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Fractional Hermite-Hadamard type inequalities for co-ordinated convex functions

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Abstract. In this paper, we first construct a new integral equality. Using this equality, we establish Hermite-Hadamard type fractional integral inequalities involving two variables via convexity.

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Contents

1 Introduction **437**

2 Preliminaries **438**

3 Main results **440**

1. Introduction

The Hermite-Hadamard integral inequality, which may be expressed as follows: for every convex function $\mathcal{S}$ on the finite interval $[k,l]$, we have

$$\mathcal{S}\left(\frac{k+l}{2}\right) \leq \frac{1}{l-k}\int_k^l \mathcal{S}(x)dx \leq \frac{\mathcal{S}(k)+\mathcal{S}(l)}{2} \tag{1.1}$$

is one of the most well-known mathematical inequalities for convex functions. (1.1) holds in the opposite way if the function $\mathcal{S}$ is concave (see [15]).

Dragomir determined the bidimentionnal analogue of (1.1) provided by in [3].

$$\begin{aligned}\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right) &\leq \frac{1}{2}\left(\frac{1}{l-k}\int_k^l \mathcal{S}(x,\tfrac{u+v}{2})dx + \frac{1}{v-u}\int_u^v \mathcal{S}(\tfrac{k+l}{2},y)dy\right)\\ &\leq \frac{1}{(l-k)(v-u)}\int_k^l\int_u^v \mathcal{S}(x,y)dydx\end{aligned}$$

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$$\le \frac{1}{4}\left(\frac{1}{l-k}\int_k^l \mathcal{S}(x,u)dx+\frac{1}{l-k}\int_k^l \mathcal{S}(x,v)dx + \frac{1}{v-u}\int_u^v \mathcal{S}(k,y)dy+\frac{1}{v-u}\int_u^v \mathcal{S}(l,y)dy\right)$$
$$\le \frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}. \tag{1.2}$$

Numerous scholars have been drawn to the inequalities (1.2), and numerous generalizations, improvements, expansions, and modifications of (1.1) have been documented in the literature (see $[1,2,4,6,7,14,16-20]$ and the references therein).

The following results was provided by Sarikaya [16].

Theorem 1.1. *Let $\mathcal{S}:\Delta\to\mathbb{R}$ partially differentiable map on $\Delta=[k,l]\times[u,v]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}\right|$ is a co-ordinated convex function on Δ, then we have*

$$\Bigg|\frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u) + J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)\right)-\mathfrak{A}\Bigg|$$
$$\le\frac{(l-k)(v-u)}{4(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,u)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,v)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,u)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,v)\right|\right),$$

where

$$\mathfrak{A}=\frac{\Gamma(\beta+1)}{4(v-u)^\beta}\left(J^\beta_{u^+}\mathcal{S}(k,v)+J^\beta_{u^+}\mathcal{S}(l,v)+J^\beta_{v^-}\mathcal{S}(k,u)+J^\beta_{v^-}\mathcal{S}(l,u)\right) + \frac{\Gamma(\alpha+1)}{4(l-k)^\alpha}\left(J^\alpha_{k^+}\mathcal{S}(l,c)+J^\alpha_{k^+}\mathcal{S}(l,v)+J^\alpha_{l^-}\mathcal{S}(k,c)+J^\alpha_{l^-}\mathcal{S}(k,v)\right). \tag{1.3}$$

Theorem 1.2. *Under the assumptions of Theorem 1.1. If $\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}\right|^q$ is a co-ordinated convex function on Δ, then we have*

$$\Bigg|\frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u) + J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)\right)-\mathfrak{A}\Bigg|$$
$$\le\frac{(l-k)(v-u)}{((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,v)\right|^q}{4}\right)^{\frac{1}{q}},$$

where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\mathfrak{A}$ is as in (1.3)*.*

The aim of this work is to establish some integral inequalities of the Hermite-Hadamard type via convexity on co-ordinates by using fractional integral operators. The obtained results are based on a new integral equality.

2. Preliminaries

Here, we revisit few definitions. We also assume throughout that $\Delta\subset\mathbb{R}^2$ with $\Delta:=[k,l]\times[u,v]$ where $k<l$ and $u<v$.

Definition 2.1. *[6] Convexity on the co-ordinates on Δ is the property of a function $\mathcal{S}:\Delta\to\mathbb{R}$ that holds when the inequality*

$$\begin{aligned}\mathcal{S}\left(gx+(1-g)\,\xi,\lambda y+(1-\lambda)\,j\right)\leq & g\lambda\mathcal{S}(x,y)+g\,(1-\lambda)\,\mathcal{S}(x,j)\\ & +(1-g)\,\lambda\mathcal{S}(\xi,y)+(1-g)\,(1-\lambda)\,\mathcal{S}(\xi,j)\end{aligned}$$

remains true for any $(x,y),(x,j),(\xi,y),(\xi,j)\in\Delta$ and $g,\lambda\in[0,1]$.

Definition 2.2. *[5] The Riemann-Liouville integrals $J^{\alpha}_{k^+}\mathcal{S}$ and $J^{\alpha}_{l^-}\mathcal{S}$ of order α are defined by:*

$$J^{\alpha}_{k^+}\mathcal{S}(\xi)=\frac{1}{\Gamma(\alpha)}\int_k^{\xi}(\xi-t)^{\alpha-1}\mathcal{S}(t)dt,\quad \xi>k,$$

$$J^{\alpha}_{l^-}\mathcal{S}(\xi)=\frac{1}{\Gamma(\alpha)}\int_{\xi}^{l}(t-\xi)^{\alpha-1}\mathcal{S}(t)dt,\quad l>\xi,$$

respectively, where $\alpha>0,k\geq 0,\mathcal{S}\in L^1[k,l]$ and $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$, is the gamma function and $J^{0}_{k^+}\mathcal{S}(\xi)=J^{0}_{l^-}\mathcal{S}(\xi)=\mathcal{S}(\xi)$.

Definition 2.3. *[5] The Riemann-Liouville integrals $J^{\alpha,\beta}_{k^+,u^+}$, $J^{\alpha,\beta}_{k^+,v^-}$, $J^{\alpha,\beta}_{l^-,u^+}$, and $J^{\alpha,\beta}_{l^-,v^-}$ of order $\alpha,\beta>0$ with $k,u\geq 0$, $k<l$ and $u<v$ are defined by*

$$J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_k^l\int_u^v(l-\xi)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi, \tag{2.1}$$

$$J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_k^l\int_u^v(l-\xi)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi \tag{2.2}$$

$$J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_k^l\int_u^v(\xi-k)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi, \tag{2.3}$$

$$J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_k^l\int_u^v(\xi-k)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi, \tag{2.4}$$

where $\mathcal{S}\in L^1(\Delta),\Gamma$ is the gamma function, and

$$J^{0,0}_{k^+,u^+}\mathcal{S}(l,v)=J^{0,0}_{k^+,v^-}\mathcal{S}(l,u)=J^{0,0}_{l^-,u^+}\mathcal{S}(k,v)=J^{0,0}_{l^-,v^-}\mathcal{S}(k,u)=\mathcal{S}(\xi,y).$$

Definition 2.4. *[16] The Riemann-Liouville integrals $J^{\alpha}_{l^-}\mathcal{S}(k,u)$, $J^{\alpha}_{k^+}\mathcal{S}(l,u)$, $J^{\beta}_{v^-}\mathcal{S}(k,u)$ and $J^{\alpha}_{u^+}\mathcal{S}(k,v)$ of order $\alpha,\beta>0$ with $k,u\geq 0$, $k<l$ and $u<v$, are defined by*

$$J^{\alpha}_{l^-}\mathcal{S}(k,u)=\frac{1}{\Gamma(\alpha)}\int_k^l(\xi-k)^{\alpha-1}\mathcal{S}(\xi,u)\,d\xi, \tag{2.5}$$

$$J^{\alpha}_{k^+}\mathcal{S}(l,u)=\frac{1}{\Gamma(\alpha)}\int_k^l(l-\xi)^{\alpha-1}\mathcal{S}(\xi,u)\,d\xi, \tag{2.6}$$

$$J_{v^-}^{\beta}\mathcal{S}(k,u)=\frac{1}{\Gamma(\beta)}\int_c^v (y-u)^{\beta-1}\mathcal{S}(k,y)\,dy, \tag{2.7}$$

$$J_{u^+}^{\alpha}\mathcal{S}(k,v)=\frac{1}{\Gamma(\beta)}\int_u^v (v-y)^{\beta-1}\mathcal{S}(k,y)\,dy, \tag{2.8}$$

where $\mathcal{S}\in L^1(\Delta)$ and Γ represents the gamma function.

3. Main results

Lemma 3.1. *Assume that $\mathcal{S}:\Delta\to\mathbb{R}$ be a partially differentiable map. If $\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\in L(\Delta)$, then we have*

$$\begin{aligned}
&\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)-\frac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}+\mathfrak{A}-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^{\alpha}(v-u)^{\beta}}\\
&\times\left(J_{k^+,u^+}^{\alpha,\beta}\mathcal{S}(l,v)+J_{l^-,u^+}^{\alpha,\beta}\mathcal{S}(k,v)+J_{k^+,v^-}^{\alpha,\beta}\mathcal{S}(l,u)+J_{l^-,v^-}^{\alpha,\beta}\mathcal{S}(k,u)\right)\\
=&\frac{(l-k)(v-u)}{4}\left(\int_0^1\int_0^1\mathcal{K}\mathcal{H}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,dF\,d\chi_2\right.\\
&-\int_0^1\int_0^1\left((1-\chi_1)^{\alpha}-\chi_1^{\alpha}\right)\left((1-\chi_2)^{\beta}-\chi_2^{\beta}\right)\\
&\times\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\lambda,
\end{aligned}\tag{3.1}$$

where

$$\mathcal{K}=\left\{\begin{cases}1 & \text{if } 0\le\chi_1<\frac{1}{2},\\ -1 & \text{if } \frac{1}{2}\le\chi_1<1,\end{cases}\right. \tag{3.2}$$

$$\mathcal{H}=\left\{\begin{cases}1 & \text{if } 0\le\chi_2<\frac{1}{2},\\ -1 & \text{if } \frac{1}{2}\le\chi_2<1,\end{cases}\right. \tag{3.3}$$

$$\begin{aligned}
\mathfrak{A}=&\frac{\Gamma(\beta+1)}{4(v-u)^{\beta}}\left(J_{u^+}^{\beta}\mathcal{S}(k,v)+J_{u^+}^{\beta}\mathcal{S}(l,v)+J_{v^-}^{\beta}\mathcal{S}(k,u)+J_{v^-}^{\beta}\mathcal{S}(l,u)\right)\\
&+\frac{\Gamma(\alpha+1)}{4(l-k)^{\alpha}}\left(J_{k^+}^{\alpha}\mathcal{S}(l,u)+J_{k^+}^{\alpha}\mathcal{S}(l,v)+J_{l^-}^{\alpha}\mathcal{S}(k,u)+J_{l^-}^{\alpha}\mathcal{S}(k,v)\right).
\end{aligned}\tag{3.4}$$

Proof. Let

$$I=\frac{(l-k)(v-u)}{4}(I_1-I_2), \tag{3.5}$$

where

$$I_1=\int_0^1\int_0^1\mathcal{K}\mathcal{H}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2,$$

$$I_2=\int_0^1\int_0^1\left((1-\chi_1)^{\alpha}-\chi_1^{\alpha}\right)\left((1-\chi_2)^{\beta}-\chi_2^{\beta}\right)\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2.$$

Clearly, we have

$$I_1=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2$$

$$
\begin{aligned}
&-\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v\right)d\chi_1d\chi_2\\
&-\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v\right)d\chi_1d\chi_2\\
&+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v\right)d\chi_1d\chi_2\\
=&\frac{1}{(l-k)(v-u)}\left(\left(\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)-\mathcal{S}\left(l,\frac{u+v}{2}\right)-\mathcal{S}\left(\frac{k+l}{2},v\right)+\mathcal{S}(l,v)\right)\right.\\
&-\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)+\mathcal{S}(k,v)-\mathcal{S}\left(\frac{k+l}{2},v\right)\\
&-\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}(l,u)+\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)-\mathcal{S}\left(l,\frac{u+v}{2}\right)\\
&\left.+\mathcal{S}(k,u)-\mathcal{S}\left(\frac{k+l}{2},u\right)-\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)\right)\\
=&\frac{4}{(l-k)(v-u)}\left(\left(\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)\right)+\frac{\mathcal{S}(l,v)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(k,u)}{4}\right.\\
&\left.-\frac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}\right).
\end{aligned}
\tag{3.6}
$$

Using the integration by parts, I_2 gives

$$
I_2=\int_0^1\left((1-\chi_2)^\beta-\chi_2^\beta\right) \tag{3.7}
$$

$$
\begin{aligned}
&\times\left(\int_0^1\left((1-\chi_1)^\alpha-\chi_1^\alpha\right)\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v\right)d\chi_1\right)d\chi_2\\
=&\frac{1}{(l-k)(v-u)}\left(\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)\right)\\
&-\frac{\beta}{(l-k)(v-u)}\left(\int_0^1(1-\chi_2)^{\beta-1}\mathcal{S}\left(k,\chi_2u+(1-\chi_2)v\right)d\chi_2\right.\\
&+\int_0^1\chi_2^{\beta-1}\mathcal{S}\left(k,\chi_2u+(1-\chi_2)v\right)d\chi_2+\int_0^1\chi_2^{\beta-1}\mathcal{S}\left(l,\chi_2u+(1-\chi_2)v\right)d\chi_2\\
&\left.+\int_0^1(1-\chi_2)^{\beta-1}\mathcal{S}\left(l,\chi_2u+(1-\chi_2)v\right)d\chi_2\right)\\
&-\frac{\alpha}{(l-k)(v-u)}\left(\int_0^1(1-\chi_1)^{\alpha-1}\mathcal{S}\left(\chi_1k+(1-\chi_1)l,u\right)d\chi_1\right.\\
&+\int_0^1\chi_1^{\alpha-1}\mathcal{S}\left(\chi_1k+(1-\chi_1)l,u\right)d\chi_1+\int_0^1\chi_1^{\alpha-1}\mathcal{S}\left(\chi_1k+(1-\chi_1)l,v\right)d\chi_1\\
&\left.+\int_0^1(1-\chi_1)^{\alpha-1}\mathcal{S}\left(\chi_1k+(1-\chi_1)l,v\right)d\chi_1\right)
\end{aligned}
$$

$$+\frac{\alpha\beta}{(l-k)(v-u)}\left(\int_0^1\int_0^1 \chi_1^{\alpha-1}\chi_2^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1\right.$$
$$+\int_0^1\int_0^1 (1-\chi_1)^{\alpha-1}\chi_2^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1$$
$$+\int_0^1\int_0^1 \chi_1^{\alpha-1}(1-\chi_2)^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1$$
$$\left.+\int_0^1\int_0^1 (1-\chi_1)^{\alpha-1}(1-\chi_2)^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1\right).$$

Combining (3.5)-(3.7) and making a changes $\xi=\chi_1 k+(1-\chi_1)l$ and $y=\chi_2 u+(1-\chi_2)v$, we get

$$I=\mathcal{S}\left(\frac{k+l}{2},\frac{u+b}{2}\right)-\frac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}$$
$$+\frac{\beta}{4(v-u)^\beta}\left(\int_u^v (y-u)^{\beta-1}\mathcal{S}(k,y)\,dy+\int_u^v (y-u)^{\beta-1}\mathcal{S}(l,y)\,dy\right.$$
$$\left.+\int_u^v (v-y)^{\beta-1}\mathcal{S}(k,y)\,dy+\int_u^v (v-y)^{\beta-1}\mathcal{S}(l,y)\,dy\right) \tag{3.8}$$
$$+\frac{\alpha}{4(l-k)^\alpha}\left(\int_k^l (\xi-k)^{\alpha-1}\mathcal{S}(\xi,u)\,d\xi+\int_k^l (\xi-k)^{\alpha-1}\mathcal{S}(\xi,v)\,d\xi\right.$$
$$\left.+\int_k^l (l-\xi)^{\alpha-1}\mathcal{S}(\xi,u)\,dx+\int_k^l (l-\xi)^{\alpha-1}\mathcal{S}(\xi,v)\,d\xi\right)$$
$$-\frac{\alpha\beta}{4(l-k)^\alpha(v-u)^\beta}\left(\int_k^l\int_u^v (l-\xi)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\right.$$
$$+\int_k^l\int_u^v (\xi-k)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi$$
$$+\int_k^l\int_u^v (l-\xi)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi$$
$$\left.+\int_k^l\int_u^v (\xi-k)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\right). \tag{3.9}$$

The proof is thus finished. ■

In what follows, we note

$$\Lambda(k,l,u,v,\mathcal{S})$$
$$=\mathcal{S}\left(\frac{k+l}{2},\frac{u+v}{2}\right)-\frac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}+\mathfrak{A}-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}$$

$$\times\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)+J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)\right),$$

where $\mathfrak{A}$ is given by (3.4).

Theorem 3.2. *For a partial differentiable map $\mathcal{S}:\Delta\to\mathbb{R}$ whose $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|$ is co-ordinated convex, we have*

$$\begin{aligned}|\Lambda(k,l,u,v,\mathcal{S})|\leq&\tfrac{(l-k)(v-u)}{4}\left(\tfrac{1}{4}+\tfrac{1}{(\alpha+1)(\beta+1)}\right)\\&\times\left(\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right).\end{aligned}\tag{3.10}$$

Proof. Using the absolute value on both sides of (3.1), we get

$$\begin{aligned}&|\Lambda(k,l,u,v,\mathcal{S})|\\\leq&\tfrac{(l-k)(v-u)}{4}\left(\int_0^1\int_0^1\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\right|d\chi_1d\chi_2\right.\\&+\int_0^1\int_0^1(1-\chi_1)^\alpha(1-\chi_2)^\beta\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^\alpha(1-\chi_2)^\beta\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1(1-\chi_1)^\alpha\chi_2^\beta\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^\alpha\chi_2^\beta\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\right|d\chi_1d\chi_2.\end{aligned}\tag{3.11}$$

Since $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|$ is co-ordinated convex, (3.11) gives

$$\begin{aligned}&|\Lambda(k,l,u,v,\mathcal{S})|\\\leq&\tfrac{(l-k)(v-u)}{4}\left(\int_0^1\int_0^1\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1(1-\chi_2)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\right.\\&+(1-\chi_1)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+(1-\chi_1)(1-\chi_2)\left.\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\\&+\int_0^1\int_0^1(1-\chi_1)^\alpha(1-\chi_2)^\beta\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1(1-\chi_2)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\\&+(1-\chi_1)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+(1-\chi_1)(1-\chi_2)\left.\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^\alpha(1-\chi_2)^\beta\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1(1-\chi_2)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\\&+(1-\chi_1)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+(1-\chi_1)(1-\chi_2)\left.\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\end{aligned}$$

$$+\int_0^1\int_0^1 (1-\chi_1)^\alpha \chi_2^\beta \left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.$$
$$\left. + (1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + (1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right) d\chi_1 d\chi_2$$
$$+\int_0^1\int_0^1 \chi_1^\alpha \chi_2^\beta \left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.$$
$$\left.\left. + (1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + (1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right) d\chi_1 d\chi_2\right)$$
$$=\frac{(l-k)(v-u)}{4}\left(\frac{1}{4}+\frac{1}{(\alpha+1)(\beta+1)}\right)$$
$$\times\left(\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right),$$

which is the desired outcome. ■

Theorem 3.3. *Suppose that all the assumptions of Theorem 3.2 hold. If* $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|^q$ *is co-ordinated convex function, then we have*

$$|\Lambda(k,l,u,v,\mathcal{S})|$$
$$\leq\frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\frac{9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{4}\right)^{\frac{1}{q}}\right),$$

where $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathfrak{A}$ *is given by* (3.4).

Proof. Using the absolute value on both sides of (3.1) and then applying Hölder's inequality, it yields

$$|\Lambda(k,l,u,v,\mathcal{S})|$$
$$\leq\frac{(l-k)(v-u)}{4}\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\left(\int_{0}^{1}\int_{0}^{1}d\chi_1 d\chi_2\right)^{\frac{1}{p}}+\left(\int_{0}^{1}\int_{0}^{1}(1-\chi_1)^{\alpha p}(1-\chi_2)^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right.$$

$$+\left(\int_{0}^{1}\int_{0}^{1}\chi_1^{\alpha p}(1-\chi_2)^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}+\left(\int_{0}^{1}\int_{0}^{1}(1-\chi_1)^{\alpha p}\chi_2^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}$$

$$\left.+\left(\int_{0}^{1}\int_{0}^{1}\chi_1^{\alpha p}\chi_2^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right)$$

$$\left.\times\left(\int_{0}^{1}\int_{0}^{1}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right)$$

$$=\frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)$$

$$\left.\times\left(\int_{0}^{1}\int_{0}^{1}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right). \tag{3.12}$$

Now, using the convexity of $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|^q$, (3.12) gives

$$|\Lambda(k,l,u,v,\mathcal{S})|$$

$$\leq \frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}} \left(\left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\chi_1 \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \chi_1 (1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q \right.\right.\right.$$
$$\left.\left. + (1-\chi_1) \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + (1-\chi_1)(1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q \right) d\chi_1 d\chi_2 \right)^{\frac{1}{q}}$$
$$+ \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \left(\chi_1 \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \chi_1 (1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q \right.\right.$$
$$\left.\left. + (1-\chi_1) \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + (1-\chi_1)(1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q \right) d\chi_1 d\chi_2 \right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{1}{2}}^{1} \int_0^{\frac{1}{2}} \left(\chi_1 \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \chi_1 (1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q \right.\right.$$
$$\left.\left. + (1-\chi_1) \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + (1-\chi_1)(1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q \right) d\chi_1 d\chi_2 \right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \left(\chi_1 \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \chi_1 (1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q \right.\right.$$
$$\left.\left. + (1-\chi_1) \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + (1-\chi_1)(1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q \right) d\chi_1 d\chi_2 \right)^{\frac{1}{q}}$$
$$+ \left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}} (\beta p+1)^{\frac{1}{p}}} \right) \left(\int_0^1 \int_0^1 \left(\chi_1 \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \chi_1 (1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q \right.\right.$$
$$\left.\left.\left. + (1-\chi_1) \chi_2 \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + (1-\chi_1)(1-\chi_2) \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q \right) d\chi_1 d\chi_2 \right)^{\frac{1}{q}} \right)$$
$$= \frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}} \left(\left(\frac{\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + 9\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q}{64} \right)^{\frac{1}{q}} \right.$$
$$+ \left(\frac{3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q + 9\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q}{64} \right)^{\frac{1}{q}}$$
$$+ \left(\frac{3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + 9\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q}{64} \right)^{\frac{1}{q}}$$
$$+ \left(\frac{9\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q + 3\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q}{64} \right)^{\frac{1}{q}}$$
$$\left. + \left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}} (\beta p+1)^{\frac{1}{p}}} \right) \left(\frac{\left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,u) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(k,v) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,u) \right|^q + \left| \frac{\partial^2 \mathcal{S}}{\partial \chi_1 \partial \chi_2}(l,v) \right|^q}{4} \right)^{\frac{1}{q}} \right).$$

The proof is over. ■

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