton is given by

$$\begin{aligned}
\mathcal{H} &= \omega_1 T(t) + \omega_2 T_s(t) + \omega_3 T_L(t) + \omega_4 u_1^2 + \omega_5 u_2^2 + \omega_6 u_3^2 \\
&+ \lambda_1 \left(\Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T - \left[\mu + \Lambda_9 \frac{T_S}{A+S} + \Lambda_1 \frac{D+H}{A+S} + \Lambda_3 \frac{T_S}{A+S} + (1-u_1)\Lambda_7 \frac{T+T_S+T_L}{A+S} \right] S \right) \\
&+ \lambda_2 \left(\left(\frac{\Lambda_1 S}{A+S} + \frac{mI}{A+I} \right)(D+H) + \pi H - \left[\Lambda_2 + \mu + \delta_1 \frac{T_S}{A} + (1-u_1)\varepsilon_1 \frac{T+T_S+T_L}{A} + (1-u_2)\varepsilon_2 \frac{T_S+T_L}{A} + (1-u_3)\varepsilon_3 \frac{T_L}{A} \right] D \right) \\
&+ \lambda_3 \left(\Lambda_3 \frac{T_S}{A+S} S + n \frac{D+H}{A+I} I - \left[\pi + \mu + \Lambda_4 + \delta_2 \frac{T_S}{A} + (1-u_1)\varepsilon_4 \frac{T+T_S+T_L}{A} + (1-u_2)\varepsilon_5 \frac{T_S+T_L}{A} + (1-u_3)\varepsilon_6 \frac{T_L}{A} \right] H \right) \\
&+ \lambda_4 \left(\Lambda_9 \frac{T_S}{A+S} S - \left[\mu + (n+m) \frac{D+H}{A+I} + (1-u_2) e \frac{T_S+T_L}{A+I} + (1-u_1) q \frac{T+T_S+T_L}{A+I} \right] I \right) \\
&+ \lambda_5 \left([hT + aT_S + l_1 T_L] \frac{D+H}{A} - \left[\mu + \eta + \Lambda_6 + (1-u_3) l_2 \frac{T_L}{A+P} + (1-u_2) b \frac{T_S+T_L}{A+P} + (1-u_1) k \frac{T+T_S+T_L}{A+P} \right] P \right) \\
&+ \lambda_6 \left((1-u_1) \left[\Lambda_7 \frac{S}{A+S} + q \frac{I}{A+I} + k \frac{P}{A+P} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A} \right] (T+T_S+T_L) - \left[\Lambda_8 + \mu + (D+H)\left(\frac{h}{A} + \frac{\delta_3}{A} \right) + (1-u_2)\beta_1 \frac{T_S+T_L}{A} + (1-u_3)\beta_2 \frac{T_L}{A} \right] T \right) \\
&+ \lambda_7 \left((1-u_2) \left[\beta_1 \frac{T}{A} + \frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e \frac{I}{A+I} + b \frac{P}{A+P} \right] (T_S+T_L) - \left[\mu + \Lambda_5 + (D+H)\left(\frac{a}{A} + \frac{\delta_4}{A} \right) + (1-u_3)\beta_3 \frac{T_L}{A} \right] T_S \right) \\
&+ \lambda_8 \left((1-u_3) \left[\beta_2 \frac{T}{A} + \beta_3 \frac{T_S}{A} + \frac{\varepsilon_3 D + \varepsilon_6 H}{A} + l_2 \frac{P}{A+P} \right] T_L - \left[\mu + (D+H)\left(\frac{l_1}{A} + \frac{\delta_5}{A} \right) \right] T_L \right)
\end{aligned}$$

where, λ_i for $i = 1, 2, 3, ..., 8$, represent the adjoint variables associated with the state variables of the model (5.1)-(5.8).

Theorem 5.2. *Let (u_1^*, u_2^*, u_3^*) be a solution of the problem of minization (5.1)-(5.11). Then, the adjoint variables are given by*

$$\begin{aligned}
\dot{\lambda}_1 &= \lambda_1 \mu + (\lambda_1 - \lambda_6) \frac{\Lambda_7 A (T+T_S+T_L)(1-u_1)}{(A+S)^2} + (\lambda_1 - \lambda_2) \frac{\Lambda_1 A (D+H)}{(A+S)^2} + (\lambda_1 - \lambda_3) \frac{\Lambda_3 A T_S}{(A+S)^2} \\
&+ (\lambda_1 - \lambda_4) \frac{\Lambda_9 A T_S}{(A+S)^2}
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_2 &= \lambda_1 \mu + (\lambda_2 - \lambda_1)\Lambda_2 + (\lambda_3 - \lambda_1) \frac{\Lambda_3 T_S S}{(A+S)^2} + (\lambda_4 - \lambda_1) \frac{\Lambda_9 T_S S}{(A+S)^2} + (\lambda_1 - \lambda_2) \frac{\Lambda_1 S (S+T+T_S+T_L)}{(A+S)^2} \\
&+ (\lambda_6 - \lambda_1) \frac{\Lambda_7 S (T+T_S+T_L)(1-u_1)}{(A+S)^2} + (\lambda_2 - \lambda_6) \frac{\varepsilon_1 (A-D)(T+T_S+T_L)(1-u_1)}{A^2} \\
&+ (\lambda_6 - \lambda_3) \frac{\varepsilon_4 H (T+T_S+T_L)(1-u_1)}{A^2} + (\lambda_6 - \lambda_4) \frac{qI (T+T_S+T_L)(1-u_1)}{(A+I)^2} \\
&+ (\lambda_6 - \lambda_5) \frac{kP (T+T_S+T_L)(1-u_1)}{(A+P)^2} + (\lambda_6 - \lambda_5) \frac{hT (T+T_S+T_L)}{A^2} \\
&+ (\lambda_4 - \lambda_2) \frac{mI (I+T+T_S+T_L)}{(A+I)^2} + (\lambda_4 - \lambda_3) \frac{nI (I+T+T_S+T_L)}{(A+I)^2} + (\lambda_7 - \lambda_4) \frac{eI (T_S+T_L)(1-u_2)}{(A+I)^2} \\
&+ (\lambda_7 - \lambda_6) \frac{\beta_1 T (T_S+T_L)(1-u_2)}{A^2} + (\lambda_2 - \lambda_7) \frac{\varepsilon_2 (A-D)(T_S+T_L)(1-u_2)}{A^2} \\
&+ (\lambda_7 - \lambda_3) \frac{\varepsilon_5 H (T_S+T_L)(1-u_2)}{A^2} + (\lambda_7 - \lambda_5) \frac{bP (T_S+T_L)(1-u_2)}{(A+P)^2} + (\lambda_7 - \lambda_5) \frac{aT_S (T+T_S+T_L)}{A^2} \\
&+ (\lambda_8 - \lambda_7) \frac{\beta_3 T_S T_L (1-u_3)}{A^2} + (\lambda_8 - \lambda_6) \frac{\beta_2 T T_L (1-u_3)}{A^2} + (\lambda_8 - \lambda_5) \frac{l_2 P T_L (1-u_3)}{(A+P)^2} \\
&+ (\lambda_8 - \lambda_5) \frac{l_1 T_L (T+T_S+T_L)}{A^2} + (\lambda_8 - \lambda_3) \frac{\varepsilon_6 H T_L (1-u_3)}{A^2} + (\lambda_2 - \lambda_8) \frac{\varepsilon_3 (A-D) T_L (1-u_3)}{A^2} \\
&+ \delta_1 \lambda_2 \frac{(A-D) T_S}{A^2} - \delta_2 \lambda_3 \frac{H T_S}{A^2} + \delta_3 \lambda_6 \frac{T (T+T_S+T_L)}{A^2} + \delta_4 \lambda_7 \frac{T_S (T+T_S+T_L)}{A^2} + \delta_5 \lambda_8 \frac{T_L (T+T_S+T_L)}{A^2}
\end{aligned}$$

$$
\begin{aligned}
\dot{\lambda}_3 &= \lambda_3\mu + (\lambda_3-\lambda_2)\pi + (\lambda_3-\lambda_1)\Lambda_4 + (\lambda_1-\lambda_2)\frac{\Lambda_1 S(S+T+T_S+T_L)}{(A+S)^2} + (\lambda_3-\lambda_1)\frac{\Lambda_3 S T_S}{(A+S)^2} \\
&+ (\lambda_4-\lambda_1)\frac{\Lambda_9 S T_S}{(A+S)^2} + (\lambda_6-\lambda_1)\frac{\Lambda_7 S(T+T_S+T_L)(1-u_1)}{(A+S)^2} + (\lambda_4-\lambda_2)\frac{mI(I+T+T_S+T_L)}{(A+I)^2} \\
&+ (\lambda_4-\lambda_3)\frac{nI(I+T+T_S+T_L)}{(A+I)^2} + (\lambda_7-\lambda_4)\frac{eI(T_S+T_L)(1-u_2)}{(A+I)^2} + (\lambda_6-\lambda_4)\frac{qI(T+T_S+T_L)(1-u_1)}{(A+I)^2} \\
&+ (\lambda_8-\lambda_5)\frac{l_2 P T_L(1-u_3)}{(A+P)^2} + (\lambda_7-\lambda_5)\frac{bP(T_S+T_L)(1-u_2)}{(A+P)^2} + (\lambda_6-\lambda_5)\frac{kP(T+T_S+T_L)(1-u_1)}{(A+P)^2} \\
&+ (\lambda_6-\lambda_2)\frac{\varepsilon_1 D(T+T_S+T_L)(1-u_1)}{A^2} + (\lambda_7-\lambda_2)\frac{\varepsilon_2 D(T_S+T_L)(1-u_2)}{A^2} + (\lambda_8-\lambda_2)\frac{\varepsilon_3 D T_L(1-u_3)}{A^2} \\
&+ (\lambda_3-\lambda_6)\frac{\varepsilon_4(A-H)(T+T_S+T_L)(1-u_1)}{A^2} + (\lambda_3-\lambda_7)\frac{\varepsilon_5(A-H)(T_S+T_L)(1-u_2)}{A^2} \\
&+ (\lambda_3-\lambda_8)\frac{\varepsilon_6(A-H)T_L(1-u_3)}{A^2} + (\lambda_6-\lambda_5)\frac{hT(T+T_S+T_L)}{A^2} + (\lambda_6-\lambda_5)\frac{aT_S(T+T_S+T_L)}{A^2} \\
&+ (\lambda_8-\lambda_5)\frac{l_1 T_L(T+T_S+T_L)}{A^2} + (\lambda_7-\lambda_6)\frac{\beta_1 T(T_S+T_L)(1-u_2)}{A^2} + (\lambda_8-\lambda_6)\frac{\beta_2 T T_L(1-u_3)}{A^2} \\
&+ (\lambda_8-\lambda_7)\frac{\beta_3 T_S T_L(1-u_3)}{A^2} - \delta_1\lambda_2\frac{DT_S}{A^2} + \delta_2\lambda_3\frac{(A-H)T_S}{A^2} + \delta_3\lambda_6\frac{T(T+T_S+T_L)}{A^2} \\
&+ \delta_4\lambda_7\frac{T_S(T+T_S+T_L)}{A^2} + \delta_5\lambda_8\frac{T_L(T+T_S+T_L)}{A^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_4 &= \lambda_4\mu + (\lambda_4-\lambda_2)\frac{mA(D+H)}{(A+I)^2} + (\lambda_4-\lambda_3)\frac{nA(D+H)}{(A+I)^2} + (\lambda_4-\lambda_6)\frac{qA(T+T_S+T_L)(1-u_1)}{(A+I)^2} \\
&+ (\lambda_4-\lambda_7)\frac{eA(T_S+T_L)(1-u_2)}{(A+I)^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_5 &= \lambda_5(\mu+\eta) + (\lambda_5-\lambda_1)\Lambda_6 + (\lambda_5-\lambda_8)\frac{l_2 A T_L(1-u_3)}{(A+P)^2} + (\lambda_5-\lambda_7)\frac{bA(T_S+T_L)(1-u_2)}{(A+P)^2} \\
&+ (\lambda_5-\lambda_6)\frac{kA(T+T_S+T_L)(1-u_1)}{(A+P)^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_6 &= -\omega_1 + \lambda_6\mu + (\lambda_6-\lambda_1)\Lambda_8 + (\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2} + (\lambda_3-\lambda_1)\frac{\Lambda_3 S T_S}{(A+S)^2} + (\lambda_4-\lambda_1)\frac{\Lambda_9 S T_S}{(A+S)^2} \\
&+ (\lambda_1-\lambda_6)\frac{\Lambda_7 S(S+D+H)(1-u_1)}{(A+S)^2} + (\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2} + (\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2} \\
&+ (\lambda_7-\lambda_4)\frac{eI(T_S+T_L)(1-u_2)}{(A+I)^2} + (\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2} + (\lambda_8-\lambda_5)\frac{l_2 P T_L(1-u_3)}{(A+P)^2} \\
&+ (\lambda_7-\lambda_5)\frac{bP(T_S+T_L)(1-u_2)}{(A+P)^2} + (\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2} + (\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2} \\
&+ (\lambda_7-\lambda_2)\frac{\varepsilon_2 D(T_S+T_L)(1-u_2)}{A^2} + (\lambda_8-\lambda_2)\frac{\varepsilon_3 D T_L(1-u_3)}{A^2} + (\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2} \\
&+ (\lambda_7-\lambda_3)\frac{\varepsilon_5 H(T_S+T_L)(1-u_2)}{A^2} + (\lambda_8-\lambda_3)\frac{\varepsilon_6 H T_L(1-u_3)}{A^2} + (\lambda_6-\lambda_5)\frac{h(A-T)(D+H)}{A^2} \\
&+ (\lambda_5-\lambda_7)\frac{aT_S(D+H)}{A^2} + (\lambda_5-\lambda_8)\frac{l_1(D+H)T_L}{A^2} + (\lambda_6-\lambda_7)\frac{\beta_1(A-T)(1-u_2)}{A^2} \\
&+ (\lambda_6-\lambda_8)\frac{\beta_2(A-T)(1-u_3)}{A^2} + (\lambda_8-\lambda_7)\frac{\beta_3 T_S T_L(1-u_3)}{A^2} - \delta_1\lambda_2\frac{DT_S}{A^2} - \delta_2\lambda_3\frac{HT_S}{A^2} \\
&+ \delta_3\lambda_6\frac{(A-T)(D+H)}{A^2} - \delta_4\lambda_7\frac{T_S(D+H)}{A^2} - \delta_5\lambda_8\frac{T_L(D+H)}{A^2}
\end{aligned}
$$

$$\begin{aligned}
\dot{\lambda}_7 \;=\; & -\omega_2+\lambda_7\mu+(\lambda_7-\lambda_1)\Lambda_5+(\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2}+(\lambda_1-\lambda_3)\frac{\Lambda_3 S(A+S-T_S)}{(A+S)^2}\\
&+\;(\lambda_1-\lambda_4)\frac{\Lambda_9 S(A+S-T_S)}{(A+S)^2}+(\lambda_1-\lambda_6)\frac{\Lambda_7(S+D+H)(1-u_1)}{(A+S)^2}+(\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2}\\
&+\;(\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2}+(\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2}+(\lambda_4-\lambda_7)\frac{eI(I+T+D+H)(1-u_2)}{(A+I)^2}\\
&+\;(\lambda_8-\lambda_5)\frac{l_2 PT_L(1-u_3)}{(A+P)^2}+(\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2}+(\lambda_5-\lambda_7)\frac{bP(P+T+D+H)(1-u_2)}{(A+P)^2}\\
&+\;(\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2}+(\lambda_2-\lambda_7)\frac{\varepsilon_2 D(T+D+H)(1-u_2)}{A^2}+(\lambda_8-\lambda_2)\frac{\varepsilon_3 DT_L(1-u_3)}{A^2}\\
&+\;(\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2}+(\lambda_3-\lambda_7)\frac{\varepsilon_5 H(T+D+H)(1-u_2)}{A^2}+(\lambda_8-\lambda_3)\frac{\varepsilon_6 HT_L(1-u_3)}{A^2}\\
&+\;(\lambda_7-\lambda_5)\frac{a(A-T_S)(D+H)}{A^2}+(\lambda_5-\lambda_6)\frac{hT(D+H)}{A^2}+(\lambda_5-\lambda_8)\frac{l_1 T_L(D+H)}{A^2}\\
&+\;(\lambda_6-\lambda_7)\frac{\beta_1 T(T+D+H)(1-u_2)}{A^2}+(\lambda_8-\lambda_6)\frac{\beta_2 TT_L(1-u_3)}{A^2}+(\lambda_7-\lambda_8)\frac{\beta_3 T_L(A-T_S)(1-u_3)}{A^2}\\
&+\;\delta_1\lambda_2\frac{D(A-T_S)}{A^2}+\delta_2\lambda_3\frac{H(A-T_S)}{A^2}-\delta_3\lambda_6\frac{T(D+H)}{A^2}+\delta_4\lambda_7\frac{(A-T_S)(D+H)}{A^2}-\delta_5\lambda_8\frac{T_L(D+H)}{A^2}
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_8 \;=\; & -\omega_3+\lambda_8\mu+(\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2}+(\lambda_3-\lambda_1)\frac{\Lambda_3 ST_S}{(A+S)^2}+(\lambda_4-\lambda_1)\frac{\Lambda_9 ST_S}{(A+S)^2}\\
&+\;(\lambda_1-\lambda_6)\frac{\Lambda_7(S+D+H)(1-u_1)}{(A+S)^2}+(\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2}+(\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2}\\
&+\;(\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2}+(\lambda_4-\lambda_7)\frac{eI(I+T+D+H)(1-u_2)}{(A+I)^2}\\
&+\;(\lambda_5-\lambda_8)\frac{l_2 P(A+P-T_L)(1-u_3)}{(A+P)^2}+(\lambda_5-\lambda_7)\frac{bP(P+T+D+H)(1-u_2)}{(A+P)^2}\\
&+\;(\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2}+(\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2}\\
&+\;(\lambda_2-\lambda_7)\frac{\varepsilon_2 D(T+D+H)(1-u_2)}{A^2}+(\lambda_2-\lambda_8)\frac{\varepsilon_3 D(A-T_L)(1-u_3)}{A^2}\\
&+\;(\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2}+(\lambda_3-\lambda_7)\frac{\varepsilon_5 H(T+D+H)(1-u_2)}{A^2}\\
&+\;(\lambda_3-\lambda_8)\frac{\varepsilon_6 H(A-T_L)(1-u_3)}{A^2}+(\lambda_8-\lambda_5)\frac{l_1(A-T_L)(D+H)}{A^2}+(\lambda_5-\lambda_6)\frac{hT(D+H)}{A^2}\\
&+\;(\lambda_5-\lambda_7)\frac{aT_S(D+H)}{A^2}+(\lambda_6-\lambda_7)\frac{\beta_1 T(T+D+H)(1-u_2)}{A^2}+(\lambda_6-\lambda_8)\frac{\beta_2 T(A-T_L)(1-u_3)}{A^2}\\
&+\;(\lambda_7-\lambda_8)\frac{\beta_3 T_S(A-T_L)(1-u_3)}{A^2}-\delta_1\lambda_2\frac{DT_S}{A^2}-\delta_2\lambda_3\frac{HT_S}{A^2}-\delta_3\lambda_6\frac{T(D+H)}{A^2}-\delta_4\lambda_7\frac{T_S(D+H)}{A^2}\\
&+\;\delta_5\lambda_8\frac{(A-T_L)(D+H)}{A^2}
\end{aligned}$$

Further, the optimal control (u_1^*, u_2^*, u_3^*) *is*

$$\begin{aligned}
u_1^* &= \max\{0, \min\{1; \tau_1^*\}\}\\
u_2^* &= \max\{0, \min\{1; \tau_2^*\}\}\\
u_3^* &= \max\{0, \min\{1; \tau_3^*\}\}
\end{aligned}$$

where

$$\tau_1^*=\frac{1}{2\omega_4}\left[\frac{(\lambda_6-\lambda_1)\Lambda_7 S}{A+S}+\frac{(\lambda_6-\lambda_2)\varepsilon_1 D}{A}+\frac{(\lambda_6-\lambda_3)\varepsilon_4 H}{A}+\frac{(\lambda_6-\lambda_4)qI}{A+I}+\frac{(\lambda_6-\lambda_5)kP}{A+P}\right](T+T_S+T_L)$$

$$\tau_2^* = \frac{1}{2\omega_5}\left[\frac{(\lambda_7-\lambda_2)\varepsilon_2 D}{A} + \frac{(\lambda_7-\lambda_3)\varepsilon_5 H}{A} + \frac{(\lambda_7-\lambda_4)eI}{A+I} + \frac{(\lambda_7-\lambda_5)bP}{A+P} + \frac{(\lambda_7-\lambda_6)\beta_1 T}{A}\right](T_S+T_L)$$

$$\tau_3^* = \frac{1}{2\omega_6}\left[\frac{(\lambda_8-\lambda_2)\varepsilon_3 D}{A} + \frac{(\lambda_8-\lambda_3)\varepsilon_6 H}{A} + \frac{(\lambda_8-\lambda_5)l_2 P}{A+P} + \frac{(\lambda_8-\lambda_6)\beta_2 T}{A} + \frac{(\lambda_8-\lambda_7)\beta_3 T_S}{A}\right]T_L$$

Proof. Following [30] and [32], we determine the differential of the Hamiltonian with respect the system variables and deduce the adjoint system

$$\begin{array}{cccc} \dot{\lambda}_1 = -\dfrac{\partial \mathcal{H}}{\partial S} & \dot{\lambda}_2 = -\dfrac{\partial \mathcal{H}}{\partial D} & \dot{\lambda}_3 = -\dfrac{\partial \mathcal{H}}{\partial H} & \dot{\lambda}_4 = -\dfrac{\partial \mathcal{H}}{\partial I} \\ \dot{\lambda}_5 = -\dfrac{\partial \mathcal{H}}{\partial P} & \dot{\lambda}_6 = -\dfrac{\partial \mathcal{H}}{\partial T} & \dot{\lambda}_7 = -\dfrac{\partial \mathcal{H}}{\partial T_S} & \dot{\lambda}_8 = -\dfrac{\partial \mathcal{H}}{\partial T_L} \end{array}$$

To obtain the optimal control formulation we solve the given equation by the Hamiltonian differential $\mathcal{H}$ with respect to (u_1, u_2, u_3). It follows that

$$u_1^* = \begin{cases} 0 & \text{if} \quad \tau_1^* \leqslant 0 \\ \tau_1^* & \text{if} \quad 0 < \tau_1^* < 1 \\ 1 & \text{if} \quad \tau_1^* \geqslant 1 \end{cases},$$

$$u_2^* = \begin{cases} 0 & \text{if} \quad \tau_2^* \leqslant 0 \\ \tau_2^* & \text{if} \quad 0 < \tau_2^* < 1 \\ 1 & \text{if} \quad \tau_2^* \geqslant 1 \end{cases}$$

and

$$u_3^* = \begin{cases} 0 & \text{if} \quad \tau_3^* \leqslant 0 \\ \tau_3^* & \text{if} \quad 0 < \tau_3^* < 1 \\ 1 & \text{if} \quad \tau_3^* \geqslant 1 \end{cases}.$$

with

$$\tau_1^* = \frac{1}{2\omega_4}\left[\frac{(\lambda_6-\lambda_1)\Lambda_7 S}{A+S} + \frac{(\lambda_6-\lambda_2)\varepsilon_1 D}{A} + \frac{(\lambda_6-\lambda_3)\varepsilon_4 H}{A} + \frac{(\lambda_6-\lambda_4)qI}{A+I} + \frac{(\lambda_6-\lambda_5)kP}{A+P}\right](T+T_S+T_L)$$

$$\tau_2^* = \frac{1}{2\omega_5}\left[\frac{(\lambda_7-\lambda_2)\varepsilon_2 D}{A} + \frac{(\lambda_7-\lambda_3)\varepsilon_5 H}{A} + \frac{(\lambda_7-\lambda_4)eI}{A+I} + \frac{(\lambda_7-\lambda_5)bP}{A+P} + \frac{(\lambda_7-\lambda_6)\beta_1 T}{A}\right](T_S+T_L)$$

$$\tau_3^* = \frac{1}{2\omega_6}\left[\frac{(\lambda_8-\lambda_2)\varepsilon_3 D}{A} + \frac{(\lambda_8-\lambda_3)\varepsilon_6 H}{A} + \frac{(\lambda_8-\lambda_5)l_2 P}{A+P} + \frac{(\lambda_8-\lambda_6)\beta_2 T}{A} + \frac{(\lambda_8-\lambda_7)\beta_3 T_S}{A}\right]T_L$$

■

5.3. Numerical simulation of the optimal control problem

The aim of the control is to determine a cost-effective control strategy. To achieve this, we use numerical simulations to study the impact of each control. In addition, we will carry out a comparative study of the impact of each control in the fight against ideological terrorism. This study will enable us to highlight the impact of the strategies linked to each control function, and thus to identify an effective counter-terrorism strategy at a lower cost in terms of time, human and financial resources.

In order to observe the impact of the values taken by the control functions, we'll use the same parameter values as in the case $R_0 = 0.0268 < 1$. Indeed, for $R_0 < 1$, the terrorist ideology is already in extinction, and for control values tending towards 1, we should be able to observe curves showing faster decay in the compartments P, T, T_S and T_L. This will show that by acting on the factors represented by the three controls, the State can significantly increase the effectiveness of the fight.

Figures Comments:

Figure 4: We assume that the government and its partners and the entire population, in a spirit of patriotism, are working to defeat terrorism. This action is reflected in continuous awareness-raising actions, the reinforcement and acquisition of increasingly efficient military equipment accompanied by solid and adapted military training. As the figure shows, the implementation of a synergy of action based on the three control functions makes it possible to defeat terrorism in all its forms in a relatively short time.

Figure 5: Among the infected classes P, T, T_s and T_L, terrorists are the class that is most in contact with the other compartments. Thus, by putting all the available means on the control u_1, we see that the measures taken in this direction reach all the other compartments. For $u_1 = 1$, we assume that the entire population has fully integrated the fact that no one should adhere to terrorist ideology. So the terrorist compartment will gradually empty out, stabilizing at zero, and there will be no more opportunities for recruitment. Then, since terrorist soldiers recruit mainly from the T class and leaders from the T and Ts classes, the extinction of the terrorist class inexorably leads to the extinction of the Ts and T_L classes.

Figure 6: The control u_2 represents the ability of DSF and HDV to respond attack and carry out preventive operations. So, for $u_2 = 1$ DFS and HDV are the DFS and HDV are well trained, equipped and qualified for combat. However, military equipment and training are designed only for the army. This can be seen in the figure. In fact, the consequences of the control u_2 are effective on soldier terrorists, but have no effect on other compartments. The soldier terrorists will certainly be eradicated, but the terrorist ideology will remain through the persistence of the T and TL compartments. The latter will always work to create the compartment of terrorist soldiers. The struggle can go on forever, which means that terrorism cannot be defeated by military action alone.

Figure 7: Putting all resources into u_3 control is probably the least effective control strategy. Leading terrorists are eliminated, but all other classes remain intact, and the curves are confounded. Eliminating the leaders will disorganize the fight and spread panic among the terrorists. However, the transition from soldier to leader allows a renewal of the leader class. The change of leaders within terrorist groups can also, against all odds, instill a new dynamic and reinvigorate the terrorists. It's worth noting that new leaders, coming from the soldier class of terrorists, because they have fought in combat, are more familiar with the context of the struggle and may prove to be more competent. These new, potentially more effective leaders can reverse the trend of the struggle and succeed in leading the terrorists to an undesirable victory for a country.

Figure 8: Joint actions on controls $u_1(t)$, $u_2(t)$ and $u_3(t)$ have a significant impact on the compartments T, T_S and T_L that we aim to stabilize at 0. This is a further proof of the importance of synergy of action (numbers, equipment, strategy and training) within the defense and ongoing awareness-raising campaigns.

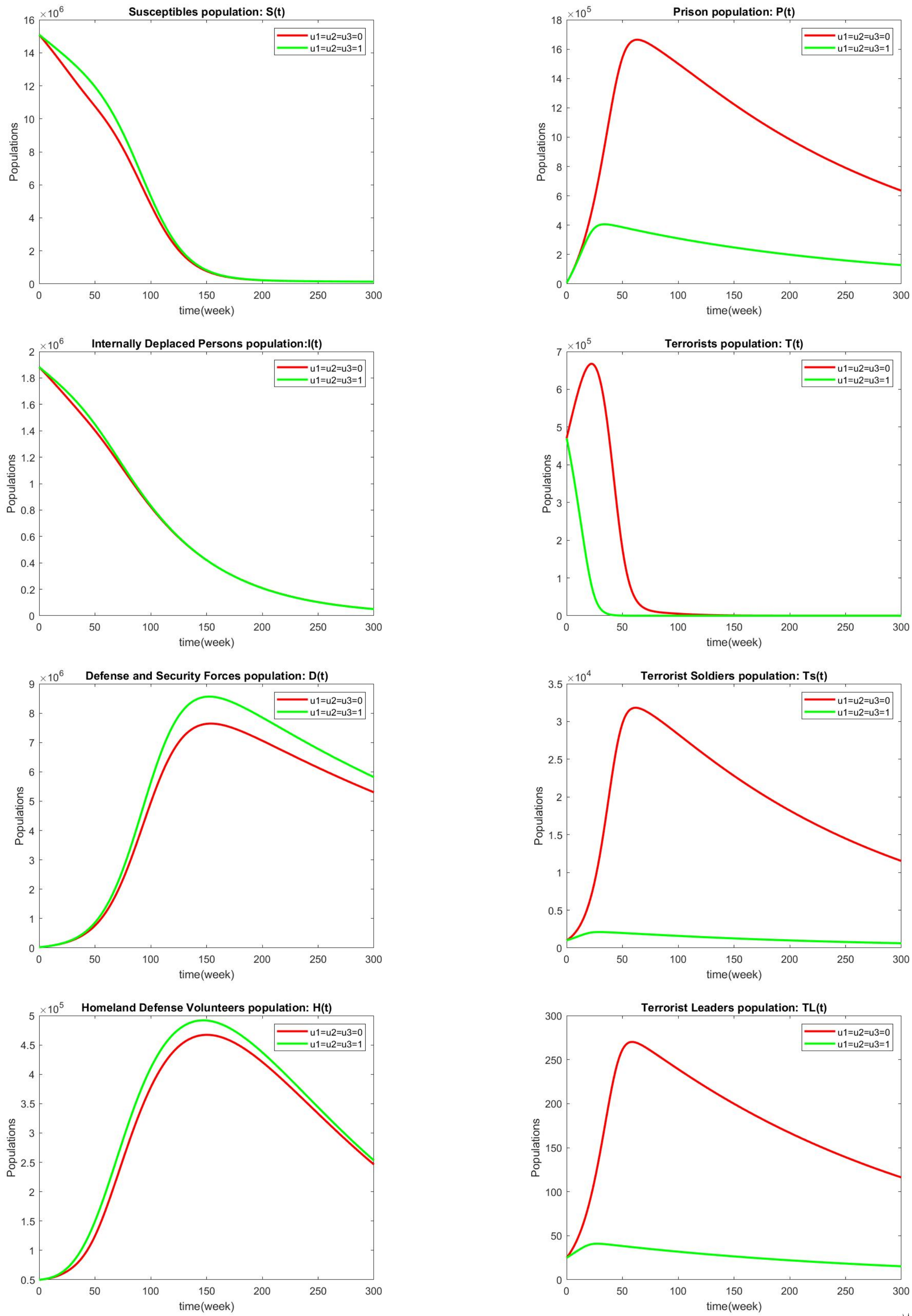

Figure 4: **Comparative evolution of the different populations, setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = u_2 = u_3 = 1$**.**

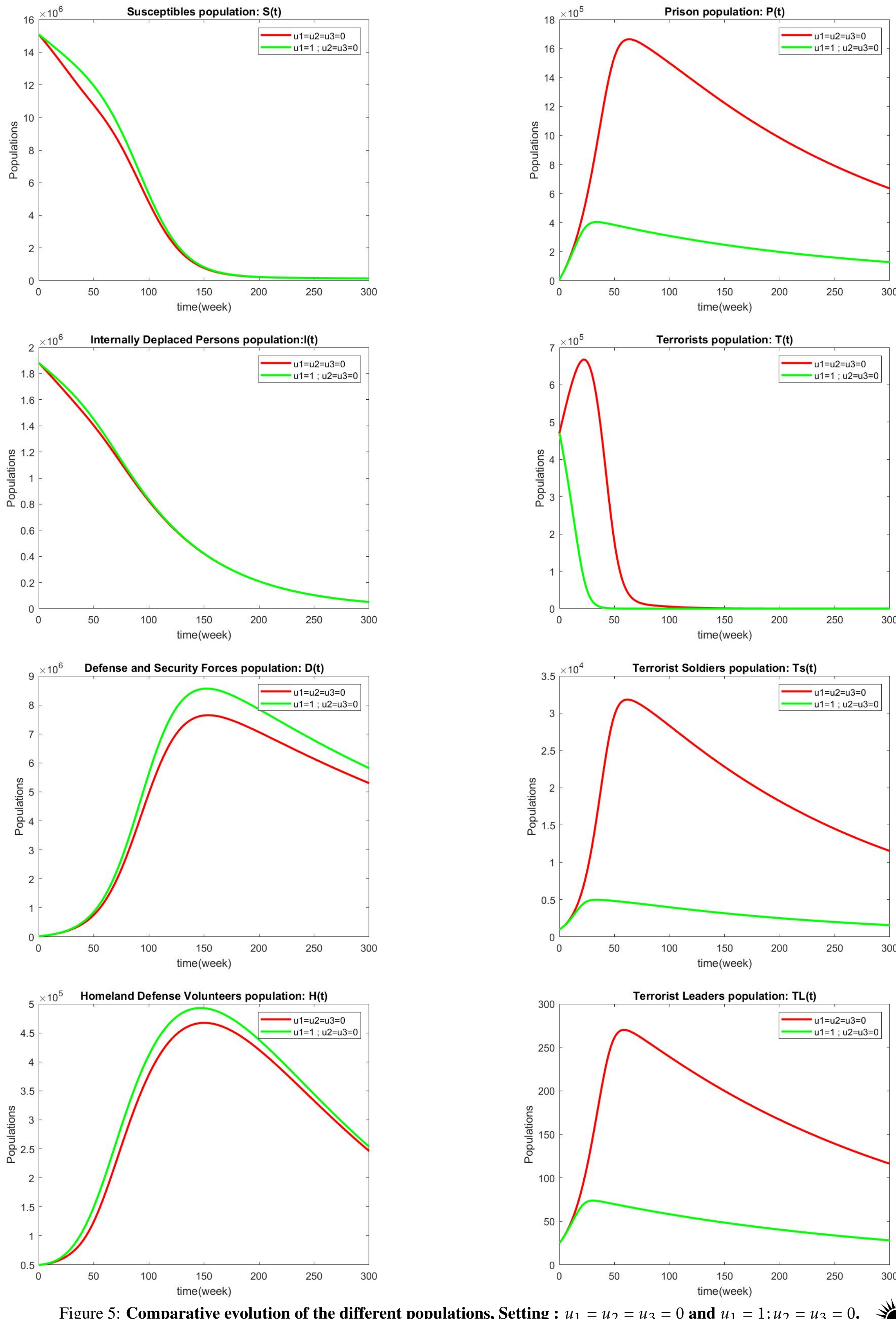

Figure 5: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = 1; u_2 = u_3 = 0$**.**

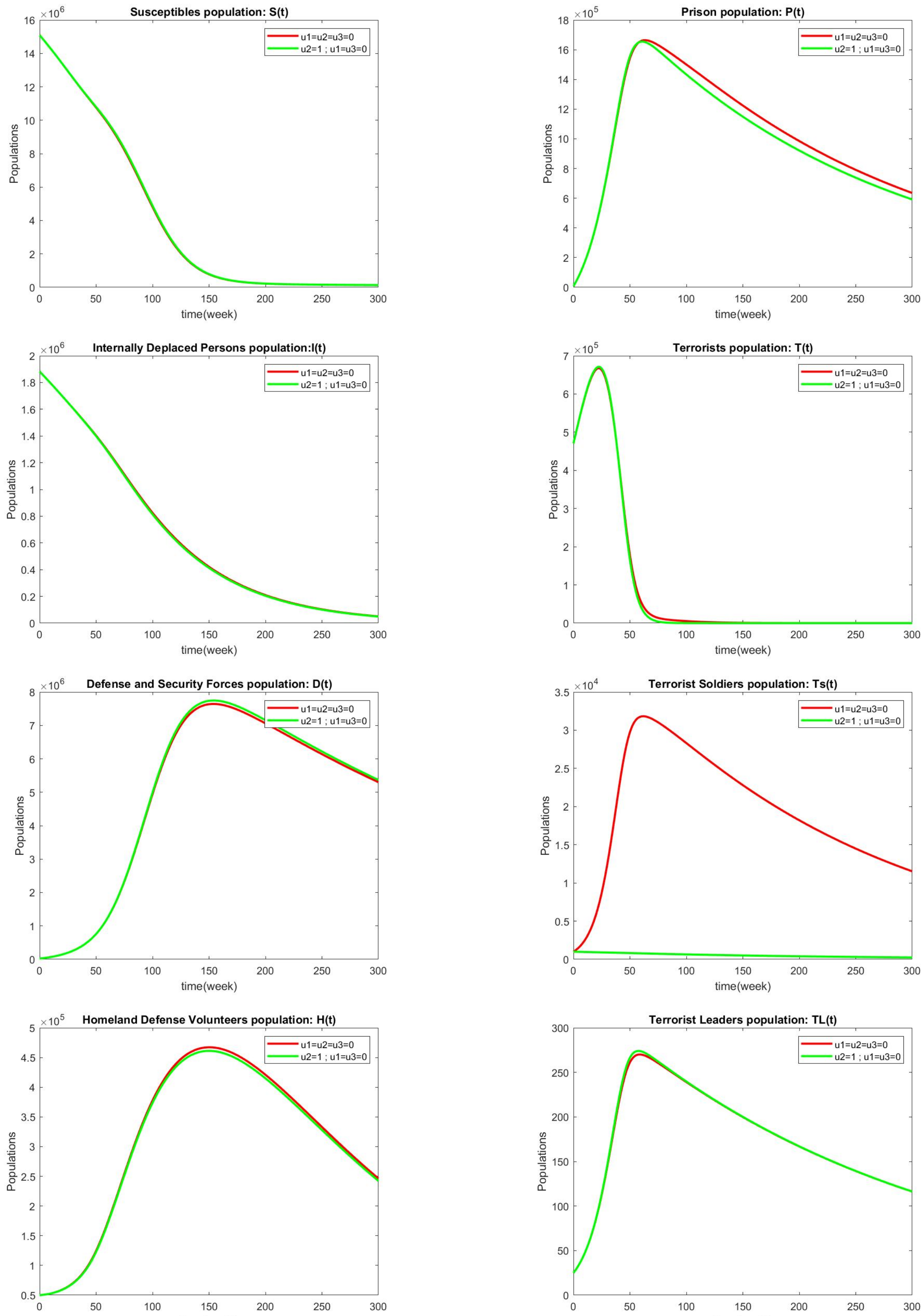

Figure 6: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_2 = 1; u_1 = u_3 = 0$.

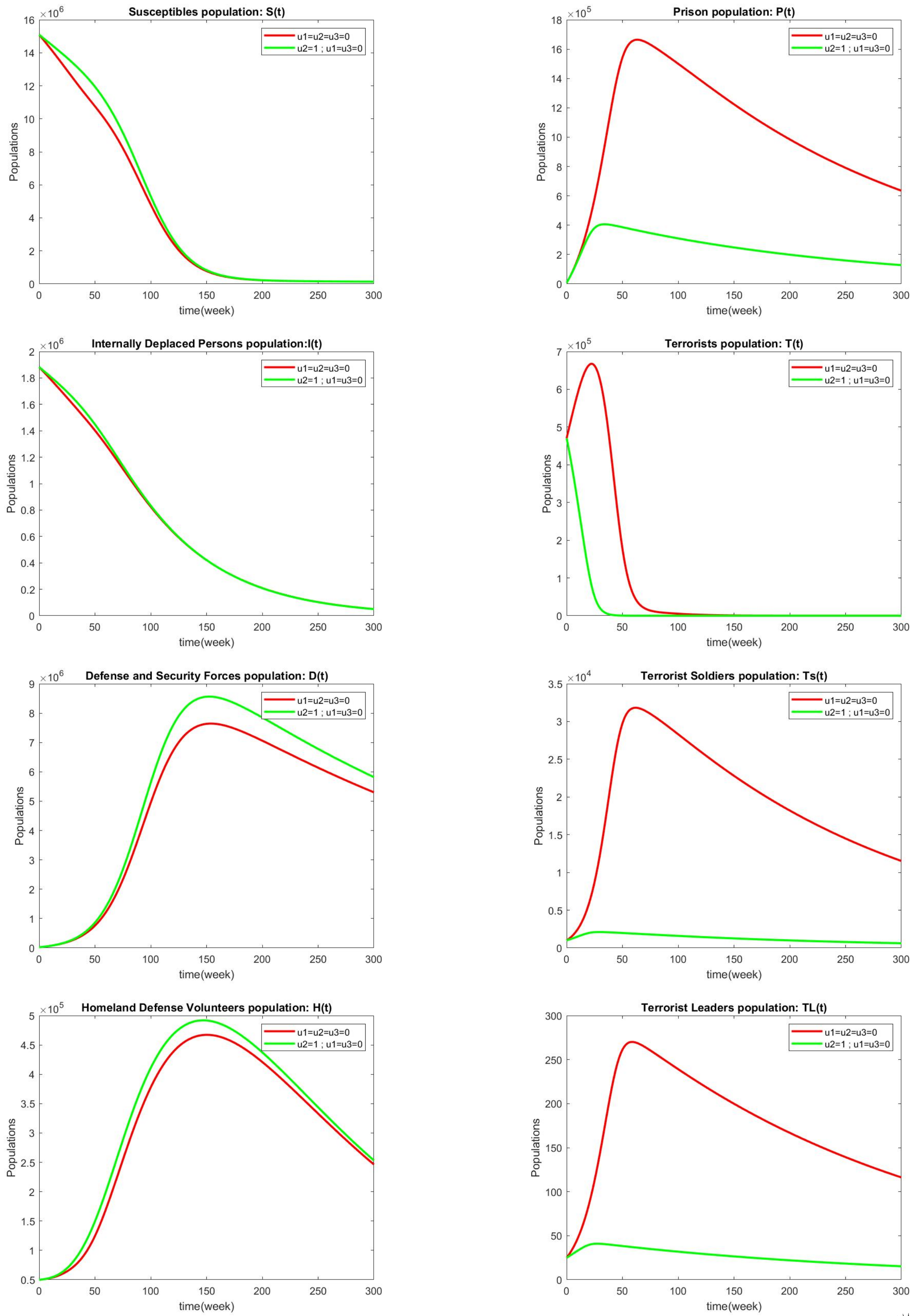

Figure 7: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_3 = 1; u_1 = u_2 = 0$**.**

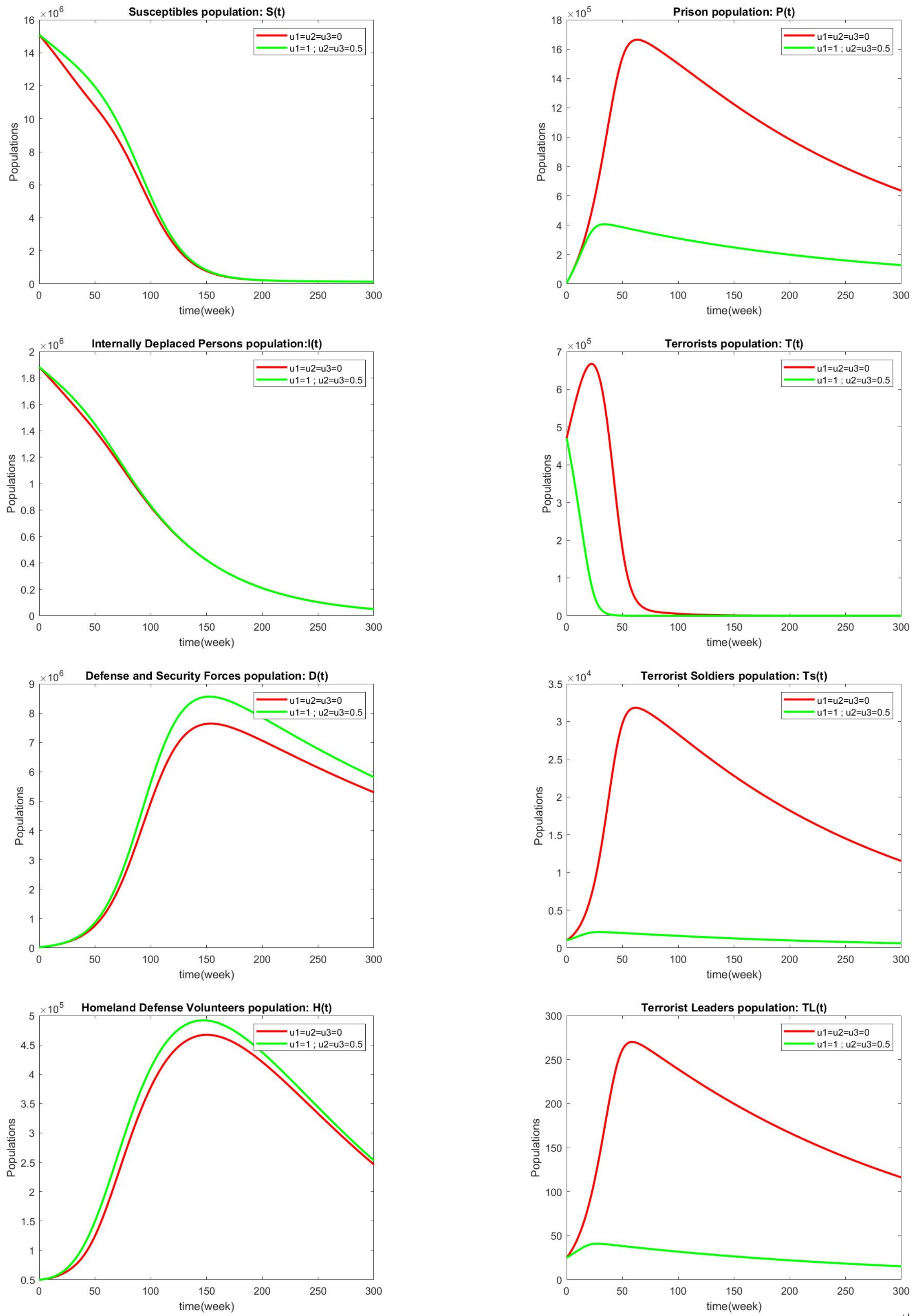

Figure 8: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = 1; u_2 = u_3 = 0.5$**.**

6. Discussing

The analysis of the optimal control problem shows that it's important that there be synergy of action in the fight against terrorism. Indeed, each of the three axes of struggle that we have identified and controlled must be considered and substantial resources must be injected into them. When all the three axes are stimulated simultaneously, terrorism can be eradicated in a time of about 300 weeks, see figure 4. On the other hand, when one of the axes is abandoned, after the same period of 300 weeks, there are still some individuals in compartments T, T_S and T_L, see figures 5, 6 and 7. This will result in a longer time of struggle, during which time uncontrolled events could change the course of the struggle. It should also be noted that the u_1 control is the most sensitive, see figure 5 where the evolution of the populations in compartments T, T_S and T_L is like in figure 4 where $u_1 = u_2 = u_3 = 1$; this means that the government, in coordination with civil organizations and religious institutions, is carrying out large-scale awareness-raising actions in order to rekindle the patriotic flame in the hearts of the people. According to our model, such actions will reduce the number of individuals in compartment T to zero. As a result, the compartments T_S and T_L will no longer be able to recruit and will be emptied; the figure 8 is supporting this idea.

7. Conclusion

This paper present a mathematical modeling and control of the dynamics of terrorist ideologies. In particular, the model takes into account the fact that military personnel, FDS and HDV, can be led to radicalize. Subdividing the population in eight compartments we have constructed a deterministic model using contacts process. The theoretical analysis of the model highlights the existence of a disease-free equilibrium that is globally asymptotically stable. Consequently the spread of the terrorist ideologies can be effectively controlled in the population, whatever the number of infectious people individuals initially introduced into the completely susceptible population. This is how we have introduced tree (03) time-dependent control $u_1(t)$, $u_2(t)$ and $u_3(t)$ with the aim of limiting or even eradicating the spreading of terrorist ideologies in the population.

Terrorism is a new challenge for our country. The fight has made progress, but the threat persists and has diversified, according to the United Nation. It is up to each State, according to its realities, to find endogenous and lasting solutions to effectively and definitively eradicate the terrorist hydra.

8. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

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