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An introduction of total dominator color class total dominating sets in graphs

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Abstract

Let G be a finite, undirected and connected graph with minimum degree at least one. In this paper we define a new graph parameter called total dominator color class total domination number of G. A proper coloring $\mathscr{C}$ of G is said to be a total dominator color class total dominating set of G if each vertex properly dominates a color class in $\mathscr{C}$ and each color class in $\mathscr{C}$ is properly dominated by a vertex in $\mathrm{V(G)}$. A total dominator color class total dominating set D of G is a minimal total dominator color class total dominating set if no proper subset of D is a total dominator color class total dominating set of G . The total dominator color class total domination number is the minimum cardinality taken over all minimal total dominator color class total dominating sets in G and is denoted by $\gamma_{\chi}^{td}(G)$. Here we obtain the characterization of lower and upper bounds of $\gamma_{\chi}^{td}(G)$, and total dominator color class total domination number of Path graph, Cycle graph and Banana graph.

Keywords

Chromatic number, Domination number, Total domination, Dominator color class dominating set, Total dominator color class total domination number

AMS Subject Classification

05C15, 05C69.

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Contents

1. Introduction

All graphs considered in this paper are finite, undirected graphs with minimum degree at least one and we follow standard definitions of graph theory as found in [8]. Let $G=(V,E)$ be a connected graph with no isolated vertices. The open neighborhood $\mathrm{N(v)}$ of a vertex $\mathrm{v} \in \mathrm{V(G)}$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $\mathrm{N[v]} = \mathrm{N(v)} \cup \{\mathrm{v}\}$. For a set $\mathrm{S} \subseteq \mathrm{V}$, the open neighborhood $\mathrm{N(S)}$ is defined to be $\mathrm{U}_{\mathrm{v}\in\mathrm{S}} N(v)$, and the closed neighborhood of S is $\mathrm{N[S]} = \mathrm{N(S)} \cup \mathrm{S}$. For any set H of vertices of G, the induced sub graph $\langle \mathrm{H} \rangle$ is the maximal sub graph of G with vertex set H. A subset S of V is called a dominating set if every vertex in $\mathrm{V}-\mathrm{S}$ is adjacent to some vertex in S.

A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(\mathrm{G})$ is the minimum cardinality taken over all minimal dominating sets of G. A γ-set of G is any minimal dominating set with cardinality γ. A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(\mathrm{G})$. A total dominator coloring of G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The total dominator chromatic number is denoted by $\chi_{td}(\mathrm{G})$. This notion was introduced by [9]. A color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that every color classes in $\mathscr{C}$ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by $\gamma_\chi(\mathrm{G})$. This notion was introduced by [4].

A dominator color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that each vertex v in G is dominated by a color class $\mathscr{C}_i \in \mathscr{C}$ and each color class $\mathscr{C}_i \in \mathscr{C}$ is dominated by a vertex in G. The dominator color class domination number of G is the minimum cardinality taken over all dominator color class dominating sets in G and is denoted by $\gamma_\chi^d(\mathrm{G})$. This notion was introduced by [5]. The join $\mathrm{G}_1+\mathrm{G}_2$ of graphs G_1 and G_2 with disjoint vertex set V_1 and V_2 and edge sets E_1 and E_2, respectively, is the graph union $G_1 \cup G_2$ together with each vertex in V_1 is adjacent to every vertices in V_2. A path on n vertices denoted by $\mathrm{P_n}$, is a connected graph with all vertices having degree two but two vertices have degree 1 and $\mathrm{V}(\mathrm{P_n})=\{u_i/1\leq \mathrm{i} \leq \mathrm{n}\}\, \& u_iu_{i+1}\in E(P_n)$ for $\mathrm{i}<\mathrm{n}$. A cycle graph is a graph on $\mathrm{n}\geq 3$ vertices containing a single cycle through all vertices and is denoted by $\mathrm{C_n}$. A banana graph $B_{m,n}$ is a graph obtained by connecting one leaf of each m copies of an n -star graph with a single root vertex that is distinct from all the stars.

2. Main Results

We introduce a new concept, Total dominator color class total dominating sets on graphs.

Definition 2.1. *Let G be a connected graph with minimum degree at least one. A proper coloring $\mathscr{C}$ of G is said to be a total dominator color class total dominating set of G if each vertex properly dominates a color class in $\mathscr{C}$ and each color class in $\mathscr{C}$ is properly dominated by a vertex in $V(G)$. A total dominator color class total dominating set D is a minimal total dominator color class total dominating set if no proper subset of D of G is a total dominator color class total dominating set of G. The total dominator color class total domination number is the minimum cardinality taken over all minimal total dominator color class total dominating sets in G and is denoted by $\gamma_\chi^{td}(G)$.*

This concept is illustrated by the following example:

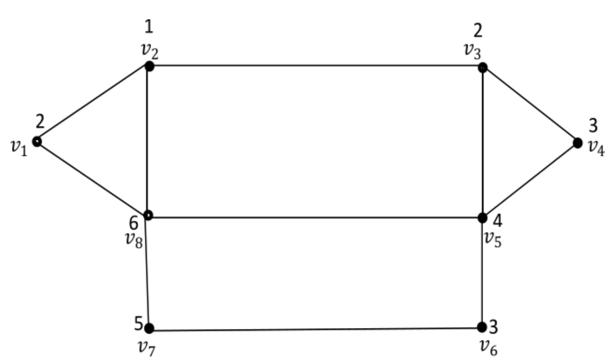

Figure 1. $\gamma_x^{td}(G)=6$

In figure 1, $\mathscr{C}_1=\{v_2\},\mathscr{C}_2=\{v_1,v_3\},\mathscr{C}_3=\{v_4,v_6\},\mathscr{C}_4=\{v_5\},\mathscr{C}_5=\{v_7\},\mathscr{C}_6=\{v_8\}$. Then the color classes $\mathscr{C}_i, 1\leq i\leq 6$ are properly dominated by vertices $\mathrm{v}_1\,(\mathrm{or_v}), \mathrm{v}_2$, v_5, (v_4 or v_6), $\mathrm{v}_8, \mathrm{v}_7$ respectively and vertices $\{v_i : 1\leq i\leq 8\}$ are properly dominated by the color classes $\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_1,\mathscr{C}_3,\mathscr{C}_4,\mathscr{C}_3,\mathscr{C}_6,\mathscr{C}_5$ respectively. So $\gamma_\chi^{\mathrm{td}}(G)=6$.

The next two theorem give the characterization of lower and upper bound of Total dominator color class total dominating sets of G.

Theorem 2.2. *Let G be a connected graph of order n . Then $\gamma_\chi^{td}(\mathrm{G})=2$ iff $\mathrm{G}\cong K_{m,n}$, for $\mathrm{m},\mathrm{n}\in\mathbb{N}$.*

Proof. Suppose $\gamma_\chi^{td}(\mathrm{G})=2$. Let $\mathscr{C}_1,\mathscr{C}_2$ be two color classes of G . Let $\mathrm{x}\in\mathscr{C}_1$. Since x cannot dominate $\mathscr{C}_1$, it should dominate $\mathscr{C}_2$. Similarly for any $\mathrm{y}\in\mathscr{C}_2$, y dominates $\mathscr{C}_1$. Also $\mathscr{C}_1$ and $\mathscr{C}_2$ are properly dominated by a vertex. Thus, G is a complete bipartite graph with partitions $\mathscr{C}_1$ and $\mathscr{C}_2$. Hence $\mathrm{G}\cong K_{m,n}$, for $\mathrm{m},\mathrm{n}\in\mathbb{N}$. The converse is obvious. □

Theorem 2.3. *Let G be a complete graph of order n . Then $\gamma_\chi^{td}(\mathrm{G})=\mathrm{n}$ iff $\mathrm{G}\cong K_n$, for $\mathrm{n}\geq 2$.*

Proof. Let G be a non-complete graph with $\delta\geq 1$. We show that $\gamma_\chi^{td}(\mathrm{G})<\mathrm{n}$. Since G is non complete, there exist $v_1v_2\notin E(G)$. We have two cases.

Case (i): Let $\delta(G)\geq 2$. We assign color 1 to the vertices $\{v_1,v_2\}$ and allot colors, say $\{2,3,\ldots,\mathrm{n}-1\}$ to the remaining $(\mathrm{n}-2)$ vertices. We attain a γ_χ^{td} - coloring of G .

Case (ii): Let $\delta(G)=1$. We consider two subcases.

Subcase (2.1): Let G has at least two end vertices. We choose v_1 and v_2 as two end vertices and proceed as in case (i), to show that $\gamma_\chi^{td}(\mathrm{G})<\mathrm{n}$.

Subcase (2.2): Let G has exactly one end vertex v_1 with support v_3. Let $v_2v_3\in E(G)$. Proceeding as before we show that $\gamma_\chi^{td}(\mathrm{G})<\mathrm{n}$. The converse is obvious.

□

Theorem 2.4. *If G is a disconnected graph with nontrivial components $G_1,G_2,\ldots\ldots,G_k, \mathrm{k}\geq 2$ then $\left(\max_{1\leq i\leq k}\gamma_\chi^{td}(G_i)\right)+2\mathrm{k}-2\leq\gamma_\chi^{td}(\mathrm{G})\leq\sum_{i=1}^{k}\gamma_\chi^{td}(G_i)$ and these bounds are sharp.*

Proof. For each $\mathrm{i}(1\leq i\leq k)$, let $\mathscr{C}_{i_1},\mathscr{C}_{i_2},\ldots\ldots,\mathscr{C}_{i_{ri}}$ be the color classes of the component $G_1,G_2,\ldots\ldots,G_k$ respectively. Then $\cup_{i=1}^{k}(\mathscr{C}_{i_1},\mathscr{C}_{i_2},\ldots\ldots,\mathscr{C}_{i_{ri}})$ is a total dominator color class total dominating sets of G. Thus, $\gamma_\chi^{td}(\mathrm{G})\leq\sum_{i=1}^{k}\gamma_\chi^{td}(G_i)$.

Now we prove the lower bound. Let G_l be a component of G with minimum total dominator color class total domination number of G. Then $\gamma_\chi^{td}(G_l)=\max_{1\leq i\leq k}\left\{\gamma_\chi^{td}(G_i)\right\}$. For each $\mathrm{i}\neq 1, G_i$ needs atleast two new colors, since each vertex in G_i properly dominates a color class and each color class is properly dominated by a vertex. This establishes the required result. The lower and upper bound is sharp if $\mathrm{G}\cong \mathrm{m}K_2$. □

Theorem 2.5. *Let G be P_n or C_n. Then for $n\geq 5$,*

$$\gamma_\chi^{td}(P_n)=\gamma_\chi^{td}(C_n)=\begin{cases}\left\lfloor\frac{2n}{3}\right\rfloor & \text{if } n\equiv 0(\bmod 3)\\ 2\left\lfloor\frac{n+2}{3}\right\rfloor & \text{if } n\not\equiv 0(\bmod 3)\end{cases}$$

Proof. Let $\mathrm{V}(\mathrm{P_n}) = \{v_i / 1 \leq \mathrm{i} \leq \mathrm{n}\} \& v_i v_{i+1} \in E(P_n)$ for $\mathrm{i} < \mathrm{n}$. Let $\mathrm{n} \geq 5$, let $\mathscr{C}$ be a γ_χ^{td} coloring of $\mathrm{P_n}$. We consider two cases.

Case (i): When $n \equiv 0(\bmod 3)$.
For $1 \leq i \leq \frac{n}{3}$, Let $\mathrm{H_i} = \langle v_{3i-2}, v_{3i-1}, v_{3i} \rangle$ be the vertex induced subgraph of $\mathrm{P_n}$ and assign 2 distinct colors, say $2\mathrm{i}-1$ and $2\,\mathrm{i}$ to the vertices $\{v_{3i-2}, v_{3i}\}$ and $\{v_{3i-1}\}$ respectively. We obtain a γ_χ^{td} coloring of $\mathrm{P_n}$. Thus $\gamma_\chi^{td}(P_n) = \left\lfloor \frac{2n}{3} \right\rfloor$

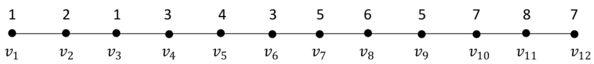

Figure 2. $\gamma_\chi^{td}(P_n) = 8$

Case (ii): When $n \not\equiv 0(\bmod 3)$. We consider two subcases.

Subcase (2.1): When $n \equiv 1(\bmod 3)$. Since $n-4 \equiv 0(\bmod 3)$ and P_n is obtained from P_{n-4} followed by $\mathrm{P_4}$. Thus $\gamma_\chi^{td}(P_n) = \gamma_\chi^{td}(P_{n-4}) + \gamma_\chi^{td}(P_4) = 2\left\lfloor \frac{n+2}{3} \right\rfloor$

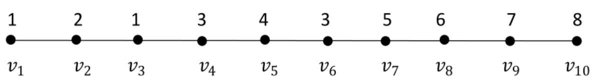

Figure 3. $\gamma_\chi^{td}(P_n) = 8$

Subcase (2.2): When $n \equiv 2(\bmod 3)$. Since $n-2 \equiv 0(\bmod 3)$ and P_n is obtained from P_{n-2} followed by $\mathrm{P_2}$. Thus $\gamma_X^{td}(P_n) = \gamma_X^{td}(P_{n-2}) + \gamma_X^{td}(P_2) = 2\left\lfloor \frac{n+2}{3} \right\rfloor$
This γ_χ^{td} - coloring is true for $\mathrm{C_n}$ also.

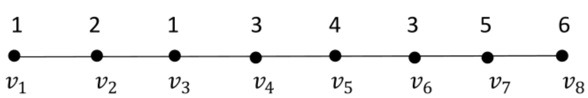

Figure 4. $\gamma_\chi^{td}(P_n) = 6$

□

Theorem 2.6. *For a banana graph $B_{m,n}$, $\gamma_\chi^{td}(B_{m,n}) = 2(\mathrm{m}+1), \forall \mathrm{m}, \mathrm{n} \geq 2$.*

Proof. Let $\mathrm{V}(B_{m,n}) = \{u\} \cup \{v_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$. $B_{m,n}$ consists of 1 vertex, say u , have degree m, m vertices say $\{v_{in} / 1 \leq i \leq m\}$ have degree 2 , m vertices, say $\{v_{i1} / 1 \leq i \leq m\}$ have degree n and $\mathrm{m}(\mathrm{n}-1)$ vertices, say $\{v_{ij} / 1 \leq i \leq n, 2 \leq j \leq n\}$ have degree 1 respectively. We assign colors 1 and 2 to the vertices $\{u\}$ and $\{v_{im} / 1 \leq i \leq n\}$ respectively. For $1 \leq i \leq m$, assign colors $2\mathrm{i}+2$ and $2\mathrm{i}+1$ to the vertices $\{v_{i1} / 1 \leq i \leq m\}$ and $\{v_{ij} / 1 \leq i \leq m, 2 \leq j \leq n\}$ respectively. We get a $\gamma_\chi^{td}-$ coloring of $B_{m,n}$.

□

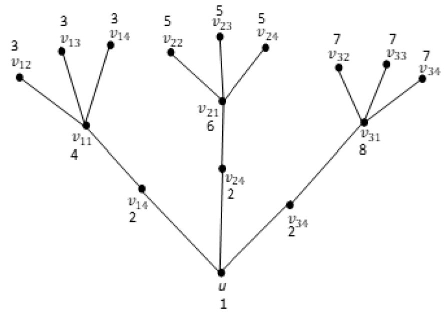

Figure 5. $\gamma_\chi^{td}(B_{3,4}) = 8$

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