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On β-γ-connectedness and $\beta_{(\gamma,\delta)}$-continuous functions

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Abstract. The purpose of this work is to present the idea of β-γ-separated sets, examine their characteristics in topological spaces and then define the notation for β-γ-connected and β-γ-disconnectedness. In addition, the study of topological qualities that involves for β-γ-connected spaces via β-γ-separated sets. An analysis is conducted on the properties of β-γ-connected spaces and how they behave under $\beta_{(\gamma,\delta)}$-continuous functions. We also provide the ideas of β-γ-components in a space X and β-γ-locally connected spaces.

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Contents

1. Introduction

One of the most significant, practical and basic notations in general topology and other high level mathematical discipline now a days is connectedness. The notation of connectedness is fruitful in computing, topology, algebraic topology and advanced calculus. Many researchers across the globe have investigated properties of connectedness ([2], [3], [4], [5], [6]) and obtained new and interesting results.

The idea of β-open set in topological spaces was first proposed by M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud in 1983. Their proof was that the set of all β-open sets in (X,τ) is finer topology on X then τ. The researchers worked on two related topologies that were tested on the same underlying structure to determine if they share the same topological properties. The basic properties of β-connectedness were obtained by Jafari and Noiri [7] in 2003. Several other forms of connectedness can be introduced and studied using it. Tahiliani [8] discussed and studied the characterisations of β-γ-open sets in topological spaces in 2011. This work presents and investigates an additional kind of connectivity that is defined on β-open sets in (X,τ) via operations. Their behavior under is $\beta_{(\gamma,\delta)}$-continuous,as well as their attributes are discussed in this study.

The procedures γ and δ are defined on the set of all β-open sets of topological spaces (X,τ) and (Y,σ) correspondingly during the conversion.For any subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ stands for the closure and interior of A, respectively, for any subset A of X.

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2. Preliminaries

Here we lay down the groundwork by defining key terms and showing key findings:

The condition that $A \subseteq \mathrm{Cl}(\mathrm{Int}(Cl(A)))$ merely indicates that subset A of topological space X is β-open [1]. A β-open sets counterpart is a β-closed set, and $\beta O(X)$ [1] is the collection of all β-open sets. $\beta\,\mathrm{Cl}(A)$ [2] is the symbol for intersection of all β-closed sets that include A, while $\beta\,\mathrm{Int}(A))$ [2] is the symbol for union of all β-open sets that contain A.

The condition $V \in V^{\gamma}$ satisfied for each $V \in \beta O(X)$ in an operation $\gamma : \beta O(X) \to P(X)$.The function $V^{id} = V$ for each set $V \in \beta O(X)$ is called the identity operation on $\beta O(X)$.

As γ and δ are always defined on the family of β-open sets in space, We always mean them as operations. From [8], we retrieve the following definitions and findings:

Definition 2.1. *(i): If there exists a β-open set U of X that contains x and $U^{\gamma} \subseteq A$, then for any point $x \in A$, a subset A of X is termed as of β-γ-open set. The β-γ-closed is counterpart of β-γ-open set. The set symbolized by $\beta O(X)\gamma$ includes all β-γ-open sets of (X, τ).*

(ii): $\beta_{\gamma}\,\mathrm{Cl}(A)$ notation represents β-γ-closure of A, which is the intersection of all β-γ-closed sets set containing A. The $\beta\gamma\,\mathrm{Int}(A)$ notation represents β-γ-interior of A, which is the union of all β-γ-open set included in A. The β-γ-boundary of a set A is represented by $\beta\gamma Bd(A)$ and is defined by $(\beta_{\gamma}\,\mathrm{Cl}(A) - \beta_{\gamma}\,\mathrm{Int}(A))$.

(iii): If, for every element x in X and each β-δ-open set V that contains $f(x)$, there exists a β-γ-open set U such that $x \in U$ and $f(U) \subseteq V$, then we say that $f : (X, \tau) \to (Y, \sigma)$ is $\beta_{(\gamma,\delta)}$-continuous.

(iv): For any β-γ-closed set A of (X, τ), the set $f(A)$ is β-δ-closed in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta_{(\gamma,\delta)}$-closed.

(v): For any β-γ-open set A of (X, τ), the set $f(A)$ is β-δ-open in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta_{(\gamma,\delta)}$-open.

Theorem 2.2. *Suppose X be a subset of a topological space and A is a subset of it. Then*

(i) $x \in \beta\gamma\,\mathrm{Cl}(A)$ *if and only if every β_{γ}-open set U containing x has non empty intersection with A.*

(ii) $\beta_{\gamma}\,\mathrm{Cl}(X - A) = X - \beta_{\gamma}\,\mathrm{Int}(A)$.

3. β-γ-connected spaces

Definition 3.1. *(i): If $(\beta\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta\,\mathrm{Cl}(B)) = \emptyset$, then the subsets A and B of a topological space (X, τ) are said to be β-separated.*

(ii): The term "β-γ-separated" is used to describe a pair of subsets A and B of a topological space (X, τ), where

$$(\beta_{\gamma}\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta_{\gamma}\,\mathrm{Cl}(B)) = \emptyset.$$

Remark 3.2. *Each two β-γ-separated sets are always disjoint, since $A \cap B \subseteq A \cap \beta_{\gamma}\,\mathrm{Cl}(B) = \emptyset$. The converse may not hold in general.*

Example 3.3. *The set $X = \{a, b, c\}$, and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ are defined as follows: $A^{\gamma} = A$ if $b \in A$, $A^{\gamma} = \mathrm{Cl}(A)$ if $b \notin A$, then $\{a, b\}$ and $\{c\}$ are disjoint subsets of X which are not β-γ-separated.*

Given that $\beta\,\mathrm{Cl}(A) \subseteq \beta_{\gamma}\,\mathrm{Cl}(A)$, for all subsets A of X, it follows that every β-γ-separated set is β-separated. The preceding example, however suggests that reverse may not be true. Both $\{a\}$ and $\{b, c\}$ are β-separated in this case,but they are not β-γ-separated.

Theorem 3.4. *The following claims hold if A and B are two non empty subsets of space X*

(1) *If A and B are β-γ-separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are also β-γ-separated.*

(2) *If A and B are disjoint and are both β-γ-closed (both β-γ-open), then A and B are β-γ-separated.*

(3) *If A and B are both β-γ-closed (both β-γ-open) then $H = A \cap (X - B)$ and $G = B \cap (X - A)$ are β-γ-separated.*

Proof. 1. Since $A_1 \subseteq A$ implies $\beta_\gamma \operatorname{Cl}(A_1) \subseteq \beta_\gamma \operatorname{Cl}(A)$ for every pair of A and A_1, $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta_\gamma \operatorname{Cl}(A_1) \cap B_1 = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B_1) \cap A_1 = \emptyset$. Hence A_1 and B_1 are β-γ-separated.

2. The equations $A = \beta_\gamma \operatorname{Cl}(A)$ and $B = \beta_\gamma \operatorname{Cl}(B)$ hold if A and B are both β-γ-closed. Hence because $A \cap B = \emptyset$, it follows that $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$, A and B are β-γ-separated. In other words, the complement of disjoint β-γ-open sets A and B are also β-γ-closed sets. Specifically X-A and X-B are β-γ-separated. If A and B are disjoint and are both, then their complements are disjoint and β-γ-closed. Furthermore, $A \subseteq \beta_\gamma \operatorname{Cl}(A) \subseteq \beta_\gamma \operatorname{Cl}(X - B) = X - B$ and $B \subseteq \beta_\gamma \operatorname{Cl}(B) \subseteq X - A$. Hence by given part (1), A and B are β-γ-separated.

3. Since A and B are β-γ-open, it follows that $X - A$ and $X - B$ are β-γ-closed. Also, $H \subseteq X - B$ means that $\beta_\gamma \operatorname{Cl}(H) \subseteq \beta_\gamma \operatorname{Cl}(X - B)$. Then because $\beta_\gamma \operatorname{Cl}(H) \cap B = \emptyset$ and it follows that $\beta_\gamma \operatorname{Cl}(H) \cap G = \emptyset$. Similarly if $H \cap \beta_\gamma \operatorname{Cl}(G) = \emptyset$. i.e. H and G are β-γ-separated. $(X - A)$ and $(X - B)$ are β-γ-open if and only if A and B are β-γ-closed. Consequently, H and G are β-γ-separated. ■

Theorem 3.5. *If there is a set U and set V in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then the subsets A and B of a space X are β-γ-separated and conversely.*

Proof. We have $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ as A and B are β-γ-separated sets. Therefore the sets $V = X - \beta_\gamma \operatorname{Cl}(A)$ and $U = X - \beta_\gamma \operatorname{Cl}(B)$ are β-γ-open, such that $A \subseteq U$, $B \subseteq V$ with $A \cap V = \emptyset$ and $B \cap U = \emptyset$. On the other hand, if U and V exists in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$, $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then $X - V$ and $X - U$ are β-γ-closed and $\beta_\gamma \operatorname{Cl}(A) \subseteq X - V \subseteq X - B$ and $\beta_\gamma \operatorname{Cl}(B) \subseteq X - U \subseteq X - A$ respectively. Hence $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ were determined. ■

Theorem 3.6. *In any topological space (X, τ), the following statements are equivalent:*

(1) *$\emptyset$ and X are the only sets which are both β-γ-open and β-γ-closed in X.*

(2) *X is not the union of two disjoint non empty β-γ-open sets.*

(3) *X is not the union of two disjoint non empty β-γ-closed sets.*

(4) *X is not the union of non empty β-γ-separated sets.*

Proof. (1)⇒(2): It is assumed that (2) is not true. Given that A and B are disjoint, non empty and are β-γ-open so let $X = A \cup B$. So $X - A = B$ is a nonempty set which is proper β-γ-open. It follows that (1) is not true, since A is non empty proper β-γ-open and β-γ-closed in X.

(2)⇒(3): Clear.

(3)⇒(4): If (4) is false, then $X = A \cup B$, where A and B are nonempty and β-γ-separated sets. Then $\beta\gamma \operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta_\gamma \operatorname{Cl}(B) \subseteq B$ and hence B is β-γ-closed. Similarly A is also β-γ-closed. i.e. (3) is false.

(4)⇒(1). Assuming that (1) is not true, assume that there is a non empty proper subset A of X, that is both β-γ-open and β-γ-closed. If A and B are β-γ-separated and $X = A \cup B$, then (4) is not true since. $B = X - A$ is non empty, β-γ-open and β-γ-closed. ■

Definition 3.7. *The condition that a subset C of a space X is β-γ-disconnected is that $C = A \cup B$, where A and B are non empty β-γ-separated or that C is β-γ-connected if there exists no non empty β-γ-separated sets A and B of X such that $C = A \cup B$.*
A pair of sets A and B is referred to as a β-γ-disconnection of C if C is β-γ-disconnected.

In Example 3.3, X is β-γ-disconnected, since $\{c\}$ and $\{a, b\}$ are β-γ-separated sets and hence there union is X.

Example 3.8. (i) *Assume X is a set comprising $\{a, b, c\}$ and $\tau = \{\theta, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let γ be an operation on $\beta O(X)$ such that $A^\gamma = A$ if $c \in A$ and $A^\gamma = \mathrm{Cl}(A)$ if $c \notin A$. Then X is β-γ-disconnected.*

(ii) *Assume X is a set comprising $\{a, b, c\}$ and $\tau = \{\theta, X, \{a\}, \{c\}, \{a, c\}\}$. Let γ be an operation on $\beta O(X)$ such that $A^\gamma = A$ if $b \in A$, $A^\gamma = X$, if $b \notin A$. So X is β-γ-connected, because there is no non empty pair A, B of non empty β-γ-separated subsets of X such that $X = A \cup B$.*

Remark 3.9. (1) *Every indiscrete space is β-γ-connected.*

(2) *Every discrete space with more than one point is β_{id}-disconnected.*

(3) *A space X is β-γ-connected if any of the conditions (1) to (4) in Theorem 3.6 holds.*

(4) *A space X is β-γ-disconnected if $X = A \cup B$, satisfies any one of the following statements:*

- (i) *A and B are disjoint, non-empty and β-γ-open sets.*
- (ii) *A and B are disjoint, non-empty and β-γ-closed sets.*
- (iii) *A and B are disjoint, non-empty and β-γ-separated sets.*

Theorem 3.10. *If there is non empty proper subset A of X which is both β-γ-open and β-γ-closed in X, then we say that space X is β-γ-disconnected.*

Proof. Follows from above remarks. ■

Theorem 3.11. *Every non empty proper subset of X must have a non-empty β-γ-boundary for a space X to be β-γ-connected.*

Proof. Let A be nonempty proper subset of X with $\beta_\gamma \,\mathrm{Bd}(A) = \emptyset$. Then $\beta_\gamma \,\mathrm{Cl}(A) = \beta_\gamma \,\mathrm{Int}(A) \cup \beta_\gamma \,\mathrm{Bd}(A)$ implies $\beta_\gamma \,\mathrm{Cl}(A) = \beta_\gamma \,\mathrm{Int}(A)$. Because A is both β-γ-open and β-γ-closed and $\beta_\gamma \,\mathrm{Int}(A) \subseteq A$ is nonempty proper subset of X, by Theorem 3.10, X is β-γ-disconnected, which is a contradictory. Due to this, A has a non-empty β-γ-boundary. On the other hand, let X be β-γ-disconnected. Next, by Theorem 3.10, X contain a valid subset A that is non empty proper subset and is both β-γ-open and β-γ-closed. i.e. $\beta_\gamma \,\mathrm{Cl}(A) = A$, $\beta_\gamma \,\mathrm{Cl}(X - A) = X - A$ and $\beta_\gamma \,\mathrm{Cl}(A) \cap \beta_\gamma \,\mathrm{Cl}(X - A) = \emptyset$. So A has empty β-γ-boundary, which is again a contradiction. Hence X is β-γ-connected. ■

Lemma 3.12. *Suppose M and N are β-γ-separated subsets of X. If $C \subseteq M \cup N$ and C is β-γ-connected, then $C \subseteq M$ or $C \subseteq N$.*

Proof. Since $C \cap M \subseteq M$ and $C \cap N \subseteq N$ then $C \cap M$ and $C \cap N$ are β-γ-separated sets. Also $C = C \cap (M \cup N) = (C \cap M) \cup (C \cap N)$. Since C is β-γ-connected, so $(C \cap M)$ and $(C \cap N)$ cannot form a β-γ-disconnection of C. Therefore, either $C \cap M = \emptyset$, so $C \subseteq N$ or $C \cap M = \emptyset$ so $C \subseteq M$. ■

Theorem 3.13. *Suppose C and C_i $(i \in I)$ are β-γ-connected but not β-γ-separated subsets of X, then $S = C \cup C_i$ is β-γ-connected for each i.*

Proof. Where M and N are β-γ-separated, then $C \cup C_i$ is equal to $S = M \cup N$ if S is β-γ-disconnected. Either $C \subseteq M$ or $C \subseteq N$ and $C_i \subseteq M$ or $C_i \subseteq N$ are required by Lemma 3.12. Assume $C \subseteq M$ without sacrificing generality. A contradiction would occur if for some i, $C_i \subseteq N$, and C and C_i would be β-γ-separated. Therefore every $C_i \subseteq M$. Therefore $N = \emptyset$. i.e. M and N are not β-γ-disconnected in S. ■

Corollary 3.14. *Assume that, C_i is β-γ-connected subset of X for every $i \in I$, and if C_i, share a point, then $C_i \cup \{C_i : i \in I\}$ is β-γ-connected.*

Proof. With $I = \emptyset$, the set $\cup C_i = \emptyset$ is obviously β-γ-connected for all i in I. In Theorem 3.13, choose $i_0 \in I$ and C_{i0} be the central set C. If I is not equal to $\emptyset$. It is not true that $C_i \cap C_{i0}$ equal to $\emptyset$ for every $i \in I$. So C_i and C_{i0} are not β-γ-separated. The β-γ-connectedness of $\cup\{C_i : i \in I\}$ is shown by Theorem 3.13. ■

Corollary 3.15. *Suppose that for all $x, y \in X$, there exists a β-γ-connected set $C_{xy} \subseteq X$ with $x, y \in C_{xy}$. Then X is β-γ-connected.*

Proof. Obviously $X = \emptyset$ is β-γ-connected. By hypothesis, there exists a β-γ-connected set C_{ay} that contains both a and y for any $y \in X$ where $X \neq \emptyset$, and let $a \in X$ be a fixed element. The β-γ-connection of $X = \cup\{C_{ay} : y \in X\}$ is established by Corollary 3.14. ■

Corollary 3.16. *Let C be a β-γ-connected subset of X and $A \subseteq X$. If $C \subseteq A \subseteq \beta_\gamma \operatorname{Cl}(C)$, then A is also β-γ-connected.*

Proof. If $a \in \beta_\gamma \operatorname{Cl}(C)$ is true for all $a \in A$, then $\{a\} \cap \beta_\gamma \operatorname{Cl}(C)$ is not equal to $\emptyset$. C and $\{a\}$ are not β-γ-separated. Thus, $A = C \cup \cup\{\{a\} : a \in A\}$ is β-γ-connected by Theorem 3.13. ■

Remark 3.17. *In particular, the β-γ-closure of a β-γ-connected set is β-γ-connected.*

Corollary 3.18. *If for every β-δ-open set V of Y, $f^{-1}(V)$ is β-γ-open in X, then function $f : X \to Y$ is $\beta_{(\gamma,\delta)}$-continuous.*

Proof. Assume that V be β-δ-open in Y. Then $Y - V$ is a set in Y that is β-δ-closed. Following the reasoning in ([8, Theorem 16(ii)]), the set $f^{-1}(Y - V)$ is β-γ-closed set in X. The reason for this is because $f^{-1}(V)$ is β-γ-open set in X, since $f^{-1}(Y - V) = X - f^{-1}(V)$.

On the other side, consider $x \in X$ and V as a β-δ-open subset of Y that contains $f(x)$. Then $x \in f^{-1}(V)$. Given x and $f(f^{-1}(V)) \subseteq V$. It may be inferred that $f^{-1}(V)$ is β-γ-open in X. Hence f is $\beta_{(\gamma,\delta)}$-continuous. ■

Theorem 3.19. *If $f : (X, \tau) \to (Y, \sigma)$ is onto $\beta_{(\gamma,\delta)}$-continuous function and X is β-γ-connected, then Y is β-δ-connected.*

Proof. Y is β-δ-disconnected if and only if A and B give a β-δ-disconnection of Y. A and B are both β-δ-open sets according to Remark 3.9. Both $f^{-1}(A)$ and $f^{-1}(B)$ are both non empty β-γ-open set in X because f is $\beta_{(\gamma,\delta)}$-continuous, according to Corollary 3.18. Now, for function f, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$. Remark 3.9 states that $f^{-1}(A)$ and $f^{-1}(B)$ are two β-γ-disconnections of X. Then Y is β-δ-disconnected is contradicted by this. ■

Theorem 3.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be an injective function. Then the following are equivalent:*

(i) *f is $\beta_{(\gamma,\delta)}$-continuous.*

(ii) *$f^{-1}(V) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(V))$ for every subset β-γ-open set V of Y.*

(iii) *$f(\beta_\gamma \operatorname{Cl}(A)) \subseteq \beta_\delta \operatorname{Cl}(f(A))$ for every subset A of X.*

(iv) *$\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(B))$ for every subset B of Y.*

(v) $f^{-1}(\beta_\delta \operatorname{Int}(B)) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(B))$ *for every subset B of Y.*

Proof. (i)⇒(ii): Let $x \in f^{-1}(V)$, where V is a β-δ-open subset of Y. Then $f(x) \in V$. Since f is $\beta_{(\gamma,\delta)}$-continuous, there exists β-γ-open set U of X containing x such that $f(U) \subseteq V$ and so $U \subseteq f^{-1}(V)$, this implies that $x \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$. Thus $f^{-1}(V) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(V))$ for every β-δ-open subset V of Y.

(ii)⇒(iii): Let A be any subset of X and $f(x) \notin \beta_\delta \operatorname{Cl}(f(A))$, then by Theorem 2.2(i), there exists a β-γ-open set V of Y containing $f(x)$ such that $V \cap f(A) = \emptyset$ and hence $f^{-1}(V) \cap A = \emptyset$. Also $f(x) \in V$ implies $x \in f^{-1}(V)$, which implies $x \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$. Hence, there exists a β-γ-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Then $U \cap A = \emptyset$ and so $x \notin \beta_\gamma \operatorname{Cl}(A)$ and hence $f(x) \notin (\beta_\gamma \operatorname{Cl}(A))$. Thus $f(\beta_\gamma \operatorname{Cl}(A)) \subseteq \beta_\delta \operatorname{Cl}(f(A))$.

(iii)⇒(iv): Let B be any subset of Y. Since $f(f^{-1}(B)) \subseteq B$, so we have $\beta_\delta \operatorname{Cl}(f(f^{-1}(B)) \subseteq \beta_\delta \operatorname{Cl}(B)$. Also $f^{-1}(B) \subseteq X$. Then by (iii), we have $f(\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq (\beta_\delta \operatorname{Cl} f(f^{-1}(B)) \subseteq \beta_\delta \operatorname{Cl}(B)$. Thus $\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(B))$.

(iv)⇒(v): Let B be any subset of Y and $x \in f^{-1}(\beta_\delta \operatorname{Int}(B))$. Then by Theorem 2.2(ii), $x \notin X - f^{-1}(\beta_\gamma \operatorname{Int}(B)) = f^{-1}(\beta_\gamma \operatorname{Cl}(Y - B)$. By (iv), $x \notin (\beta_\gamma \operatorname{Cl}(f^{-1}(Y - B)) = X - (\beta_\gamma \operatorname{Int}(f^{-1}(B))$ and hence $x \in \beta_\gamma \operatorname{Int} f^{-1}(B)$. Thus $f^{-1}(\beta_\delta \operatorname{Int}(B)) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(B))$.

(v)⇒(i): Let $x \in X$ and V be any β-δ-open set of Y containing $f(x)$. Since $V \cap (Y - V) = \emptyset$, we have $f(x) \notin \beta_\gamma \operatorname{Cl}(Y - V) = Y - \beta_\gamma \operatorname{Int}(V))$ and hence $f(x) \notin \beta_\gamma \operatorname{Cl}(Y - B) = Y - \beta_\gamma \operatorname{Int}(V)$ and so $f(x) \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$, which implies that $x \in f^{-1}(\beta_\delta \operatorname{Int}(v))$. By (v), we obtain that $x \in \beta_\gamma(\operatorname{Int} f^{-1}(V))$. This means that there exists a β-γ-open set U of X containing x such that $U \subseteq f^{-1}(V)$ and so $f(U) \subseteq V$. This shows that f is $\beta_{(\gamma,\delta)}$-continuous. ■

Corollary 3.21. *Let $f : X \to Y$ be a $\beta_{(\gamma,\delta)}$-continuous and injective function. If K is β-γ-connected in X, then $f(K)$ is β-δ-connected in Y.*

Proof. Suppose that $f(K)$ is β-δ-disconnected in Y. Then there exists two β-δ-separated sets P and Q of Y such that $f(K) = P \cup Q$. Let $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P$ is not empty, so is $K \cap f^{-1}(P)$. Hence A and B are non empty. Now $A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = K \cap (f^{-1}(P) \cup f^{-1}(Q)) = K \cap (f^{-1}(P \cup Q)) = K \cap (f^{-1}(f(K)) = K$. Since f is $\beta_{(\gamma,\delta)}$-continuous, then by Theorem 3.20, $\beta_\gamma \operatorname{Cl}(f^{-1}(Q)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(Q))$ and this together with $B \subseteq f^{-1}(Q)$, implies $\beta_\delta \operatorname{Cl}(B) \subseteq f^{-1}(\beta_\gamma \operatorname{Cl}(Q))$. Since $P \cap \beta_\gamma \operatorname{Cl}(Q) = \emptyset$, $A \cap \beta_\gamma \operatorname{Cl}(B) \subseteq A \cap f^{-1}(\beta_\gamma \operatorname{Cl}(Q)) \subseteq f^{-1}(P) \cap f^{-1}(\beta_\gamma \operatorname{Cl}(Q)) = \emptyset$. i.e. $A \cap \beta_\gamma \operatorname{Cl}(B) = \emptyset$. Similarly $B \cap \beta_\gamma \operatorname{Cl}(A) = \emptyset$. Thus A and B are β-γ-separated, therefore K is a β-γ-disconnected, a contradiction. Hence $f(K)$ is β-δ-connected. ■

Theorem 3.22. *A space X is β-γ-disconnected if and only if there exists an $\beta_{(\gamma,\mathrm{id})}$-continuous function from X onto discrete space $\{0, 1\}$.*

Proof. Suppose that X is β-γ-disconnected. Then, there exists disjoint β-γ-open sets G_1 and G_2 of X such that $X = G_1 \cup G_2$. Define a function $f : X \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in G_1, \\ 1 & \text{if } x \in G_2. \end{cases}$$

Now,the only β_{id}-open sets in $\{0, 1\}$ are $\emptyset, \{0\}, \{1\}, \{0, 1\}$. So, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$ and $f^{-1}(\{0, 1\}) = X$, which are β-γ-open sets in X. Thus by Corollary 3.18, f is $\beta_{(\gamma,\mathrm{id})}$-continuous function from X onto discrete space $\{0, 1\}$. Conversely, let the hypothesis holds and if possible suppose that X is β-γ-connected. Therefore by Theorem 3.19, $\{0, 1\}$ is β_{id}-connected which is a contradiction by Remark 3.9. So X must be β-γ-disconnected. ■

Theorem 3.23. *A space X is β-γ-connected if and only if every $\beta_{(\gamma,\mathrm{id})}$-continuous function from space X to the discrete space $\{0,1\}$ is constant.*

Proof. Consider X be β-γ-connected and consider any $\beta_{(\gamma,\mathrm{id})}$-continuous function $f : X \to \{0,1\}$. Since the space $\{0,1\}$ is discrete, we may say that $\{y\}$ is both β_{id}-open and β_{id}-closed in space $\{0,1\}$. If we let $y \in f(X) \subseteq \{0,1\}$, then $\{y\} \subseteq \{0,1\}$. For any y in Y, $f^{-1}(\{y\})$ is both β-γ-open and β-γ-closed in X according to Corollary 3.18 and ([8, Theorem 16(ii)]) since f is $\beta_{(\gamma,\mathrm{id})}$-continuous function. We may deduce that $f(x) = y$ for every $x \in X$ because $y \in f(X)$, so x is a function of $f^{-1}(\{y\})$. Therefore $f^{-1}(\{y\})$ does not include empty set. If $f^{-1}(\{y\})$ is not equal to X, then $f^{-1}(\{y\})$ is a non empty subset of X which is both β-γ-open and β-γ-closed in X. So there is a contradiction as, X is β-γ-connected. By Theorem 3.10. Therefore if $f^{-1}(\{y\}) = X$, then $f(X) = \{y\}$. This indicates that f is constant since for each $x \in X$, $f(x) = y$. ■

Definition 3.24. *A set C is called maximal β-γ-connected set if it is β-γ-connected and if D is β-γ-connected such that $C \subseteq D \subseteq X$, then $C = D$. A maximal β-γ-connected subset C of a space X is called a β-γ-component of X, if X itself β-γ-connected, then X is only β-γ-component of X.*

Theorem 3.25. *For β-γ-component of X containing x, for each $x \in X$, there is exactly one β-γ-component of X containing x.*

Proof. For any $x \in X$, let $C_x = \cup\{A : x \in A \subseteq X$ and A is β-γ-connected$\}$. Then $\{x\} \in C_x$, since C_x is union of β-γ-connected sets each containing x, is β-γ-connected by Corollary 3.14. If $C_x \subseteq D$ and D is β-γ-connected, then D was one of the sets A in the collection whose union defined C_x. So $D \subseteq C_x$ and therefore $C_x = D$. Therefore C_x is a β-γ-component of X containing x. ■

Corollary 3.26. *Two β-γ-components either are disjoint or coincide.*

Proof. Let C_x and C_y be two β-γ-components and C_x not equal to C_y. If they are not disjoint, let $p \in C_x \cap C_y$. Then by Corollary 3.14, $C_x \cup C_y$ would be a β-γ-connected set strictly larger then C_x. Therefore $C_x \cap C_y = \emptyset$. ■

Theorem 3.27. *Each β-γ-connected subset of X is contained in exactly one β-γ-component of X.*

Proof. Let A be a β-γ-connected subset of X which is not in exactly one β-γ-component of X. Suppose that C_1 and C_2 are β-γ-component of X such that, $A \subseteq C_1$ and $A \subseteq C_2$. Since C_1 and C_2 are not disjoint and by Corollary 3.14, $C_1 \cup C_2$ is another β-γ-connected subset which contain C_1 and C_2, a contradiction to the fact that C_1 and C_2, are β-γ-components. This proves that A is contained in exactly one β-γ-component of X. ■

Theorem 3.28. *A β-γ-component is a non empty β-γ-connected subset of X that is both β-γ-open and β-γ-closed.*

Proof. Assume that A be a β-γ-connected subset of X which is both β-γ-open and β-γ-closed. A is included in precisely one β-γ-component C of X, according to Theorem 3.27. It is contradictory because if A is proper subset of C, then equation $C = (C \cap A) \cup (C \cap (X - A))$ results in a β-γ-disconnection of C. Thus, $A = C$. ■

Theorem 3.29. *Every β-γ-component of X is β-γ-closed.*

Proof. Assume that C be a β-γ-component of X. according to Remark 3.17, $\beta_\gamma\,\mathrm{Cl}(C)$ is a β-γ-connected which appropriately includes the β-γ-component C of X. C is therefore β-γ-closed as $C = \beta_\gamma\,\mathrm{Cl}(C)$. ■

Definition 3.30. *For every point $x \in X$ and every β-γ-open set U containing x, there exists a β-γ-open β-γ-connected set V such that $x \in V \subseteq U$, we say that X is said to be β-γ-locally connected at x.*

Theorem 3.31. *Let $f : X \to Y$ be a $\beta_{(\gamma,\delta)}$-continuous, $\beta_{(\gamma,\delta)}$-open and bijective. If X is β-γ-locally connected, then Y is β-δ-locally connected.*

Proof. By $y \in Y$, find an element $x \in X$ such that y is equal to $f(x)$. Let U be a β-δ-open set of Y that contains y. According to Corollary 3.18, $f^{-1}(U)$ is β-γ-open in X containing x, because f is $\beta_{(\gamma,\delta)}$-continuous. There is a β-γ-open β-γ-connected set V that contains x such that $x \in V \subseteq f^{-1}(U)$ because X is β-γ-locally connected. This means that $f(x) \in f(V) \subseteq f(f^{-1}(U)) = U$ or $y \in f(V) \subseteq U$. The reason for $f(V)$ is also β-δ-open because f is $\beta_{(\gamma,\delta)}$-open. In addition according to Corollary 3.21, $f(V)$ is β-δ-connected. This establishes that Y is β-δ-locally connected. ■

4. Concluding Remarks and Acknowledgements

Our research in this study focused on β-γ-connected and β-γ-locally connected spaces and we also presented the concept of β-γ-separated sets. There is much scope of further work based on operational approach and variants of open sets. The authors would like to express their profound gratitude to the referees who helped us to enhance the paper quality and the findings.

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