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Some results on compact fuzzy strong b-metric spaces

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Abstract. In this paper, the concept of compactness on fuzzy strong b-metric space is introduced. On the other hand some basic results are developed on compactness, completeness and totally boundedness.

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Contents

1. Introduction

Several authors generalized metric spaces and fuzzy metric spaces (for reference please see [1,3,4,5,7,9]) in different ways and studied various topological properties on such spaces (please see [2,6,8,10]).
In this paper, we have considered fuzzy strong b-metric space introduced by T. Oner[7] and explore some new concepts such as compactness, totally boundedness to develop some basic results on such spaces.
The organization of the paper is as follows:
In Section 2, some preliminary results are given to be used in this paper. In Section 3, an idea of compact fuzzy strong b-metric space is introduced. Definitions of closed and bounded sets are given and some basic results are studied. The concept of $\alpha - \epsilon -$net and $\alpha -$totally bounded set is introduced and some fundamental results are developed in Section 4.

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2. Preliminaries

In this section, some preliminary results are given which are used in this paper.

Definition 2.1. *([3]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions;*
1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a * 1 = a \quad \forall a \in [0,1]$,
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Definition 2.2. *([9]) An ordered triple (X, D, K) is called strong b-metric space, D is called strong b-metric on X if X is a nonempty set, $K \geq 1$ is a given real number and*
$D : X \times X \to [0,\infty)$ satisfies the following conditions $\forall x, y, z \in X$
1) $D(x,y) = 0$ iff $x = y$,
2) $D(x,y) = D(y,x)$,
3) $D(x,z) \leq D(x,y) + KD(y,z)$.

Definition 2.3 (7)**.** *Let X be a nonempty set, $K > 1$, $*$ is a continuous t-norm and M be a fuzzy set on $X \times X \times (0,\infty)$ such that $\forall x, y, z \in X$ and $t, s > 0$,*
1) $M(x,y,t) > 0$,
2) $M(x,y,t) = 1$ iff $x = y$,
3) $M(x,y,t) = M(y,x,t)$,
4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+Ks)$,
5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.
Then M is called a fuzzy strong b-metric on X and $(X, M, *, K)$ is called fuzzy strong b-metric space.

3. Compact fuzzy strong b-metric space

In this section some definitions are given and basic results are studied.

Definition 3.1. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. A is said to be compact if every sequence in A has a convergent subsequence which converges to some point in A.*

Theorem 3.2. *Every compact fuzzy strong b-metric space is complete if $1 < K < 2$.*

Proof. Let $(X, M, *, K)$ be a compact fuzzy strong b-metric space.
Let $\{x_n\}$ be a Cauchy sequence in X.
Let r and t be arbitrary real numbers such that $r \in (0,1)$ and $t > 0$. Then $\exists r_0 \in (0,1)$ such that $(1-r_0)*(1-r_0)*(1-r_0) > 1-r$. (Since $*$ is a continuous t-norm)
Since $\{x_n\}$ is a Cauchy sequence, thus for $r_0 \in (0,1)$ and $t > 0$, there exists a natural number n_0 such that

$$M(x_n, x_{n_0}, \frac{t}{3}) > 1 - r_0 \quad \forall n \geq n_0. \tag{3.1}$$

Since X is compact, $\exists$ a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ which converges to some $x \in X$.
Thus, for $\frac{2t}{3K^2} - \frac{t}{3K} (> 0)$ and $r_0 \in (0,1)$, $\exists m \in N$ such that

$$M(x_{k_m}, x, \frac{2t}{3K^2} - \frac{t}{3K}) > 1 - r_0 \quad \forall m \geq n_0. \tag{3.2}$$

Since $k_m \geq m \geq n_0$, we have from (3.1), we have

$$M(x_{k_m}, x_{n_0}, \frac{t}{3}) > 1 - r_0. \tag{3.3}$$

Now for $n \geq n_0$, from (3.1), (3.2) and (3.3), we get

$$\begin{aligned} M(x_n, x, t) &= M(x_n, x, \frac{t}{3} + K.\frac{2t}{3K}) \\ &\geq M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x, \frac{2t}{3K}) \\ &= M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x, \frac{t}{3} + K(\frac{2t}{3K^2} - \frac{t}{3K})) \\ &\geq M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x_{K_m}, \frac{t}{3}) * M(x_{K_m}, x, \frac{2t}{3K^2} - \frac{t}{3K}) \\ &> (1 - r_0) * (1 - r_0) * (1 - r_0). \end{aligned}$$

Thus for $t > 0$ and $r \in (0, 1)$ we have
$M(x_n, x, t) > 1 - r \quad \forall n \geq n_0$
$\Rightarrow \lim_{n\to\infty} x_n = x$.
$\Rightarrow X$ is complete. ■

Note 3.1. Converse of the result may not be true. We justify it by the following example.
Example 3.1. Let $X = R$. Define $M_b(x, y, t) = \frac{t}{t+D(x,y)}$ for $t > 0$
and $x, y \in X$ where $D(x, y) = |x - y| \quad \forall x, y \in X$.
By using Example 2.2[7] it is enough to prove that (X, D, K) is a strong b-metric space to show that $(X, M_D, *, K)$ is a fuzzy strong b-metric space induced by D where $*$= product t-norm.
Solution . First we show that (X, D, K) is a strong b-metric space.
1. $D(x, y) = |x - y| = 0$ iff $x = y$
2. $D(x, y) = |x - y| = |y - x| = D(y, x)$
3. $D(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| \leq |x - y| + K|y - z|, K > 1$
$\therefore D(x, z) \leq D(x, y) + KD(y, z) \quad \forall x, y, z \in X$.
Thus (X, D, K) is a strong b-metric space.
So, $(X, M_D, *, K)$ is a fuzzy strong b-metric space.
Next we show that $(X, M_D, *, K)$ is complete.
Suppose $\{x_n\}$ is a Cauchy sequence in X.
We choose $\epsilon = \frac{tr}{1-r}(> 0)$ arbitrarily where $t > 0, r \in (0, 1)$.
Now for $t > 0$ and $r \in (0, 1)$, there exists n_0,
such that $M_D(x_n, x_m, t) = \frac{t}{t+|x_n - x_m|} > 1 - r, \quad \forall n, m \geq n_0$.
$\Rightarrow |x_n - x_m| < t(\frac{1}{1-r} - 1) = \frac{tr}{1-r} = \epsilon \quad \forall n, m \geq n_0$.
$\Rightarrow |x_n - x_m| < \epsilon \quad \forall n, m \geq n_0$.
So $\{x_n\}$ is a Cauchy sequence in R. Since R is complete, there exists $x \in R$ such that $x_n \to x$.
Now, $M_D(x_n, x, t) = \frac{t}{t+|x_n - x|} \quad \forall t > 0$.

$$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = \frac{t}{t + \lim_{n\to\infty} |x_n - x|} \quad \forall t > 0.$$

$$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = \frac{t}{t + 0} = 1 \quad \forall t > 0.$$

$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = 1 \quad \forall t > 0$.
Thus $x_n \to x$, for some $x \in X$.
So, $(X, M_D, *, K)$ is complete.
If possible suppose that $(X, M_D, *, K)$ is compact.
Let $\{x_n\}$ be a sequence in X such that $x_n = n \quad \forall n$.
Since X is compact, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \to y$, for some $y \in X$.
Now, $M_D(y_n, y, t) = \frac{t}{t+|y_n - y|} \quad \forall t > 0$.

$\lim_{n\to\infty} M_D(y_n, y, t) = \dfrac{t}{t + \lim_{n\to\infty} |y_n - y|} \quad \forall t > 0.$
$\Rightarrow 1 = \frac{t}{t+ \lim_{n\to\infty} |y_n - y|}$
$\Rightarrow \lim_{n\to\infty} |y_n - y| + t = t$
$\Rightarrow \lim_{n\to\infty} |y_n - y| = 0$
$\Rightarrow y_n \to y$, for some $y \in R$.
Which is a contradiction since the sequence of all natural numbers has no convergent sequence in R. w.r.t. usual metric.
Thus $(X, M_D, *, K)$ is not compact.

Definition 3.3. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space. A subset A of X is said to be bounded if $\exists t > 0$, $r \in (0, 1)$*
such that $M(x, y, t) > 1 - r \quad \forall x, y \in A$.

Definition 3.4. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space. A subset F of X is said to be closed if for any sequence $\{x_n\}$ in F such that $x_n \to x$ implies $x \in F$.*
i.e. $\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \forall t > 0$ implies $x \in F$.

Proposition 3.5. *Every compact subset of a fuzzy strong b-metric space is closed and bounded.*

Proof. Let $(X, M, *, K)$ be a fuzzy strong b-metric space and A be a subset of X.
If possible suppose that A is not closed. So $\exists$ a sequence $\{x_n\}$ in A such that $x_n \to x$ but $x \notin$ A.
Since A is compact, so $\exists$ a subsequence $\{x_{n_K}\}$ of $\{x_n\}$ which converges to some point in A.
Since $x_n \to x$ thus $\{x_{n_K}\} \to x$ and hence $x \in A$.
Which is a contradiction. Thus A is closed.
Now we show that A is bounded.
If possible suppose that A is unbounded. Fix $x_0 \in A$.
Choose a sequence $\{\alpha_n\} \in (0, 1) \quad \forall n$ such that $\alpha_n \to 1$ as $n \to \infty$.
Thus for a given $t > 0$, for each n, $\exists x_n \in A$ such that
$M(x_0, x_n, t) \leq 1 - \alpha_n$.
Now we obtain a sequence $\{x_n\}$ in A. Since A is compact, $\exists$ a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which converges to some point $x \in A$.
Now we have $M(x_0, x_{n_l}, t) \leq 1 - \alpha_{n_l}$
We have, $1 - \alpha_{n_l} \geq M(x_0, x_{n_l}, t)$
$= M(x_0, x_{n_l}, \frac{t}{2} + \frac{Kt}{2K})$
$\geq M(x_0, x, \frac{t}{2}) * M(x, x_{n_l}, \frac{t}{2K})$
$\Rightarrow \lim_{n\to\infty} (1 - \alpha_{n_l}) \geq \lim_{n\to\infty} M(x_0, x, \dfrac{t}{2}) * \lim_{n\to\infty} M(x, x_{n_l}, \dfrac{t}{2K})$
$\Rightarrow \quad 0 \geq M(x_0, x, \frac{t}{2}) * 1 = M(x_0, x, \frac{t}{2})$
$\Rightarrow M(x_0, x, \frac{t}{2}) = 0$ which contradict the condition (3.1). ■

Note 3.2. Converse of the above result may not be true. We justify it by the following example.
Example 3.2. Let $X = l_2$.
Define $D(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|^2)^{\frac{1}{2}}$ where $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$
Then it is easy to verify that (X, D) is a strong b-metric space for $K \geq 1$
Again define $M_b(x, y, t) = \frac{t}{t+D(x,y)} \quad \forall t \in (0, \infty)$.
Then by using Example 2.2[7], it follows that $(X, M_b, *, K)$ is a fuzzy strong b-metric space w.r.t. the t-norm $*$=product.

Choose $A = \{(1,0,0,....),(0,1,0,...),(0,0,1,0,...),....\}$ subset of l_2.
For $x, y \in A$ with $x \neq y$ we get $M_b(x,y,t) = \frac{t}{t+\sqrt{2}}$.
Take $t = \sqrt{2}+1$ and $\alpha = \frac{1}{2}$.
Then $\forall x, y(x \neq y) \in A$ we get,
$M_b(x,y,\sqrt{2}+1) = \frac{\sqrt{2}+1}{\sqrt{2}+1+\sqrt{2}}$
Now, $\frac{\sqrt{2}+1}{\sqrt{2}+1+\sqrt{2}} - \frac{1}{2} = \frac{2\sqrt{2}+2-2\sqrt{2}-1}{2(2\sqrt{2}+1)}$
$= \frac{1}{2(2\sqrt{2}+1)} > 0$
Thus $M_b(x,y,\sqrt{2}+1) > 1-\alpha = 1-\frac{1}{2} \qquad \forall x, y(x \neq y) \in A$
Also for $x = y$, $M_b(x,y,\sqrt{2}+1) = 1 > 1-\frac{1}{2}$.
Thus A is bounded.
On the other hand if we consider the sequence $\{x_n\}$ in A where $x_n = (0,0,0,...,1(n^{th} place),0,...)$.
Clearly A is closed and since neither the sequence $\{x_n\}$ nor its any subsequence converges to some element in A, so A is not compact.

Proposition 3.6. *Every finite subset in a fuzzy strong b-metric space is bounded.*

Proof. Let $(X,M,*,K)$ be a fuzzy strong b-metric space and A be a finite subset of X containing n elements $x_1, x_2, ...x_n$.
Choose $t_0 > 0$ fixed. Let $\min_{i,j} M(x_i,x_j,t_0) = \beta \quad i,j = 1,2,....n$.
Clearly $\beta \in (0,1)$.
Choose $\alpha \in (0,1)$ such that $\min_{i,j} M(x_i,x_j,t_0) > 1-\alpha$
$\Rightarrow M(x_i,x_j,t_0) > 1-\alpha \qquad \forall x_i, x_j \in A$.
$\Rightarrow A$ is bounded. ∎

4. Totally bounded set in fuzzy strong b-metric space

In this section the concept of $\alpha-$totally bounded set is introduced and some fundamental results on $\alpha-$ totally bounded sets are developed.

Definition 4.1. *Let $(X,M,*,K)$ be a fuzzy strong b-metric space and $A \subset X$ and $\alpha \in (0,1)$ be given. Let $\epsilon > 0$ be a positive number. A set $B \subset X$ is said to be an $\alpha-\epsilon$-net for the set A if for any $x \in A$, $\exists y \in B$ such that*
$M(x,y,\frac{\epsilon}{K}) > 1-\alpha$.
B may be finite or infinite.

Definition 4.2. *A set A in a fuzzy strong b-metric space $(X,M,*,K)$ is said to be α-totally bounded for a given $\alpha \in (0,1)$, if for any $\epsilon > 0$, there exists a finite $\alpha-\epsilon$-net for the set A.*

Theorem 4.3. *Let $(X,M,*,K)$ be a fuzzy strong b-metric space and $A \subset X$ be α-totally bounded for some $\alpha \in (0,1)$. Then A is bounded.*

Proof. Since A is α-totally bounded, so for each $\epsilon > 0$, there exists a finite $\alpha-\epsilon$-net B for the set A.
Choose $\epsilon_0 > 0$. Then for each $x \in A$, there exists $y \in B$ such that $M(x,y,\frac{\epsilon_0}{K}) > 1-\alpha$.
Since B is finite thus B is bounded. (by Proposition 3.6).
So $\exists \epsilon_1 > 0$ and $\alpha_0 \in (0,1)$ such that
$M(y_1,y_2,\epsilon_1) > 1-\alpha_0 \quad \forall y_1, y_2 \in B$.

Now, for arbitrary $x_1, x_2 \in A$ we have,

$$\begin{aligned} M(x_1,x_2,\epsilon_1+2\epsilon_0) &= M(x_1,x_2,\epsilon_1+K.\frac{\epsilon_0}{K}+K.\frac{\epsilon_0}{K}) \\ &\geq M(x_1,y_2,\epsilon_1+K\frac{\epsilon_0}{K})*M(y_2,x_2,\frac{\epsilon_0}{K}) \\ &\geq M(x_1,y_1,\frac{\epsilon_0}{K})*M(y_1,y_2,\epsilon_1)*M(x_2,y_2,\epsilon_0). \end{aligned} \tag{4.1}$$

Now $M(x_1,y_1,\frac{\epsilon_0}{K}) > 1-\alpha,\ \ M(y_1,y_2,\epsilon_1) > 1-\alpha_0$ and $M(x_2,y_2,\frac{\epsilon_0}{K}) > 1-\alpha$.

Choose $\beta \in (0,1)$ such that (since $*$ is continuous).
$(1-\alpha)*(1-\alpha_0)*(1-\alpha) > 1-\beta$.
From (4.1), we get
$M(x_1,x_2,\epsilon_1+2\epsilon_0) \geq (1-\alpha)*(1-\alpha_0)*(1-\alpha) > 1-\beta$.
$\Rightarrow M(x_1,x_2,\epsilon_1+2\epsilon_0) > 1-\beta. \quad \forall x_1,x_2 \in A$.
$\Rightarrow$ A is bounded.

■

Note 4.1. The converse of the theorem is not true. We can prove it by the following example.

Example 4.2. Let $X = l_2$. Define $D(x,y) = (\sum_{i=1}^{\infty}|x_i-y_i|^2)^{\frac{1}{2}}$ where $x = (x_1,x_2,x_3,.....)$ and $y = (y_1,y_2,y_3,.....)$.
Then it is easy to verify that (X,D) is a strong b-metric space for $K \geq 1$
Again define $M_b(x,y,t) = \frac{t}{t+D(x,y)} \quad \forall t > 0, \forall x,y \in X$.

Then by using Example 2.2[7], it follows that $(X, M_b, *, K)$ is a fuzzy strong b-metric space w.r.t. the t-norm $*$=product.
Consider $A = \{(1,0,0,...),(0,1,0...),(0,0,1,0,...),..\}$. Then $A \subset X$. It is proved that A is bounded (by previous Example 3.2).
Now, we show that there is no $\alpha-\epsilon-$net for A. Choose $\epsilon = \frac{\sqrt{2}}{(1+K)}, \alpha = 1-\frac{1}{\sqrt{2}}$ and if possible suppose that N is a finite $\alpha-\epsilon$-net for A. Then for $x_i, x_j, (i \neq j)\epsilon A$, there exist y_i, y_j from N such that $M_b(x_i,y_i,\epsilon) > 1-\alpha$ and $M_b(x_j,y_j,\epsilon) > 1-\alpha$.
Now, $M_b(x_i,x_j,\epsilon+K\epsilon) \geq M_b(x_i,y_i,\epsilon).M_b(x_j,y_j,\epsilon)$
$> (1-\alpha).(1-\alpha)$
$= (1-\alpha)^2$
$\Rightarrow \frac{(1+K)\epsilon}{(1+K)\epsilon+\sqrt{2}} > (1-\alpha)^2$
$\Rightarrow \frac{\sqrt{2}}{2\sqrt{2}} > (1-\alpha)^2$
$\Rightarrow \frac{1}{2} > \frac{1}{2}$.
Which is a contradiction. So, A is not $\alpha-\epsilon-$bounded.

Definition 4.4. *Let $(X,M,*,K)$ be a fuzzy strong b-metric space and $\alpha \in (0,1)$.*
(i) A sequence $\{x_n\}$ is said to be α-convergent and converges to x if
$\lim_{n\to\infty} M(x_n,x,t) > 1-\alpha \quad \forall t > 0$.
(ii) A sequence $\{x_n\}$ in X is said to be α-Cauchy sequence if
$\lim_{m,n\to\infty} M(x_n,x_m,t) > 1-\alpha \quad \forall t > 0$.
(iii) A subset A of X is said to be α-compact if every sequence in A has an α-convergent subsequence converges to some element in A.
If the converging point belongs to X not to A then we say that A is α-compact in X.

Definition 4.5. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A(\subset X)$ be a nonempty subset of X. Then $\alpha-$ diameter of A is defined as*
$\alpha - \delta(A) = \bigvee_{x,y\in A} \bigwedge\{t > 0 : M(x, y, t) > 1 - \alpha\}, \quad 0 < \alpha < 1.$

Theorem 4.6. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$.*
(1) if A is compact then A is α-totally bounded $\forall\alpha \in (0, 1)$.
(2) If X is α-complete and A is α-totally bounded $\forall\alpha \in (0, 1)$ then A is α-compact in X $\forall\alpha \in (0, 1)$ w.r.t. the t-norm $* = \min$.

Proof. (1) We assume that A is compact. Choose $\alpha \in (0, 1)$ and $\epsilon > 0$ be arbitrary. Let x_1 be an arbitrary element of X.
If $M(x, x_1, \frac{\epsilon}{K}) > 1 - \alpha \quad \forall x \in A$, then a finite $\alpha - \epsilon-$net B exists for A. i.e. $B = \{x_1\}$.
If not, $\exists$ a point $x_2 \in A$ such that $M(x_1, x_2, \frac{\epsilon}{K}) \leq 1 - \alpha$. If for every point $x \in A$ either $M(x, x_1, \frac{\epsilon}{K}) > 1 - \alpha$ or $M(x, x_2, \frac{\epsilon}{K}) > 1 - \alpha$ then a finite ϵ-net B exists for A.
i.e. $B = \{x_1, x_2\}$.
If, however, this is not true, then there exists $x_3 \in A$ such that $M(x_3, x_1, \frac{\epsilon}{K}) \leq 1-\alpha$ and $M(x_3, x_2, \frac{\epsilon}{K}) \leq 1-\alpha$.
Then a finite $\alpha - \epsilon-$net $B = \{x_1, x_2, x_3\}$ exists for A.
Continuing in this way, we obtain points $x_1, x_2, ..., x_n$; $x_1 \in X$ and $x_i \in A$, $2 \leq i \leq n$ for which
$M(x_i, x_j, \frac{\epsilon}{K}) \leq 1 - \alpha \quad$ for $i \neq j$.
There are two cases may arise.
Case I. The procedure stops after k th step.
Then we obtain points $x_1, x_2, ..., x_k$ such that for every $x \in A$ at least one of the inequalities
$M(x_i, x, \frac{\epsilon}{K}) > 1 - \alpha, \ \ i = 1, 2, ..., k$ holds and then $B = \{x_1, x_2, ...x_k\}$ is a finite $\alpha - \epsilon-$net for A and here A is α-totally bounded.
Case II. The procedure continues indefinitely.
Then we obtain an infinite sequence $\{x_n\}$, $x_1 \in X$ and $x_i \in A$ for $i > 1$ such that
$M(x_i, x_j, \frac{\epsilon}{K}) \leq 1 - \alpha \quad$ for $i \neq j$.
If possible suppose there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x.
Now $M(x_{n_k}, x, \frac{\epsilon}{2K}) * M(x, x_{n_{k+1}}, \frac{\epsilon}{2K^2}) \leq M(x_{n_k}, x_{n_{k+1}}, \frac{\epsilon}{K}) \leq 1 - \alpha$
$\Rightarrow \lim_{k\to\infty} M(x_{n_k}, x, \frac{\epsilon}{2K}) * \lim_{k\to\infty} M(x, x_{n_{k+1}}, \frac{\epsilon}{2K^2}) \leq 1 - \alpha$
$\Rightarrow 1 * 1 \leq 1 - \alpha$
$\Rightarrow 1 \leq 1 - \alpha$ which is a contradiction.
Thus Case II does not arise.
Hence A is α-totally bounded. Since $\alpha \in (0, 1)$ is arbitrary thus A is α-totally bounded $\forall\alpha \in (0, 1)$.
2. We assume that X is α-complete and α-totally bounded for each $\alpha \in (0, 1)$.
So for every $\epsilon > 0$ and each $\alpha \in (0, 1)$, there exists a finite $\alpha - \epsilon-$net for A. Let $\alpha \in (0, 1)$ be given. We choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and $\epsilon_n > 0$ $\forall n$ and $\epsilon_{n+1} < \epsilon_n$ and construct for each $n = 1, 2, ...$ a finite $\alpha - \epsilon_n-$ net
$[x_1^{(n)}, x_2^{(n)}, ..., x_{k_n}^{(n)}]$ for the set A. Let $T = \{x_n\}$ be an arbitrary sequence of elements from A. Without loss of generality we may assume that $x_i \neq x_j$ if $i \neq j$ and T is the infinite set with elements x_n.
Around every point of the $\alpha - \epsilon_1$-net $[x_1^{(1)}, x_2^{(1)}, ..., x_{k_1}^{(1)}]$, we construct closed balls with radius ϵ_1. It is clear that each element of $\{x_n\}$ belongs to one or more of these balls.
Since the number of balls is finite, there exists at least one ball containing an infinite subset $T_1 \subset T$ (say $B[x_1^{(1)}, \alpha, \epsilon_1]$).
Now we show that $\alpha - \delta(T_1) \leq 2\frac{\epsilon_1}{K}$.
Let $x, y \in T_1$. Then $M(x, x_i^{(1)}, \frac{\epsilon_1}{K}) > 1 - \alpha$ and $M(y, x_i^{(1)}, \frac{\epsilon_1}{K}) > 1 - \alpha$ $(1 \leq i \leq k_1)$.
Now $M(x, y, 2\epsilon_1) = M(x, y, \epsilon_1 + K \cdot \frac{\epsilon_1}{K}) \geq M(x, x_i^{(1)}, \epsilon_1) * M(y, x_i^{(1)}, \frac{\epsilon_1}{K})$
$\qquad \geq M(x, x_i^{(1)}, \frac{\epsilon_1}{K}) * M(y, x_i^{(1)}, \frac{\epsilon_1}{K})$

$> (1-\alpha) * (1-\alpha) = 1-\alpha.$
$\Rightarrow \bigwedge\{t > 0 : M(x, y, t) > 1-\alpha\} \leq 2\epsilon_1$
$\Rightarrow \bigvee_{x,y \in T_1} \bigwedge\{t > 0 : M(x, y, t) > 1-\alpha\} \leq 2\epsilon_1$
$\Rightarrow \alpha - \delta(T_1) \leq 2\epsilon_1.$
Next, around every point of the $\alpha - \epsilon_2$-net $[x_1^{(2)}, x_2^{(2)}, ..., x_{k_2}^{(2)}]$
we construct closed sphere with radius ϵ_2.
By the same argument as above, there exists an infinite subset $T_2 \subset T_1$ and
$\alpha - \delta(T_2) \leq 2\epsilon_2.$
Continuing in this process, we obtain a sequence of infinite subsets $T \supset T_1 \supset T_2 \supset ... \supset T_n \supset ...$ where $\alpha - \delta(T_n) \leq 2\epsilon_n \quad \forall n.$
We now choose a point $x_{p_1} \in T_1$, a point $x_{p_2} \in T_2$ different from x_{p_1}, a point $x_{p_3} \in T_3$ different from x_{p_1} and x_{p_2} and so on.
We have $x_{p_n} \in T_n$, $x_{p_m} \in T_m$ and for $n > m$, $T_n \subset T_m$.
Thus for $n > m$, $x_{p_n}, x_{p_m} \in T_m$.
So $\bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} \leq \alpha - \delta(T_m) \leq 2\epsilon_m.$
$\Rightarrow \lim_{n,m\to\infty} \bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} = 0$
Thus for a given $\epsilon > 0$, there exists a natural number say n_0 such that
$\bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} < \epsilon \quad \forall m, n \geq n_0.$
$\Rightarrow M(x_{p_n}, x_{p_m}, \epsilon) > 1-\alpha \quad \forall m, n \geq n_0.$
$\Rightarrow \lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, \epsilon) \geq 1-\alpha$
Since $\epsilon > 0$ is arbitrary, thus
$\Rightarrow \lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, t) \geq 1-\alpha \quad \forall t > 0$
Choose $\beta \in (0,1)$ such that $1-\alpha > 1-\beta$.
So $\lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, t) > 1-\beta.$
Thus $\{x_{p_n}\}$ is a β-Cauchy sequence in A and hence in X. Since X is β-complete, thus there exists $x \in X$ such that
$\lim_{n\to\infty} M(x_{p_n}, x_{p_m}, t) > 1-\beta \quad \forall t > 0.$
Hence A is β-compact in X.
Since $\alpha \in (0,1)$ is arbitrary, thus $\beta \in (0,1)$ is also arbitrary and hence the proof is complete. ■

Definition 4.7. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$.*
The closure of A is denoted by $\bar{A}$ and is defined by $\bar{A} = A \cup A'$ where A' denotes the derived set of A.

Proposition 4.8. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. For $x \in \bar{A}$, for each $\epsilon > 0$ and $\alpha \in (0,1)$, there exists $y \in A$ such that*
$M(x, y, \epsilon) > 1-\alpha$.

Proof. Let $x \in \bar{A}$. So $x \in A \cup A'$.
Case I . $x \in A$. Then we choose $y = x$ and we have
$M(x, y, \epsilon) = M(x, x, \epsilon) = 1 > 1-\alpha$ for each $\epsilon > 0$ and $\alpha \in (0,1)$.
Case II . $x \ notin$ A and $x \in A'$.
Thus for each $\epsilon > 0$ and $\alpha \in (0,1)$, there exists $y \in A$ such that
$y \in B(x, \epsilon, \alpha)$.
i.e. $M(x, y, \epsilon) > 1-\alpha$. ■

Proposition 4.9. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. If A is compact then $\bar{A}$ is compact.*

Proof. Let $\{y_n\}$ be a sequence in $\bar{A}$.
Choose $\epsilon > 0$ be arbitrary and $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \to 0$ as $n \to \infty$.
Now by Proposition 4.8, for each y_n, there exists $x_n \in A$ such that
$M(x_n, y_n, \frac{\epsilon}{2}) > 1 - \alpha_n$......(i)
Thus we obtain a sequence $\{x_n\}$ in A. Since A is compact, thus there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges to some point $x \in A$.
So $\lim\limits_{r\to\infty} M(x_{n_r}, x, t) = 1 \quad \forall t > 0$
i.e. $\lim\limits_{r\to\infty} M(x_{n_r}, x, \dfrac{\epsilon}{2K}) = 1$.........(ii)
Now $M(y_{n_r}, x, \epsilon) = M(y_{n_r}, x, \frac{\epsilon}{2} + K.\frac{\epsilon}{2K})$
$\qquad \geq M(y_{n_r}, x_{n_r}, \frac{\epsilon}{2}) * M(x_{n_r}, x, \frac{\epsilon}{2K})$
$\Rightarrow \lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) \geq \lim\limits_{r\to\infty} M(y_{n_r}, x_{n_r}, \dfrac{\epsilon}{2}) * \lim\limits_{r\to\infty} M(x_{n_r}, x, \dfrac{\epsilon}{2K}) = 1$........(iii)
From (i) we get $M(x_{n_r}, y_{n_r}, \frac{\epsilon}{2}) > 1 - \alpha_{n_r}$
$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, y_{n_r}, \dfrac{\epsilon}{2}) \geq 1 - \lim\limits_{r\to\infty} \alpha_{n_r} = 1$
$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, y_{n_r}, \dfrac{\epsilon}{2}) = 1$...........(iv)
Using (ii) and (iv), from (iii) we have
$\lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) \geq 1 * 1 = 1$
$\Rightarrow \lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) = 1$
Since $\epsilon > 0$ is arbitrary, thus $\lim\limits_{r\to\infty} M(y_{n_r}, x, t) = 1 \quad \forall t > 0$.
Thus the subsequence $\{y_{n_r}\}$ of $\{y_n\}$ converges to x. Hence $\bar{A}$ is compact. ■

Note 4.1. Converse of the result is not true. We justify it by the following example.
Example 4.1. Let $X = R$. Define $M(x,y,t) = e^{-\frac{D(x,y)}{t}} \quad \forall t > 0; \forall x, y \in X$.
We write $D(x,y) = |x-y| \ \ \forall x, y \in X$. Then it is verified that (X, D, K) is a strong b-metric space (by previous Example 3.2).
Now, we shall prove that $(X, M, *, K)$ is a fuzzy strong b-metric space. Where $*$ is the product t-norm and $K > 1$.
1. $M(x,y,t) = e^{-\frac{D(x,y)}{t}} > 0 \ \ \forall x, y \in X$ and $\forall t > 0$.
2. $M(x,y,t) = 1 \ \ \forall x, y \in X$ and $\forall t > 0$.
$\Leftrightarrow e^{-\frac{D(x,y)}{t}} = 1 = e^0$
$\Leftrightarrow -\frac{D(x,y)}{t} = 0 \ \ \forall t > 0$.
$\Leftrightarrow D(x,y) = 0$
$\Leftrightarrow x = y$.
3. $M(x,y,t) = e^{-\frac{D(x,y)}{t}} = e^{-\frac{D(y,x)}{t}} \quad \forall t > 0$.
$\qquad = M(y,x,t) \ \ \forall x, y \in X$
4. Now, $\forall x, y, z \in X$,
$D(x,z) \leq D(x,y) + KD(y,z) \quad K > 1$.
$\frac{D(x,z)}{t+KS} \leq \frac{D(x,y)+KD(y,z)}{t+KS}; \ \ t, s > 0$.
$e^{\frac{D(x,z)}{t+KS}} \leq e^{\frac{D(x,y)+KD(y,z)}{t+KS}}$
$e^{\frac{D(x,z)}{t+KS}} \leq e^{\frac{D(x,y)}{t+KS}} \cdot e^{\frac{D(y,z)}{\frac{t}{K}+s}}$
$\qquad \leq e^{\frac{D(x,y)}{t}} \cdot e^{\frac{D(y,z)}{s}}$
$e^{\frac{D(x,z)}{t+KS}} \leq e^{(\frac{D(x,y)}{t} + \frac{D(y,z)}{t})}$
$e^{-\frac{D(x,z)}{t+KS}} \geq e^{-(\frac{D(x,y)}{t} + \frac{D(y,z)}{t})}$
$e^{-\frac{D(x,z)}{t+KS}} \geq e^{-\frac{D(x,y)}{t}} \cdot e^{-\frac{D(y,z)}{s}}$
$\therefore M(x, z, t+Ks) \geq M(x,y,t) \cdot M(y,z,s)$
5. This is clear that $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous. Thus $(X, M, \cdot, K)$ is a fuzzy strong b-metric

space.
Let $A=(0,1)$. Then $\bar{A}=[0,1]$.
Firstly, we will show that A is not compact in X. If possible suppose that A is compact. Let $\{x_n\}$ be a sequence in A where $x_n=\frac{1}{n+1}\quad\forall n\geq 1$.
Let $\{x_{k_n}\}$ be a sequence in A such that $x_{k_n}\to y$ for some $y\in A$.
$M(x_{k_n},y,t)=e^{-\frac{D(x_{k_n},y)}{t}}\quad\forall t>0$.

$\lim_{n\to\infty}M(x_{k_n},y,t)=\lim_{n\to\infty}e^{-\frac{D(x_{k_n},y)}{t}}=e^{-\lim_{n\to\infty}e^{\frac{D(x_{k_n},y)}{t}}}$
$\Rightarrow e^0=1=e^{-\lim_{n\to\infty}e^{\frac{D(x_{k_n},y)}{t}}}\qquad\forall t>0$.
$\Rightarrow\lim_{n\to\infty}\dfrac{D(x_{k_n},y)}{t}=0\quad\forall t>0$.
$\Rightarrow\lim_{n\to\infty}D(x_{k_n},y)=0$
$\Rightarrow\lim_{n\to\infty}|x_{k_n}-y|=0$
$\Rightarrow y=0$.
$\Rightarrow y\notin$ A.
Which is a contradiction.
So, A is not complete.
Now we prove that $\bar{A}=[0,1]$ is compact.
By Hine-Borel theorem, $\bar{A}=[0,1]$ is compact in R w.r.t. usual norm given by $\|x\|=|x|\quad\forall x\in R$.
Let $\{x_n\}$ be a sequence in $\bar{A}$. So,there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges in some point $x\in\bar{A}$.
i.e. $|x_{n_r}-x|\to 0$ as $r\to\infty$ and $x\in\bar{A}$.
i.e. $D(x_{n_r},x)\to 0$ as $r\to\infty$ and $x\in\bar{A}$.
Now $M(x_{n_r},x,t)=e^{-\frac{D(x_{n_r},x)}{t}}$

$\Rightarrow\lim_{r\to\infty}M(x_{n_r},x,t)=\lim_{r\to\infty}e^{-\frac{D(x_{n_r},x)}{t}}=e^{-\lim_{r\to\infty}e^{\frac{D(x_{n_r},x)}{t}}}$.
Since $D(x_{n_r},x)\to 0$ as $r\to\infty$, from above we have,
$\Rightarrow\lim_{r\to\infty}M(x_{n_r},x,t)=1\quad\forall t>0$.
$\Rightarrow x_{n_r}\to x$ in $(X,M,*,K)$.
Since $\{x_n\}$ is an arbitrary sequence in $\bar{A}$, thus $\bar{A}$ is a compact subset in $(X,M,*,K)$.

5. Conclusion

The concept of fuzzy strong b-metric space is relatively a new idea by modifying the triangle inequality in fuzzy setting. In this paper, we explore an idea of compactness and totally boundedness on fuzzy strong b-metric spaces and establish some basic results. We think that the researchers will be enriched with serendipitous findings by this research work.

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