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Existence and trajectory controllability for the conformable fractional evolution systems

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Abstract. This article established sufficient conditions for the existence and trajectory controllability for the conformable fractional evolution equation with non-local and classical conditions. These conditions are established through the concept of the operator semi-group, nonlinear functional analysis, Banach fixed point principle and Gronwall's inequality. At last, examples in finite and infinite dimensional Banach spaces were given to validate the obtained results.

AMS Subject Classifications: 34K35, 34K45, 93B05, 93C25.

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Contents

1. Introduction and Background

Fractional order semilinear evolution equations are abstract representations for many engineering and scientific problems, see [8]. Further, the researchers discovered that the non-local condition performs better than the classical condition while dealing with physical situations. Hence, the fractional evolution system with non-local conditions is considered by many researchers, see [5], [17], [4] and [23].

A control system consists of interconnected components arranged in a way that ensures the system produces the desired response.Controllability is one of the essential qualitative properties of dynamical systems in which one has to find a suitable controller for the system that steers the system's state from an arbitrary initial state to

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the desired final state [24]. Kalman introduced a new approach named state-space analysis to find the suitable controller for the linear dynamical system on the finite-dimensional space. The various types of controllability of finite and infinite dimensional systems for the functions (linear, non-linear and semi-linear) through the concepts of operator theory can be seen in the monographs [18], [22] and articles [13], [15], [9], [25], [3], [21], [6], [2] and references therein.

To analyze the exact or complete controllability of a system, it is necessary to identify a controller capable of transitioning the system from its initial state to a specified final state. However, this type of controller might not always be economical. Hence, [10] introduced the Trajectory Controllability (TC) problems.Rather than moving the system directly from an initial state to a desired final state, the focus shifts to identifying a control that guides the system along a predefined trajectory toward the final state. For example, launching a rocket into space requires both a specific path and a target destination to ensure cost efficiency. Hence, TC is investigated by many authors, see [16], [12], [1] and [11]. Sandilya et al.investigated the TC of a semilinear parabolic system[19] . The approximate and TC of the fractional neutral system is studied using the resolvent family and Krasnoselskii's fixed point theorem by [7]. Recently, the TC of Hilfer fractional system is investigated through operator semi-group and Gronwall's inequality by [20]. However, the TC of conformable fractional systems is an untreated topic in the literature.

This article is devoted to study the existence and trajectory controllabiliy for the conformable fractional evolution equation

$$\frac{d^\alpha x}{dt^\alpha} = Ax(t) + f(t, x(t)) + w(t)$$

of order $0 < \alpha < 1$ over the finite interval $\mathbb{I} := [0, T_0]$ in the Banach space $\mathcal{X}$ with classical condition $x(0) = x_0$ and non-local condition $x(0) = h(x)$. Fractional differential operators like defined in the form of integral are not satisfies the properties of multiplication and division like integer order derivatives but, this conformable fractional differential operator satisfies these properties. This gives the motivation to established existence, uniqueness and TC for the system governed by Conformable fractional evolution system.

2. Preliminaries

This section introduces the necessary definitions and properties of conformable fractional differential and fractional integral operator, definitions and necessary properties of Laplace transform of order $0 < \alpha \leq 1$ and and various forms of controllability which will be used throughout the article.

[14] generalized the definition of the classical definition of derivative of integer order to fractional order. They named this derivative as conformable fractional derivative. Unlike R-L and Caputo fractional differential operators, the conformable fractional differential operator satisfies properties for multiplications and divisions.

The definition of conformable fractional derivative of order α is defined as:

Definition 2.1. *[14] [Conformable fractional derivative] The conformable fractional derivative of the order $\alpha \in (0, 1]$ on the function $f : [0, \infty) \to \mathbb{R}$ is defined as:*

$$\frac{d^\alpha f}{dt^\alpha} = \lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}$$

provided the limit on the right exist.

The conformable fractional differential operator have following properties:

Theorem 2.2. *[14] Let,*

$$\mathcal{C}^\alpha([0, \infty), \mathbb{R}) = \left\{ f : [0, \infty) \to \mathbb{R}; \frac{d^\alpha f(t)}{dt^\alpha} exits\ and\ continuous \right\}.$$

For all $f, g \in \mathcal{C}^\alpha$ and scalars a, b then,

(1) $\frac{d^\alpha}{dt^\alpha}\big(af(t)+bg(t)\big) = a\frac{d^\alpha f(t)}{dt^\alpha} + b\frac{d^\alpha g(t)}{dt^\alpha}$.

(2) $\frac{d^\alpha}{dt^\alpha}\big(f(t)g(t)\big) = g(t)\frac{d^\alpha f(t)}{dt^\alpha} + f(t)\frac{d^\alpha g(t)}{dt^\alpha}$.

(3) $\frac{d^\alpha}{dt^\alpha}\lambda = 0$, *for all constants* λ.

(4) *If* $f \in \mathcal{C}^\alpha$ *then* f *is continuous.*

(5) f *is differentiable then* $\frac{d^\alpha f(t)}{dt^\alpha} = t^{1-\alpha}\frac{df(t)}{dt}$

Definition 2.3. *[14][Conformable fractional integral operator] The conformable fractional integral of order* α *for a given function is expressed as follows:*

$$\mathcal{I}^\alpha f(t) = \int_0^t s^{\alpha-1} f(s)ds.$$

The fractional integral operator possesses the following properties:

(1) If f is continuous then, $\frac{d^\alpha(\mathcal{I}^\alpha f(t))}{dt^\alpha} = f(t)$.

(2) If $f \in \mathcal{C}^\alpha([0,\infty),\mathbb{R})$ then, $\mathcal{I}^\alpha\frac{d^\alpha f(t)}{dt^\alpha} = f(t) - f(0)$.

The Laplace transform is not directly compatible with the conformable fractional derivative. Therefore, to solve the fractional differential equations we used modified Laplace transform.

Definition 2.4. *[6][Laplace transform of order* $0 < \alpha \le 1$*] Let* $f(t)$ *be the function defined on the interval* $[0,\infty)$ *then, Laplace transform of order* α *is expressed as:*

$$\mathbb{L}_\alpha(f(t))(s) = \int_0^\infty t^{\alpha-1} exp\bigg(\frac{-st^\alpha}{\alpha}\bigg) f(t)dt,$$

This definition holds true if the improper integral on the right converges.

Remark 2.5. *The Laplace transform of order* $0 < \alpha \le 1$ *satisfies the following properties:*

(1) $\mathbb{L}_\alpha(af(t)+bg(t)) = a\mathbb{L}_\alpha(f(t)) + b\mathbb{L}_\alpha(g(t))$, *where* a *and* b *are scalars, and the functions* $f(t)$ *and* $g(t)$ *have Laplace transforms of order* α.

(2) $\mathbb{L}_\alpha\left(f\left(t^\alpha/\alpha\right)\right) = \mathbb{L}(f(t))$, *where* $\mathbb{L}$ *denotes the classical Laplace transform.*

(3) *If* $f \in \mathcal{C}^\alpha$, *then* $\mathbb{L}_\alpha\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = s\mathbb{L}_\alpha(f(t)) - f(0)$.

Definition 2.6. *[20][Complete Controllability] The system is considered fully controllable over the interval* $\mathbb{I}$ *if, for any* $x_0, x_1 \in \mathcal{X}$, *there exists a control function* $w(t)$ *in the control space* $\mathbb{U}$ *such that the system's state can be driven from the initial state* x_0 *at* $t = 0$ *to the desired final state* x_1 *at* $t = T_0$.

Definition 2.7. *The system is said to be fully controllable over the interval* $\mathbb{I}$ *if it is controllable over all of its sub-intervals* $[t_k, t_{k+1}]$.

Let $\mathcal{C}_\mathcal{T}$ be the set of all functions $y(\cdot)$ defined on the interval $\mathbb{I}$ that satisfy the initial condition $y(0) = x_0$ and the final condition $y(T_0) = x_1$. This set $\mathcal{C}_\mathcal{T}$ is called set of all feasible trajectories. Controller obtained from the concept of complete and total controllability for the linear system will be optimal but for the semilinear or nonlinear system may not be optimal. To overcome this situation one has to design a trajectory having optimum energy or cost and define a controller in such a way that state of the system steers along this trajectory. Finding the controller which steers the system on the prescribed optimal trajectory from initial state to desire final state is called TC.

Definition 2.8 (TC)**.** *The evolution system is callede trajectory controllable (T- Controllable) if for any trajectory $y \in \mathcal{C}_{\mathcal{T}}$, there exist L^2 control function $w \in \mathbb{U}$ such that the state of the system $x(t)$ satisfy $x(t) = y(t)$ almost everywhere over the interval $\mathbb{I}$.*

In total controllability (TC), the goal is to determine the controller that guides the system from an arbitrary initial state to the desired final state along a specified trajectory. Therefore, TC is strongest amongst all the form of controllability.

3. Existence and TC with classical conditions

This section is devoted to existence and TC of the system governed by the conformable fractional evolution system

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= Ax(t) + f(t, x(t)) + w(t) \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

of order $0 < \alpha \leq 1$ over the interval $\mathbb{I}$. For each t, the system's state $x(t)$ resides in the Banach space $\mathcal{X}$, where A is the linear operator acting on $\mathcal{X}$. The function $f : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$ satisfies the given hypotheses, and $w(\cdot)$ is the trajectory controller for the system described by equation (3.1). Let $\mathcal{C}(\mathbb{I}, \mathbb{X})$ denote the Banach space of continuous functions from $\mathbb{I}$ to $\mathcal{X}$, equipped with the norm $\|x\| = \sup_{t \in \mathbb{I}} \|x(t)\|$.

Definition 3.1. *The function $x(t)$ is called the mild solution of the evolution system* (3.1) *over the interval $\mathbb{I}$ if it satisfies the following integral equation:*

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)x_0 + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, x(s)) + w(s)]ds \tag{3.2}$$

where $T(\cdot)$ is the infinitesimal operator semi-group generated by the linear part A. The following assumptions are considered to examine the existence and uniqueness of the mild solution, as well as the total controllability (TC) of the evolution equation (3.1)*.*

- *(A1) Linear operator $A : \mathcal{X} \to \mathcal{X}$ in the system* (3.1) *generates C_0 semi-group $T(\cdot)$ therefore there exist a constant $M > 0$ which satisfies $\|T(t)\| \leq M$, $\forall\, t \in \mathbb{I}$.*
- *(A2) The nonlinear function $f : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$ is measurable with respect to first argument and continuous with respect to second argument. Moreover, there exist a function $l_f : \mathbb{R}^+ \to \mathbb{R}^+$ which is non-decreasing and positive real number r_0 such that*

  $$\|f(t, x_1) - f(t, x_2)\| \leq t^{1-\alpha} l_f(r)\|x_1 - x_2\|$$

  for all $x_1, x_2 \in B_r(\mathcal{X}), r < r_0$ and $t \in \mathbb{I}$.

Following theorem discusses the sufficient conditions for the existence and uniqueness of mild solution for the conformable fractional evolution system (3.1) over the interval $\mathbb{I}$.

Theorem 3.2. *If the conditions (A1) and (A2) are satisfied, then the system* (3.1) *has a unique mild solution for any measurable function $w(t)$ defined over the interval $\mathbb{I}$.*

Proof. Let $w(t)$ be a measurable function defined on $\mathbb{I}$. Then, we define an operator $\mathcal{F}$ acting on the Banach space $\mathcal{X}$ as follows:

$$\mathcal{F}v = T\left(\frac{t^\alpha}{\alpha}\right)x_0 + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, v(s)) + w(s)]ds$$

The equation (3.1) has unique mild solution (3.2) if $\mathcal{F}$ has unique fixed point. To show the operator $\mathcal{F}$ has unique fixed point it is enough to display $\mathcal{F}^{(n)}$ is contraction for at least one $n > 1$.
Let, $t \in \mathbb{I}$ and $x, y \in B_r(\mathcal{X})$ then for $n = 1$,

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \int_0^t s^{\alpha-1}\|T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|\|f(s, x(s)) - f(s, y(s))\|ds$$

Applying the assumptions (A1)-(A2)

$$\|\mathcal{F}x - \mathcal{F}y\| \leq Ml_f(r)\int_0^T s^{\alpha-1}s^{1-\alpha}\|x - y\|ds \leq Ml_f(r)T\|x - y\|$$

For $n = 2$

$$\|\mathcal{F}^{(2)}x - \mathcal{F}^{(2)}y\| \leq \int_0^t s^{\alpha-1}\|T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|\|f(s, \mathcal{F}x(s)) - f(s, \mathcal{F}y(s))\|ds$$

Applying assumptions (A1)-(A2)

$$\begin{aligned}\|\mathcal{F}^{(2)}x - \mathcal{F}^{(2)}y\| &\leq Ml_f(r)\int_0^t s^{\alpha-1}\|\mathcal{F}x - \mathcal{F}y\|ds \\ &\leq M^2[l_f(r)]^2\int_0^t\int_0^\tau \|x - y\|dsd\tau \\ &\leq M^2[l_f(r)]^2\int_0^T\int_0^T \|x - y\|dsd\tau \leq \frac{M^2[l_f(r)]^2T^2}{2!}\|x - y\|\end{aligned}$$

Continuing this process for $n = 3, 4, \cdots m$ and applying assumptions (A1)-(A2) to get:

$$\begin{aligned}\|\mathcal{F}^{(m)}x - \mathcal{F}^{(m)}y\| &\leq M^n\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{m-1}}(l_f(t))^m\|x - y\|dsd\tau_{m-1}\cdots d\tau_1 \\ &\leq \int_0^T\int_0^T\cdots\int_0^T(l_f(t))^m\|x - y\|dsd\tau_{m-1}\cdots d\tau_1 \\ &\leq \frac{M^m[l_f(r)]^mT^m}{m!}\|x - y\| = \rho\|x - y\|\end{aligned}$$

Taking maximum over interval $\mathbb{I}$, the quantity ρ tends to zero as $m \to \infty$ for fixed T. Hence, there exist m with $\mathcal{F}^m$ is contraction on $B_r(\mathcal{X})$. Therefore, the evolution system (3.1) has unique mild solution for any measurable function $w(t)$ over the interval $\mathbb{I}$ by general Banach contraction theorem . ■

Remark 3.3. *To established the conditions for the existence and uniqueness the generalized Banach fixed point theorem is used. This is because this fixed point theorem gives the guarantee about uniqueness of the solution.*

Theorem 3.4. *If the conditions (A1) and (A2) hold, then the system* (3.1) *is trajectory controllable over the interval* $\mathbb{I}$.

Proof. Let $u(t)$ be any trajectory from $\mathcal{C}_\mathcal{T}$ and define feed-back control of the system as:

$$w(t) = \frac{d^\alpha u(t)}{dt^\alpha} - Au(t) - f(t, u(t)) \tag{3.3}$$

Putting feedback control $w(t)$ from equation (3.3) in equation (3.1) and simplifying we have,

$$\frac{d^\alpha}{dt^\alpha}[x(t) - u(t)] = A[x(t) - u(t)] + f(t, x(t)) - f(t, u(t)) \tag{3.4}$$

Choosing $z(t) = x(t) - u(t)$, equation (3.4) yields:

$$\begin{aligned} \frac{d^\alpha z(t)}{dt^\alpha} &= Az(t) + [f(t, x(t)) - f(t, u(t))] \\ z(0) &= 0. \end{aligned} \tag{3.5}$$

The solution of the system (3.5) is

$$z(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, x(s)) - f(s, u(s))]ds$$

Therefore,

$$\|z(t)\| \leq \int_0^t s^{\alpha-1} \|T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\| \|f(s, x(s)) - f(s, u(s)\|ds$$

Assuming (A1)-(A2),

$$\|z(t)\| \leq M \int_0^t s^{\alpha-1} s^{1-\alpha} l_f(r) \|x(s) - u(s)\|ds \leq M l_f(r) \int_0^t \|z(s)\|ds$$

Using Gronwall's inequality $z(t) = 0\, a.e.$. Therefore, $x(t) = u(t) a.e.$ over the interval $\mathbb{I}$. Therefore, the system (3.1) is trajectory controllable over the interval $\mathbb{I}$.

■

Example 3.5. *Consider the evolution system*

$$\begin{aligned} \frac{d^\alpha x_1(t)}{dt^\alpha} &= -2x_1(t) + w_1(t) \\ \frac{d^\alpha x_2(t)}{dt^\alpha} &= -3x_2(t) + \frac{1}{2} t^{1-\alpha} e^{-x_2(t)} + w_2(t) \end{aligned} \tag{3.6}$$

Consider the interval $[0, 1]$ *with initial conditions* $x_1(0) = 1$*,* $x_2(0) = 0$*, and* $\alpha = 0.75$*. Let* $x(t) = [x_1(t), x_2(t)]^T$*,* $w(t) = [w_1(t), w_2(t)]^T$*, and* $f(t, x) = \left[0, \frac{1}{2} t^{1-\alpha} e^{-x_2(t)}\right]^T$*.and* $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ *the system* (3.6) *represented as:*

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)) + w(t) \tag{3.7}$$

with initial condition $x(0) = x_0 = [1, 0]'$*. Clearly the linear operator* A *in the system* (3.7) *generates* C_0 *semigroup* $T(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$*. Also,* $f(t, x) = [0, t^{1-\alpha} \frac{1}{2} e^{-x_2(t)}]'$*is measurable with respect to t and Lipschitz continuous in* x*. Consequently, by Theorem 3.2, the equation* (3.7) *admits a unique mild solution for any measurable function* $w(t)$ *on the interval* $[0, 1]$*. Furthermore, by Theorem 3.4, the system* (3.7) *is trajectory controllable on the interval* $[0, 1]$*.*

The system's state over the interval $[0, 1]$ *without trajectory controller is shown in the Figure-1. Let,* $u(t) = [(t^2 + 1), 2t]'$ *be the trajectory along which the state of the system has to steer from initial state* $x_0 = [1, 0]'$ *at time* $t = 0$ *to desired final state* $x_1 = [2, 2]'$*. Figure-2 shows the trajectory along which the state of the system has to steer from initial state to desired final state. Plugging the trajectory controller* $w(t) = [w_1(t), w_2(t)]'$

$$\begin{aligned} w_1(t) &= 2t^{2-\alpha} + 2t^2 + 2 \\ w_2(t) &= 2t^{1-\alpha} + 6t - \frac{1}{2} t^{1-\alpha} e^{-2t} \end{aligned} \tag{3.8}$$

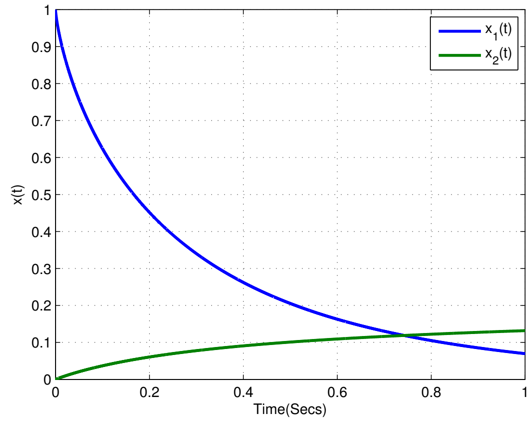

Figure 1: System (3.6) without control

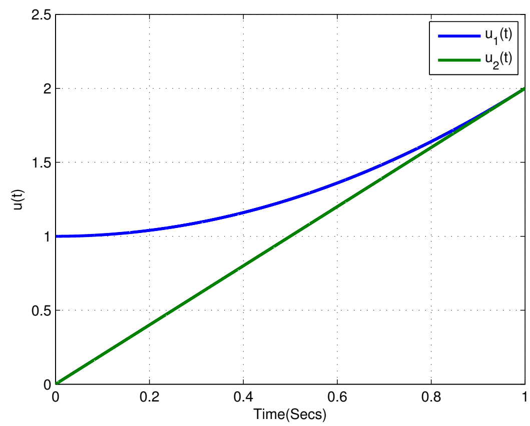

Figure 2: Trajectory u(t) of the system (3.6)

in the system (3.6)*, the system*

$$\begin{aligned}\frac{d^\alpha x_1(t)}{dt^\alpha} &= -2x_1(t) + 2t^{2-\alpha} + 2t^2 + 2 \\ \frac{d^\alpha x_2(t)}{dt^\alpha} &= -3x_2(t) + \frac{1}{2}t^{1-\alpha}e^{-x_2(t)} + 2t^{1-\alpha} + 6t - \frac{1}{2}t^{1-\alpha}e^{-2t}\end{aligned} \tag{3.9}$$

steers from the state $x_0 = [1,0]'$ *at* $t = 0$ *to state* $x_1 = [2,2]'$ *at* $t = 1$ *along trajectory* $u(t) = [t^2+1, 2t]$*. Figure-3 and Figure-4 show the controlled state of the system* $x(t)$ *on desired trajectory* $u(t)$ *and behavior of the controller* $w(t)$ *over the interval* $[0,1]$*.*

Example 3.6. *Let* $\mathcal{X} = L^2([0,1],\mathbb{R})$ *and consider the partial differential equation*

$$\frac{\partial^\alpha y(t,x)}{\partial t^\alpha} = y_{xx}(t,x) + t^{1-\alpha}e^{-y(t,x)} + w(t) \tag{3.10}$$

with the initial condition $u(0,x) = u_0(x)$ *and the boundary conditions* $y(t,0) = y(t,1) = 0$*.*

Define an operator A *as* $Ay = y_{xx}$ *over the domain* $D(A) = H^2(0,1) \cap H^1(0,1)$ *in* $\mathcal{X}$ *satisfies conditions of Hile-Yosida theorem. Therefore, by the Hille-Yosida theorem, the operator* A *serves as the infinitesimal generator of the strongly continuous semigroup of operators* $T(t)$*, which is defined by*

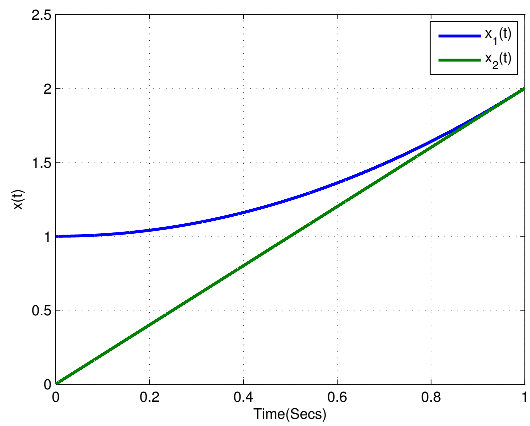

Figure 3: System (3.6) with control w(t)

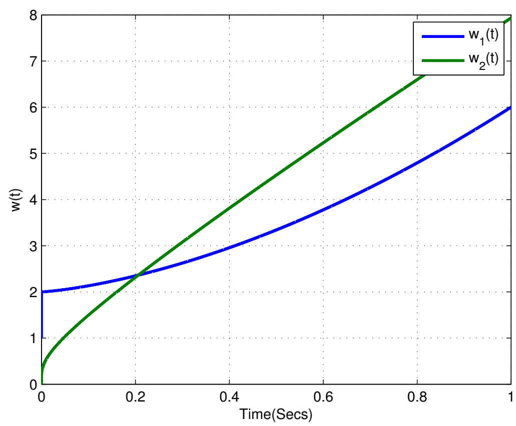

Figure 4: Trajectory controller w(t) of the system (3.6)

$$T(t)z = \sum_{n=1}^{\infty} exp(-n^2\pi^2 t) < z, \phi_n > \phi_n$$

where, ϕ_n are orthonormal Fourier basis for $\mathcal{X}$.

The equation (3.10) *can be expressed in an abstract form in the Hilbert space $\mathcal{X} = L^2([0,1], \mathbb{R})$ as:*

$$\begin{aligned} \frac{d^\alpha z}{dt^\alpha} &= Az(t) + f(t, z(t)) + w(t) \\ z(0) &= y_0 \end{aligned} \tag{3.11}$$

Clearly, $f(t,z) = t^{1-\alpha}e^{-z}$ is continuous function and there exist $l_f(r) = 1$ on $B_r(\mathcal{X})$ satisfying $\|f(t,z_1) - f(t,z_2)\| \leq \|z_1 - z_2\|$. Therefore, by Theorem-3.2, the system (3.1) *possesses a unique mild solution for any measurable function $w(t)$ over the interval $\mathbb{I}$, and according to Theorem-3.4, the system* (3.1) *is trajectory controllable over the interval $\mathbb{I}$.*

4. Existence and Controllability with Nonlocal Condition

There are many mathematical models having nonlocal initial conditions give more fruit full information than models having classical initial conditions. This motivates to study existence and TC of the system governed by the conformable fractional evolution system

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= Ax(t) + f(t, x(t)) + w(t) \\ x(0) &= h(x) \end{aligned} \tag{4.1}$$

over the interval $\mathbb{I}$. In this context, the state of the system, $x(t)$, is an element of the Banach space $\mathcal{X}$ for each value of t. A is the linear operator defined on the Banach space $\mathcal{X}$, $f : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{X}$ satisfy the conditions of the hypothesis and $w(\cdot)$ is trajectory controller for the system (4.1).

Definition 4.1. *The function $x(t)$ is considered a mild solution of the evolution system* (4.1) *on the interval $\mathbb{I}$ if it satisfies the following integral equation:*

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)h(x) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, x(s)) + w(s)]ds \tag{4.2}$$

where $T(\cdot)$ is the infinitesimal generator of the linear operator A.

The following assumptions are made in order to analyze the existence and uniqueness of the mild solution, as well as the trajectory controllability (TC) of the evolution equation (4.1).

(B1) Linear operator $A : \mathcal{X} \to \mathcal{X}$ in the system (4.1) generates C_0 semi-group $T(\cdot)$ therefore there exist a constant $M > 0$ which satisfies $\|T(t)\| \leq M$ for all $t \in \mathbb{I}$.

(B2) The nonlinear function $f : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$ is measurable with respect to first argument and continuous with respect to second argument. Moreover, there exist a constant $0 < l_f < 1$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq t^{1-\alpha} l_f \|x_1 - x_2\|$$

for all $x_1, x_2 \in B_r(\mathcal{X}), r < r_0$ and $t \in \mathbb{I}$.

(B3) The function $h : \mathcal{X} \to \mathcal{X}$ is continuous with respect to x and a constant $0 < l_h < 1$ exists which satisfies

$$\|h(x_1) - h(x_2)\| \leq l_h \|x_1 - x_2\|$$

for all $x_1, x_2 \in B_r(\mathcal{X}), r < r_0$.

Theorem 4.2. *Let the assumptions (B1)-(B3) holds, then the system* (4.1) *has a unique mild solution for any measurable function $w(t)$ over $\mathbb{I}$.*

Proof. Let $w(t)$ be any measurable function over $\mathbb{I}$ then define an operator $\mathcal{F}$ on the Banach space $\mathcal{X}$ by

$$\mathcal{F}v = T\left(\frac{t^\alpha}{\alpha}\right)h(v) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, v(s)) + w(s)]ds$$

The equation (4.1) has unique mild solution (4.2) if $\mathcal{F}$ has unique fixed point. To show the operator $\mathcal{F}$ has unique fixed point it is sufficient to show $\mathcal{F}$ is contraction. Let, $t \in \mathbb{I}$ and $x, y \in B_r(\mathcal{X})$ then

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \|T(t^\alpha/\alpha)\| \|h(x) - h(y)\| + \int_0^t s^{\alpha-1} \left\|T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\right\| \|f(s, x(s)) - f(s, y(s))\| ds$$

Applying the assumptions (B1)-(B3)

$$\|\mathcal{F}x - \mathcal{F}y\| \leq Ml_h\|x-y\| + Ml_f\int_0^T s^{\alpha-1}s^{1-\alpha}\|x-y\|ds \leq M(l_h + l_fT)\|x-y\|$$

Therefore, the operator $\mathcal{F}$ is contraction if $M(l_h + l_fT) < 1$. Hence equation (4.1) has unique mild solution for any measurable function $w(t)$ over interval $\mathbb{I}$ if $M(l_h + l_fT) < 1$. ■

Theorem 4.3. *If assumptions (B1)-(B3) are satisfied then the system* (4.1) *is trajectory controllable over the interval* $\mathbb{I}$ *provided* $L < 1$.

Proof. Let $u(t)$ be any trajectory from $\mathcal{C}_{\mathcal{T}}$ and define feed-back control of the system as:

$$w(t) = \frac{d^\alpha u(t)}{dt^\alpha} - Au(t) - f(t,u(t)) \tag{4.3}$$

Putting feedback control $w(t)$ from equation (4.3) in equation (4.1) and simplifying we have,

$$\frac{d^\alpha}{dt^\alpha}[x(t)-u(t)] = A[x(t)-u(t)] + f(t,x(t)) - f(t,u(t)) \tag{4.4}$$

Choosing $z(t) = x(t) - u(t)$, equation (3.4) yields:

$$\begin{aligned} \frac{d^\alpha z(t)}{dt^\alpha} &= Az(t) + [f(t,x(t)) - f(t,u(t))] \\ z(0) &= h(x) - h(u). \end{aligned} \tag{4.5}$$

The solution of the system (4.5) is represented as follows:

$$z(t) = T\left(\frac{t^\alpha}{\alpha}\right)[h(x)-h(u)] + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s,x(s)) - f(s,u(s))]ds$$

Clearly, $\|z(t)\| \geq 0$ for all $t \in \mathbb{I}$ therefore to show z is identically equal to 0 over the interval $\mathbb{I}$, it is sufficient to show $\|z(t)\| \leq 0$. Consider,

$$\|z(t)\| \leq \|T\left(\frac{t^\alpha}{\alpha}\right)\|\|h(x(t)) - h(u(t))\| + \int_0^t s^{\alpha-1}\|T\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|\|f(s,x(s)) - f(s,u(s)\|ds$$

Assuming (B1)-(B3),

$$\begin{aligned} \|z(t)\| &\leq M\|h(x(s)) - h(u(s))\| + M\int_0^t s^{\alpha-1}s^{1-\alpha}l_f\|x(s)-u(s)\|ds \\ &\leq M[l_h + l_fT]\|z(s)\| \end{aligned}$$

Define $L = M[l_h + l_fT]$ then, $\|z(t)\| \leq 0$ provided $L < 1$. Thus, $z(t) = 0$ $a.e.$ over the interval $\mathbb{I}$ provided $L < 1$. Hence, system (4.1) is trajectory controllable over the interval $\mathbb{I}$ provided $L < 1$. ■

Example 4.4. *Let* $\mathcal{X} = L^2([0,1],\mathbb{R})$ *and consider the partial differential equation*

$$\frac{\partial^\alpha y(t,x)}{\partial t^\alpha} = y_{xx}(t,x) + \frac{1}{10}t^{1-\alpha}cos(y(t,x)) + w(t) \tag{4.6}$$

with initial condition $y(0,x) = h(y)$,$h(y(x)) = \sum_{i=1}^2 \frac{1}{3^i}y(1/i,x)$ *and boundary conditions* $y(t,0) = y(t,1) = 0$. *Proceeding in the same manner as in example-3.6, the equation* (4.6) *converted into abstract equation in* $\mathcal{X} = L^2([0,1],\mathbb{R})$ *as:*

$$\begin{aligned} \frac{d^\alpha z}{dt^\alpha} &= Az(t) + f(t,z(t)) + w(t) \\ z(0) &= h(z) \end{aligned} \tag{4.7}$$

Clearly, $f(t,z) = t^{1-\alpha}e^{-z}$ *is smooth function therefore there exist* $l_f(r) = 1/20$ *on* $B_r(\mathcal{X})$ *for all* $r < r_0$ *satisfying* $\|f(t,z_1) - f(t,z_2)\| \leq \frac{1}{20}\|z_1 - z_2\|$. *Moreover, there exist a constant* $l_h = 3/4r$ *such that* $\|h(z_1) - h(z_2)\| \leq 3/4$ *on* $B_r(\mathcal{X})$ *for all* $r < r_0$. *Hence, the system* (4.1) *has unique mild solution for all measurable function* $w(t)$ *by Theorem-4.2, and the system* (4.1) *is trajectory controllable over the interval* $\mathbb{I}$ *by Theorem-4.3* .

5. Conclusion

This article considered the system governed by conformable fractional evolution systems with classical and non-local conditions respectively and established the existence of mild solution and TC of the systems. These conditions for the establishing the results gives guaranteed about the existence and uniqueness and TC for the evolution systems. Application in terms of illustrations were included to demonstrate the obtained results.

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