MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(04)**(2023), 447–456.
http://doi.org/10.26637/mjm1104/008

On Kenmotsu metric spaces satisfying some conditions on the W_7 -curvature tensor

Pakize Uygun*[1] and Mehmet Atçeken[2]

[1,2] *Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, 68100, Aksaray, Turkey.*

Received 11 July 2023; *Accepted* 29 September 2023

Abstract. This research article is about the geometry of the Kenmotsu manifold. Some important properties such as the $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^\star = 0$ curvature conditions of the Kenmotsu manifold have been investigated.

AMS Subject Classifications: 53C25, 53C35

Keywords: Kenmotsu manifold, η-Einstein manifold, Einstein manifold, Riemannian curvature tensor.

Contents

1 Introduction and Background 447

2 Preliminaries 448

3 Some curvature characterizations on Kenmotsu metric spaces 449

1. Introduction and Background

In 1963, K. Kobayashi and K. Nomizu demonstrated that Any two complete Riemannian manifolds that are simply connected and have a constant curvature k, are isometric to one another. [7]. Following that, several scholars, including [8–10], explored manifolds curvature in various methods to varying degrees.

According to D. B. Abdussattar's research, tensor $\widetilde{C}$ must disappear identically in order for a space time to be conharmonic to a flat space time. If a space time is conharmonically flat, it is either empty, in which case it is flat, or filled with a distribution defined by an energy momentum tensor T that has an electromagnetic field's algebraic structure while also conforming to a flat space time [1].

Let M be an $n-$dimensional differentiable manifold of differentiability class C^{r+1} with a (1,1) tensor field ϕ, the connected vector field ξ, a contact form η and the related Riemannian metric g. Kenmotsu described the differential geometric features of class manifolds in 1972. The structure developed is known as the Kenmotsu structure. A Sasakian structures are distinct from Kenmotsu structures. [6].

This study aims to examine a Kenmotsu metric manifold's curvature tensor's characteristics. In addition, we take research $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^\star = 0$ where W_5, W_6, W_7, W_8, W_9, and $W_0^\star$ denote the curvature tensors of Kenmotsu manifold, respectively.

***Corresponding author.** Email addresses: **pakizeuygun@hotmail.com** (Pakize Uygun) and **mehmetatceken@aksaray.edu.tr** (Mehmet Atçeken)

2. Preliminaries

We have collected some fundamental information regarding contact metric manifold in this part. With a $(2n+1)$-dimensional linked structure, let M be an almost contact metric manifold. (φ, ξ, η, g), that is, φ is an $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and the Riemanniann metric g satisfying

$$\varphi^2(\theta_1) = -\theta_1 + \eta(\theta_1)\xi, \quad \eta(\varphi\theta_1) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0 \tag{2.2}$$

for all $\theta_1, \theta_2 \in \Gamma(TM)$[11]. Let g be Riemannian metric compatible with (φ, ξ, η),
that is

$$g(\varphi\theta_1, \varphi\theta_2) = g(\theta_1, \theta_2) - \eta(\theta_1)\eta(\theta_2), \tag{2.3}$$

or equivalently,

$$g(\theta_1, \varphi\theta_2) = -g(\varphi\theta_1, \theta_2) \quad \text{and} \quad g(\theta_1, \xi) = \eta(\theta_1) \tag{2.4}$$

for all $\theta_1, \theta_2 \in \Gamma(TM)$[4]. If in addition to above relations

$$(\nabla_{\theta_1}\varphi)\theta_2 = -\eta(\theta_2)\varphi\theta_1 - g(\theta_1, \varphi\theta_2)\xi, \tag{2.5}$$

and

$$\nabla_{\theta_1}\xi = \theta_1 - \eta(\theta_1)\xi, \tag{2.6}$$

where g holds Riemannian connection is indicated by the symbol, the manifold $(M, \varphi, \xi, \eta, g)$ is referred to as an almost Kenmotsu manifold. In a Kenmotsu manifold M, the following relation holds[5, 6]:

$$(\nabla_{\theta_1}\eta)\theta_2 = g(\theta_1, \theta_2) - \eta(\theta_1)\eta(\theta_2), \tag{2.7}$$

$$R(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 - \eta(\theta_2)\theta_1, \tag{2.8}$$

$$R(\xi, \theta_1)\theta_2 = \eta(\theta_2)\theta_1 - g(\theta_1, \theta_2)\xi, \tag{2.9}$$

$$S(\theta_1, \xi) = -2n\eta(\theta_1), \tag{2.10}$$

$$Q\xi = -2n\xi, \tag{2.11}$$

where r is scalar curvature of the connection ∇, As defined by $S(\theta_1, \theta_2) = g(Q\theta_1, \theta_2)$, where Q is the Ricci operator, S is the Ricci tensor, and R is the Riemannian curvature tensor. It submits to

$$S(\varphi\theta_1, \varphi\theta_2) = S(\theta_1, \theta_2) + 2n\eta(\theta_1)\eta(\theta_2). \tag{2.12}$$

Unknown Kenmotsu manifold if M's Ricci tensor S has the following structure, M is allegedly η-Einstein manifold.

$$S(\theta_1, \theta_2) = ag(\theta_1, \theta_2) + b\eta(\theta_1)\eta(\theta_2) \tag{2.13}$$

in which a and b are functions on (M^{2n+1}, g) for any arbitrary vector fields θ_1, θ_2. $\eta-$ Einstein manifold becomes Einstein manifold if $b = 0$ [6, 14]. Let M be a Kenmotsu manifold of dimension $(2n+1)$. According to the relationship, the curvature tensor R of M is determined by

$$\widetilde{R}(\theta_1, \theta_2)\theta_3 = \widetilde{\nabla}_{\theta_1}\widetilde{\nabla}_{\theta_2}\theta_3 - \widetilde{\nabla}_{\theta_2}\widetilde{\nabla}_{\theta_1}\theta_3 - \widetilde{\nabla}_{[\theta_1, \theta_2]}\theta_3. \tag{2.14}$$

Following that, in a Kenmotsu manifold, we arrive

$$\widetilde{R}(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 + g(\theta_2, \theta_3)\theta_1 - g(\theta_1, \theta_3)\theta_2, \tag{2.15}$$

where $R(\theta_1, \theta_2)\theta_3 = \nabla_{\theta_1}\nabla_{\theta_2}\theta_3 - \nabla_{\theta_2}\nabla_{\theta_1}\theta_3 - \nabla_{[\theta_1,\theta_2]}\theta_3$, is the curvature tensor of M with respect to the connection ∇ [15, 16, 19]. The idea that W_5-curvature tensor was explained by [13]. W_5-curvature tensor,

W_6-curvature tensor, W_7-curvature tensor, W_8-curvature tensor, W_9-curvature tensor and $W_0^{\star}$-curvature tensor of a $(2n+1)$-dimensional Riemannian manifold are, respectively, specified as

$$W_5(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 - \frac{1}{2n}[S(\theta_1, \theta_3)\theta_2 - g(\theta_1, \theta_3)Q\theta_2], \tag{2.16}$$

$$W_6(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2, \theta_3)\theta_1 - g(\theta_1, \theta_2)Q\theta_3], \tag{2.17}$$

$$W_7(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2, \theta_3)\theta_1 - g(\theta_2, \theta_3)Q\theta_1], \tag{2.18}$$

$$W_8(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 - \frac{1}{2n}[S(\theta_2, \theta_3)\theta_1 - S(\theta_1, \theta_2)\theta_3], \tag{2.19}$$

$$W_9(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 + \frac{1}{2n}[S(\theta_1, \theta_2)\theta_3 - g(\theta_2, \theta_3)Q\theta_1], \tag{2.20}$$

$$W_0^{\star}(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 + \frac{1}{2n}[S(\theta_2, \theta_3)\theta_1 - g(\theta_1, \theta_3)Q\theta_2], \tag{2.21}$$

for all $\theta_1, \theta_2, \theta_3 \in \Gamma(TM)$[12, 13].

3. Some curvature characterizations on Kenmotsu metric spaces

The key findings for this article are presented in this section.

When we designate the W_5 curvature tensor from (2.16) and assume that M is a $(2n+1)$-dimensional Kenmotsu metric manifold, we obtain for subsequent consideration.

$$W_5(\theta_1, \theta_2)\xi = 2\eta(\theta_1)\theta_2 - \eta(\theta_2)\theta_1 + \frac{1}{2n}\eta(\theta_1)Q\theta_2. \tag{3.1}$$

Adding $\theta_1 = \xi$ to (3.1)

$$W_5(\xi, \theta_2)\xi = 2\theta_2 - \eta(\theta_2)\xi + \frac{1}{2n}Q\theta_2. \tag{3.2}$$

In (2.17) choosing $\theta_3 = \xi$ and using (2.8), we obtain

$$W_6(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 - g(\theta_1, \theta_2)\xi. \tag{3.3}$$

In (3.3), it follows

$$W_6(\xi, \theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.4}$$

From (2.18) and (2.8), we arrive

$$W_7(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 + \frac{1}{2n}\eta(\theta_2)Q\theta_1. \tag{3.5}$$

Setting $\theta_1 = \xi$, in (2.18)

$$W_7(\xi, \theta_2)\theta_3 = \eta(\theta_3)\theta_2 - 2g(\theta_2, \theta_3)\xi - \frac{1}{2n}S(\theta_2, \theta_3)\xi, \tag{3.6}$$

and

$$W_7(\xi, \theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.7}$$

The same applies, putting $\theta_3 = \xi$ in (2.19) and using (2.8), we have

$$W_8(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 + \frac{1}{2n}S(\theta_1, \theta_2)\xi. \tag{3.8}$$

In (3.8), using $\theta_1 = \xi$, we get

$$W_8(\xi, \theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.9}$$

Choosing $\theta_3 = \xi$, in (2.20), we obtain

$$W_9(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 - \eta(\theta_2)\theta_1 + \frac{1}{2n}(S(\theta_1, \theta_2)\xi - \eta(\theta_2)Q\theta_1). \tag{3.10}$$

In (3.10) it follows

$$W_9(\xi, \theta_2)\xi = \theta_2 - \eta(\theta_2)\xi. \tag{3.11}$$

In (2.21), choosing $\theta_3 = \xi$ and using (2.8), we obtain

$$W_0^\star(\theta_1, \theta_2)\xi = \eta(\theta_1)\theta_2 - 2\eta(\theta_2)\theta_1 - \frac{1}{2n}\eta(\theta_1)Q\theta_2. \tag{3.12}$$

Setting $\theta_1 = \xi$, in (3.12)

$$W_0^\star(\xi, \theta_2)\xi = \theta_2 - 2\eta(\theta_2)\xi - \frac{1}{2n}Q\theta_2. \tag{3.13}$$

Theorem 3.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_5 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that M is a $W_7 \cdot W_5 = 0$. This implies that

$$\begin{aligned}(W_7(\theta_1, \theta_2)W_5)(\theta_4, \theta_5)\theta_3 = {} & W_7(\theta_1, \theta_2)W_5(\theta_4, \theta_5)\theta_3 - W_5(W_7(\theta_1, \theta_2)\theta_4, \theta_5)\theta_3 \\ & -W_5(\theta_4, W_7(\theta_1, \theta_2)\theta_5)\theta_3 \\ & -W_5(\theta_4, \theta_5)W_7(\theta_1, \theta_2)\theta_3 = 0, \end{aligned} \tag{3.14}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Taking $\theta_1 = \theta_3 = \xi$ in (3.14), with the usage of (3.6) and (3.7), for $p_1 = \frac{1}{2n}$, we have

$$\begin{aligned}(W_7(\xi, \theta_2)W_5)(\theta_4, \theta_5)\xi = {} & W_7(\xi, \theta_2)(2\eta(\theta_4)\theta_5 - \eta(\theta_5)\theta_4 + p_1\eta(\theta_4)Q\theta_5) \\ & -W_5(\eta(\theta_4)\theta_2) - 2g(\theta_2, \theta_4)\xi - p_1S(\theta_2, \theta_4)\xi, \theta_5)\xi \\ & -W_5(\theta_4, \eta(\theta_5)\theta_2 - 2g(\theta_2, \theta_5)\xi - p_1S(\theta_2, \theta_5)\xi)\xi \\ & -W_5(\theta_4, \theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.15}$$

While considering (3.1), (3.2), (3.6) in (3.15), we obtain

$$\begin{aligned} & -W_5(\theta_4, \theta_5)\theta_2 - 4\eta(\theta_4)g(\theta_5, \theta_2)\xi - 2\eta(\theta_4)S(\theta_5, \theta_2)\xi \\ & +\eta(\theta_5)S(\theta_2, \theta_4)\xi - 2np_1\eta(\theta_4)\eta(\theta_5)\theta_2 - p_1\eta(\theta_4)S(\theta_2, Q\theta_5)\xi \\ & +2p_1g(\theta_2, \theta_4)Q\theta_5 + 2p_1S(\theta_2, \theta_4)\theta_5 - p_1\eta(\theta_5)S(\theta_2, \theta_4)\xi \\ & -p_1\eta(\theta_4)\eta(\theta_5)Q\theta_2 - 4g(\theta_2, \theta_5)\theta_4 + 2\eta(\theta_4)g(\theta_2, \theta_5)\xi \\ & -2p_1g(\theta_2, \theta_5)Q\theta_4 - 2p_1S(\theta_2, \theta_5)\theta_4 - p_1^2S(\theta_2, \theta_5)Q\theta_4 = 0. \\ & +4g(\theta_2, \theta_4)\theta_5 + p_1^2S(\theta_2, \theta_4)Q\theta_5 = 0. \end{aligned} \tag{3.16}$$

Using the formulas (2.16), (2.4), (2.11), choosing the value $\theta_5 = \xi$ for the product that is contained on both sides of (3.16) by $\xi \in \chi(M)$, we arrive

$$\begin{aligned} & [1 + p_1 - 2np_1^2]S(\theta_2, \theta_4) = [1 - 4 + 4np_1]g(\theta_2, \theta_4) \\ & +[(2np_1)^2 + 4n^2p_1 - 4np_1 - 8n + 5]\eta(\theta_4)\eta(\theta_2) = 0. \end{aligned} \tag{3.17}$$

and from (3.17) and using (2.10), we conclude

$$S(\theta_2,\theta_4) = -g(\theta_2,\theta_4) + (8-6n)\eta(\theta_2)\eta(\theta_4).$$

M is a -Einstein manifold as a result. On the other hand, consider $M^{2n+1}(\varphi,\xi,\eta,g)$ as $\eta-$Einstein manifold, i.e. $S(\theta_2,\theta_4) = -g(\theta_2,\theta_4) + (8-6n)\eta(\theta_2)\eta(\theta_4)$, then from equations (3.17), (3.16), (3.15) and (3.14), we obtain $W_7 \cdot W_5 = 0$. Which verifies our assertion. ■

Theorem 3.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_6 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Let us say M is a $W_7 \cdot W_6 = 0$. This gives way to

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_6)(\theta_4,\theta_5)\theta_3 &= W_7(\theta_1,\theta_2)W_6(\theta_4,\theta_5)\theta_3 - W_6(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3\\ &\quad -W_6(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3\\ &\quad -W_6(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.18}$$

for any $\theta_1,\theta_2,\theta_3,\theta_4,\theta_5 \in \Gamma(TM)$. Taking $\theta_1 = \theta_3 = \xi$ in (3.18) and using (3.3), (3.6), (3.7), for $p_1 = -\frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_6)(\theta_4,\theta_5)\xi &= W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 - g(\theta_4,\theta_5)\xi)\\ &\quad -W_6(\eta(\theta_4)\theta_2 - 2g(\theta_4,\theta_2)\xi + p_1 g(\theta_2,\theta_4)\xi,\theta_5)\xi\\ &\quad -W_6(\theta_4,\eta(\theta_5)\theta_2 - 2S(\theta_5,\theta_2)\xi + p_1 g(\theta_2,\theta_5)\xi)\xi\\ &\quad -W_6(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.19}$$

and we arrive

$$\begin{aligned}&\eta(\theta_4)W_7(\xi,\theta_2)\theta_5 - g(\theta_4,\theta_5)W_7(\xi,\theta_2)\xi - \eta(\theta_4)W_6(\theta_2,\theta_5)\xi\\ &+2g(\theta_2,\theta_4)W_6(\xi,\theta_5)\xi - p_1 S(\theta_4,\theta_2)W_6(\xi,\theta_5)\xi\\ &-\eta(\theta_5)W_6(\theta_4,\theta_2)\xi + 2g(\theta_2,\theta_5)W_6(\theta_4,\xi)\xi\\ &-p_1 S(\theta_5,\theta_2)W_6(\theta_4,\xi)\xi - W_6(\theta_4,\theta_5)\theta_2 + \eta(\theta_2)W_6(\theta_4,\theta_5)\xi = 0. \end{aligned} \tag{3.20}$$

Taking into account that (3.6), (3.4) and (3.3) in (3.20), we get

$$\begin{aligned}&-W_6(\theta_4,\theta_5)\theta_2 - S(\theta_5,\theta_4)\theta_2 + \eta(\theta_4)g(\theta_5,\theta_2)\xi\\ &+2p_4 g(\theta_4,\theta_2)\theta_5 - 2\eta(\theta_5)g(\theta_2,\theta_4)\xi - p_1 S(\theta_2,\theta_4)\theta_5\\ &+p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi + \eta(\theta_5)g(\theta_2,\theta_4)\xi\\ &-g(\theta_2,\theta_5)\theta_4 + p_1 S(\theta_5,\theta_2)\theta_4 = 0. \end{aligned} \tag{3.21}$$

Putting $\theta_5 = \xi$, using (2.17) and using the inner product on both sides of (3.21) by $\theta_3 \in \chi(M)$, and lastly $\theta_4 = \xi$, we draw a conclusion

$$S(\theta_2,\theta_5) = 2ng(\theta_2,\theta_5) - 4n\eta(\theta_2)\eta(\theta_5).$$

M is therefore $\eta-$Einstein manifold. Let $M^{2n+1}(\varphi,\xi,\eta,g)$ instead be $\eta-$Einstein manifold, i.e. $S(\theta_2,\theta_5) = 2ng(\theta_2,\theta_5) - 4n\eta(\theta_2)\eta(\theta_5)$, then from equations (3.21), (3.20), (3.19) and (3.18), we obtain $W_7 \cdot W_6 = 0$. This completes of the proof. ■

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_7 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that M is a $W_7 \cdot W_7 = 0$. This conforms to

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_7)(\theta_4,\theta_5)\theta_3 =& W_7(\theta_1,\theta_2)W_7(\theta_4,\theta_5)\theta_3 - W_7(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_7(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_7(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.22}$$

for any $\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_3 \in \Gamma(TM)$. Taking $\theta_1 = \theta_3 = \xi$ in (3.22) and using (3.5), (3.7), (3.6), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_7)(\theta_4,\theta_5)\xi =& W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 + p_1\eta(\theta_5)Q\theta_4) \\ & -W_7(\eta(\theta_4)\theta_2 - 2g(\theta_2,\theta_4)\xi - p_1S(\theta_2,\theta_4)\xi,\theta_5)\xi \\ & -W_7(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_2,\theta_5)\xi - p_1S(\theta_2,\theta_5)\xi)\xi \\ & -W_7(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.23}$$

and we have

$$\begin{aligned}&\eta(\theta_4)W_7(\xi,\theta_2)\theta_5 + p_1\eta(\theta_5)W_7(\xi,\theta_2)Q\theta_4 - \eta(\theta_4)W_7(\theta_2,\theta_5)\xi \\ &+2g(\theta_2,\theta_4)W_7(\xi,\theta_5)\xi + p_1S(\theta_2,\theta_4)W_7(\xi,\theta_5)\xi - W_7(\theta_4,\theta_5)\theta_2 \\ &-\eta(\theta_5)W_7(\theta_4,\theta_2)\xi + g(\theta_2,\theta_5)W_7(\theta_4,\xi)\xi + p_1S(\theta_2,\theta_5)W_7(\theta_4,\xi)\xi \\ &+\eta(\theta_2)W_7(\theta_4,\theta_5)\xi = 0. \end{aligned} \tag{3.24}$$

Taking into account that (3.5) and (3.6) in (3.24), we get

$$\begin{aligned}&-W_7(\theta_4,\theta_5)\theta_2 - 2np_1\eta(\theta_5)\eta(\theta_4)\theta_2 - p_1^2\eta(\theta_5)S(Q\theta_4,\theta_2)\xi \\ &-2p_1\eta(\theta_5)S(\theta_4,\theta_2)\xi - p_1\eta(\theta_5)\eta(\theta_4)Q\theta_2 + 2g(\theta_4,\theta_2)\theta_5 \\ &+2g(\theta_2,\theta_4)\eta(\theta_5)\xi + p_1S(\theta_4,\theta_2)\theta_5 - p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi \\ &-2g(\theta_2,\theta_5)\theta_4 - p_1S(\theta_2,\theta_5)\theta_4 = 0. \end{aligned} \tag{3.25}$$

Choosing $\theta_4 = \xi$, making use of (3.5) and inner product both sides of (3.25) by $\theta_3 \in \chi(M)$ and using $\theta_5 = \xi$,we get

$$p_1S(\theta_2,\theta_3) = -2np_1g(\theta_2,\theta_3) + [2np_1 - 4n^2p_1^2 - 1]\eta(\theta_2)\eta(\theta_3) = 0. \tag{3.26}$$

From (3.26) and by using (2.10), we set

$$S(\theta_2,\theta_3) = -2n\left(g(\theta_2,\theta_3) + \eta(\theta_2)\eta(\theta_3)\right).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $S(\theta_2,\theta_3) = -2n\left(g(\theta_2,\theta_3) + \eta(\theta_2)\eta(\theta_3)\right)$, then from equations (3.26), (3.25), (3.24), (3.23) and (3.22) we obtain $W_7 \cdot W_7 = 0$. Which verifies our assertion. ■

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_8 = 0$ if and only if M is an $\eta-$Einstein manifold..*

Proof. If M is a $W_7 \cdot W_8 = 0$, that is. As a result,

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_8)(\theta_4,\theta_5)\theta_3 =& W_7(\theta_1,\theta_2)W_8(\theta_4,\theta_5)\theta_3 - W_8(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_8(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_8(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.27}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.27) and making use of (3.8), (2.8), (2.9), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_8)(\theta_4,\theta_5)\xi = {} & W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 + p_1 S(\theta_4,\theta_5)\xi) \\ & -W_8(\eta(\theta_4)\theta_2 - 2g(\theta_4,\theta_2)\xi - p_1 S(\theta_2,\theta_4)\xi,\theta_5) \\ & -W_8(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_5,\theta_2)\xi - p_1 S(\theta_2,\theta_5)\xi)\xi \\ & -W_8(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.28}$$

Using of (3.8), (3.9), (3.6)and (3.28), we get

$$\begin{aligned} & -W_8(\theta_4,\theta_5)\theta_2 + p_1 S(\theta_5,\theta_4)\theta_2 - p_1\eta(\theta_4)S(\theta_5,\theta_2)\xi \\ & +2g(\theta_2,\theta_4)\theta_5 - 2\eta(\theta_5)S(\theta_2,\theta_4)\xi + p_1 S(\theta_2,\theta_4)\theta_5 \\ & -2p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi - 2g(\theta_5,\theta_2)\theta_4 - p_1 S(\theta_2,\theta_5)\theta_4 = 0. \end{aligned} \tag{3.29}$$

Inner product both sides of (3.29) by $\xi \in \chi(M)$, using $\theta_4 = \xi$ and putting (2.11), we have

$$3p_1 S(\theta_5,\theta_2) = -g(\theta_5,\theta_2) + [-1 - p_1]\eta(\theta_2)\eta(\theta_5) = 0. \tag{3.30}$$

From (3.30) and by using (2.10), we set

$$S(\theta_5,\theta_2) = -\frac{2n}{3}g(\theta_5,\theta_2) - \left(\frac{2n+1}{3}\right)\eta(\theta_5)\eta(\theta_2).$$

M is an $\eta-$Einstein manifold, hence. Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an η-Einstein manifold in contrast, i.e. $S(\theta_5,\theta_2) = -\frac{2n}{3}g(\theta_5,\theta_2) - \left(\frac{2n+1}{3}\right)\eta(\theta_5)\eta(\theta_2)$, then from equations (3.30), (3.29), (3.28) and (3.27), we obtain $W_7 \cdot W_8 = 0$. This completes of the proof. ∎

Theorem 3.5. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_9 = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Let us say M is a $W_7 \cdot W_9 = 0$. It follows that

$$\begin{aligned}(W_7(\theta_1,\theta_2)W_9)(\theta_4,\theta_5,\theta_3) = {} & W_7(\theta_1,\theta_2)W_9(\theta_4,\theta_5)\theta_3 - W_9(W_7(\theta_1,\theta_2)\theta_4,\theta_5)\theta_3 \\ & -W_9(\theta_4,W_7(\theta_1,\theta_2)\theta_5)\theta_3 \\ & -W_9(\theta_4,\theta_5)W_7(\theta_1,\theta_2)\theta_3 = 0, \end{aligned} \tag{3.31}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.31) and making use of (3.10), (3.6), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi,\theta_2)W_9)(\theta_4,\theta_5)\xi = {} & W_7(\xi,\theta_2)(\eta(\theta_4)\theta_5 - \eta(\theta_5)\theta_4 + p_1 S(\theta_4,\theta_5)\xi \\ & -p_1\eta(\theta_5)Q\theta_4) - W_9(\eta(\theta_4)\theta_2 - 2g(\theta_4,\theta_2)\xi \\ & -p_1 S(\theta_2,\theta_4)\xi,\theta_5)\xi - W_9(\theta_4,\eta(\theta_5)\theta_2 - 2g(\theta_5,\theta_2)\xi \\ & -p_1 g(\theta_2,\theta_5)\xi)\xi - W_9(\theta_4,\theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0. \end{aligned} \tag{3.32}$$

Using (3.6) and (3.11) in (3.32), we get

$$\begin{aligned} & -W_9(\theta_4,\theta_5)\theta_2 + 2\eta(\theta_5)g(\theta_4,\theta_2)\xi + p_1 S(\theta_4,\theta_5)\theta_2 + 2np_1\eta(\theta_5)\eta(\theta_4)Q\theta_5 \\ & +p_1^2\eta(\theta_5)S(\theta_2,Q\theta_4)\xi + 2g(\theta_2,\theta_4)\theta_5 - 2g(\theta_4,\theta_2)\eta(\theta_5)\xi \\ & +p_1 S(\theta_4,\theta_2)\theta_5 + p_1\eta(\theta_5)S(\theta_2,\theta_4)\xi - 2g(\theta_2,\theta_5)\theta_4 \\ & -p_1 S(\theta_2,\theta_5)\theta_4 - p_1\eta(\theta_4)S(\theta_2,\theta_5)\xi + p_1\eta(\theta_5)\eta(\theta_4)Q\theta_2 = 0. \end{aligned} \tag{3.33}$$

Utilizing (2.20), picking $\theta_4 = \xi$ and the inner product on both sides of (3.33) by $\xi \in \chi(M)$, we have

$$2p_1 S(\theta_2, \theta_5) = -2g(\theta_2, \theta_5) + [4n^2 p_1^2 - 4n^2 p_1 - 8np_1 + 2]\eta(\theta_5)\eta(\theta_2) \tag{3.34}$$

from which, we conclude

$$S(\theta_2, \theta_5) = -2ng(\theta_2, \theta_5) - (1 + 2n)\eta(\theta_2)\eta(\theta_5).$$

As a result, M is an η-Einstein manifold. On the other hand, consider $M^{2n+1}(\varphi, \xi, \eta, g)$ as an η-Einstein manifold, i.e. $S(\theta_2, \theta_5) = -2ng(\theta_2, \theta_5) - (1 + 2n)\eta(\theta_2)\eta(\theta_5)$, then from equations (3.34), (3.33), (3.32) and (3.31), we obtain $W_7 \cdot W_9 = 0$. Which verifies our assertion. ■

Theorem 3.6. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then M is a $W_7 \cdot W_0^\star = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Consider M to be a $W_7 \cdot W_0^\star = 0$. This means that

$$\begin{aligned}(W_7(\theta_1, \theta_2)W_0^\star)(\theta_4, \theta_5, \theta_3) = \; & W_7(\theta_1, \theta_2)W_0^\star(\theta_4, \theta_5)\theta_3 - W_0^\star(W_7(\theta_1, \theta_2)\theta_4, \theta_5)\theta_3 \\ & -W_0^\star(\theta_4, W_7(\theta_1, \theta_2)\theta_5)\theta_3 \\ & -W_0^\star(\theta_4, \theta_5)W_7(\theta_1, \theta_2)\theta_3 = 0,\end{aligned} \tag{3.35}$$

for any $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Gamma(TM)$. Setting $\theta_1 = \theta_3 = \xi$ in (3.35) and making use of (3.12), (3.6), (3.7), for $p_1 = \frac{1}{2n}$, we obtain

$$\begin{aligned}(W_7(\xi, \theta_2)W_0^\star)(\theta_4, \theta_5)\xi = \; & W_7(\xi, \theta_2)(\eta(\theta_4)\theta_5 - 2\eta(\theta_5)\theta_4 - p_1\eta(\theta_4)Q\theta_5) \\ & -W_0^\star(\eta(\theta_4)\theta_2 - 2g(\theta_2, \theta_4)\xi - p_1 S(\theta_2, \theta_4)\xi, \theta_5)\xi \\ & -W_0^\star(\theta_4, \eta(\theta_5)\theta_2 - 2g(\theta_2, \theta_5)\xi - p_1 S(\theta_2, \theta_5)\xi)\xi \\ & -W_0^\star(\theta_4, \theta_5)(\theta_2 - \eta(\theta_2)\xi) = 0.\end{aligned} \tag{3.36}$$

Using (3.12) and (3.13) in (3.36), we get

$$\begin{aligned}& -W_0^\star(\theta_4, \theta_5)\theta_2 - 2\eta(\theta_4)g(\theta_2, \theta_5)\xi + 2\eta(\theta_5)g(\theta_2, \theta_4)\xi + 2np_1\eta(\theta_4)\eta(\theta_5)\theta_2 \\ & -p_1^2\eta(\theta_4)S(\theta_2, Q\theta_5)\xi - p_1\eta(\theta_4)\eta(\theta_2)Q\theta_5 + 2g(\theta_2, \theta_4)\theta_5 - 4g(\theta_2, \theta_4)\eta(\theta_5) \\ & +p_1 S(\theta_2, \theta_4)\theta_5 - p_1^2 S(\theta_2, \theta_4)Q\theta_5 + p_1(\theta_4)\eta(\theta_5)Q\theta_2 - 2g(\theta_2, \theta_5)\theta_4 \\ & +4\eta(\theta_4)g(\theta_2, \theta_5)\xi + 2p_1 g(\theta_2, \theta_5)Q\theta_4 - p_1 S(\theta_2, \theta_5)\theta_4 + 2p_1\eta(\theta_4)S(\theta_2, \theta_5)\xi \\ & +p_1^2 S(\theta_2, \theta_5)Q\theta_4 - p_1(\theta_4)\eta(\theta_2)Q\theta_5 - 2p_1 g(\theta_2, \theta_4)Q\theta_5 = 0.\end{aligned} \tag{3.37}$$

Making use of (2.21), using $\theta_2 = \theta_4 = \xi$ and inner product both sides of (3.37) by $\theta_3 \in \chi(M)$, we have

$$2np_1^2 S(\theta_3, \theta_5) = -g(\theta_3, \theta_5) + [5 - 4n^2 p_1^2 + 2n - 4n^2 p_1]\eta(\theta_3)\eta(\theta_5). \tag{3.38}$$

Finally, from (2.10) and (3.38), we arrive

$$S(\theta_3, \theta_5) = -2ng(\theta_3, \theta_5) + 8n\eta(\theta_3)\eta(\theta_5).$$

This indicates that M is an $\eta-$Einstein manifold. Let M instead be an η-Einstein manifold, i.e. $S(\theta_3, \theta_5) = -2ng(\theta_3, \theta_5) + 8n\eta(\theta_3)\eta(\theta_5)$, then from (3.38), (3.37), (3.36) and (3.35), we have $W_7 \cdot W_0^\star = 0$. This completes of the proof. ■

Conclusion 3.7. *Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6 than we have. Assume that $M^{2n+1}(\varphi, \xi, \eta, g)$ is a Kenmotsu manifold. M is thus $W_7 \cdot W_5 = 0$, $W_7 \cdot W_6 = 0$, $W_7 \cdot W_7 = 0$, $W_7 \cdot W_8 = 0$, $W_7 \cdot W_9 = 0$ and $W_7 \cdot W_0^\star = 0$ if and only if M is an $\eta-$Einstein manifold.*

References

[1] ABDULSATTAR, D.B, On conharmonic transformations in general relativity, *Bull Calcutta Math. Soc.,* **41**(1966), 409–416.

[2] ATÇEKEN, M., UYGUN, P, Characterizations for totally geodesic submanifolds of (k, μ)-paracontact metric manifolds, *Korean J. Math.,* **28**(2020), 555–571.

[3] AYAR,G., CHAUBEY, S. K, M-projective Curvature Tensor Over Cosymplectic Manifolds, Differential Geometry, *Dynamical Systems*, **21**(2019), 23–33.

[4] BLAIR, D.E, *Contact Manifolds in Riemannian Geometry*, Lecture Note in Mathematics, Springer-Verlag, Berlin-New York, **509**, 1976.

[5] DE, U. C., PATHAK, G, On 3-dimensional Kenmotsu manifolds, *Indian J. Pure Appl. Math.*, **35**(2004), 159–165.

[6] KENMOTSU, K, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24**(1972), 93–103.

[7] KOBAYASHI, K., NOMIZU, K, *Foundations of Differential Geometry*, Wiley-Interscience, New York, 1963.

[8] MERT, T, Characterization of some special curvature tensor on Almost $C(a)$-manifold, *Asian Journal of Mathematics and Computer Research*, **29**(1)(2022), 27–41.

[9] MERT, T., ATÇEKEN, M., UYGUN, P, Semi-symmetric almost $C(a)$ -manifold on some curvature tensors, *Gulf Journal of Mathematics*, **14**(2)(2023), 101–114.

[10] MERT, T., ATÇEKEN, M., UYGUN, P., PANDEY, S, Almost n-Ricci Solitons on the Pseudosymmetric Lorentzian Para-Kenmotsu Manifolds $C(a)$ -manifold on some curvature tensors, *Earthline Journal of Mathematical Sciences*, (2023).

[11] MISHRA, R. S, *Structure on a Differentiable Manifold and Their Applications*, Chandrama Prakashan, 50-A Bairampur House Allahabad, (1984).

[12] POKHARIYAL, G. P, Relativistic significance of curvature tensors, *Internat. J. Math. Sci.*, **5**(1)(1982), 133–139.

[13] POKHARIYAL, G. P., MISHRA, R. S, Curvature tensors and their relativistic significance II, *Yokohama Math. J.*, **19**(2)(1971), 97–103.

[14] SASAKI, S, *Almost Contact Manifolds, 1,2,3*, A Lecture note, Tohoku University, (1965, 1967, 1968).

[15] TANAKA, N, On non-degenerate real hypersurfaces, graded Lie algebra and Cartan connections, *Japanese Journal of Mathematics: New series*, **2**(1)(1976), 131–190.

[16] TANNO, S, Variational problems on contact Riemannian manifolds, *Transactions of the American Mathematical Society*, **314**(1)(1989), 349–379.

[17] TRIPATHI, M. M., GUPTA, P, $T-$curvature tensor on a semi-Riemannian manifold, *J. Adv. Math. Studies*, **4**(1)(2011),117-129.

[18] UYGUN, P., ATÇEKEN, M, On (k,μ) -paracontact metric spaces satisfying some conditions on the $W_0^{\star}-$curvature tensor, *NTMSCI*, **9**(2)(2021), 26–37.

[19] WEBSTER, S. M, Pseudo-Hermitian structures on a real hypersurface, *Journal of Differential Geometry*, **13**(1)(1978), 25-41.