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On generalized pseudo conformally symmetric manifolds

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Abstract. In this paper, a type of Riemannian manifold, namely generalized pseudo conformally symmetric manifold is studied. Several geometric properties of such spaces are studied. By imposing different restrictions on the conformal curvature tensor, we have obtained several properties. If the conformal curvature tensor is harmonic, then the form of the scalar curvature is obtained. Also, the relations among the 1-forms under various conditions are obtained.

AMS Subject Classifications: 53C20, 53C21, 53C44.

Keywords: Pseudo symmetry, Second Bianchi Identity, Conformal curvature tensor, Harmonic curvature tensor.

Contents

1. Introduction

The geometry of a space mainly depends on the curvature of the space. One of the most important geometric properties of a space is symmetry. Cartan began the study of local symmetry of Riemannian spaces and studied elaborately ([2], [3]). According to him, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$. During the last sixty years, the notion of locally symmetric manifolds has been generalized by many authors in a weaker sense. They have weakened in different directions with several defining conditions by giving some curvature restrictions. Various weaker symmetries are studied as generalizations or extensions of Cartan's notion, such as recurrent manifolds by Walker [18], semi-symmetric manifolds by Szabó ([17]), pseudosymmetric manifolds in the sense of Deszcz [7], pseudosymmetric manifolds in the sense of Chaki [4], generalized pseudosymmetric manifolds by Chaki [6], weakly symmetric manifolds by Selberg [11] and weakly symmetric manifolds by Támassy and Binh [16].
According to Chaki, a Riemannian manifold is said to be pseudo symmetric if

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= 2\alpha(X)R(Y,Z,U,V)\\ &+ \alpha(Y)R(X,Z,U,V) + \alpha(Z)R(Y,X,U,V)\\ &+ \alpha(U)R(Y,Z,X,V) + \alpha(V)R(Y,Z,U,X)\end{aligned} \tag{1.1}$$

where and α is a 1-form, $X,Y,Z,U,V \in \chi(M)$.

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Pseudo symmetric manifolds are studied by many authors ([5] , [6], [12]). Ishii [8], introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. $\overline{C}$, the conharmonic curvature tensor of type (0,4) on an (M^n, g) is defined as follows

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{1.2}$$

which remains invariant under conharmonic transformation where R and S are the Riemannian curvature and Ricci curvature tensor respectively. $g \wedge S$ is the Kulkarni-Nomizu product [14].

In [13], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. They also studied it ellaborately [15]. The conharmonic curvature tensor has many applications in the theory of general relativity.
The conformal curvature tensor of type(0,4) is defined by

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}r_{il} - g_{ik}r_{jl} + g_{il}r_{jk} - g_{jl}r_{ik}) + \frac{s}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) \tag{1.3}$$

It should be noted that the conformal curvature tensor C_{ijkl} remains invariant under conformal transformation.

In 2021, Ali, Khan and Vasiulla [1] introduced generalized pseudo symmetric manifold and studied various properties. Also in 2017, Kim introduced pseudo semiconformally symmetric manifolds [10] and studied various properties. According to him, a Riemannian manifold (M^n, g) is said to be pseudo semiconformally symmetric if

$$P_{ijkl;m} = 2A_m P_{ijkl} + A_i P_{mjkl} + A_j P_{imkl} + A_k P_{ijml} + A_l P_{ijkm}, \tag{1.4}$$

where P is the semiconformal curvature tensor [9]and A is a non zero 1-form.
Motivating by the above studies in this paper, I would like to introduce generalized pseudo conformally symmetric manifold, which is defined by

$$\begin{aligned}(\nabla_X C)(Y, Z, U, V) &= 2\alpha(X)C(Y, Z, U, V) \\ &+ \beta(Y)C(X, Z, U, V) + \gamma(Z)C(Y, X, U, V) \\ &+ \delta(U)C(Y, Z, X, V) + \eta(V)C(Y, Z, U, X)\end{aligned} \tag{1.5}$$

where and α, β, γ, δ, η are 1-forms.
In terms of local coordinates

$$C_{ijkl;m} = 2\alpha_m C_{ijkl} + \beta_i C_{mjkl} + \gamma_j C_{imkl} + \delta_k C_{ijml} + \eta_l C_{ijkm} \tag{1.6}$$

2. Generalized pseudo conformally symmetric manifolds

Definition 2.1. *The conformal curvature tensor is said to be harmonic if the divergence of the curvature tensor C^i_{jkl} of type (1,3) vanishes, i.e.,*

$$C^h_{jkl;h} = 0. \tag{2.1}$$

By virtue of second Bianchhi Identity we have

$$R^h_{jkl;h} = r_{jk;l} - r_{jl;k}. \tag{2.2}$$

And then

$$r^k_{l;k} = \frac{1}{2} s; l. \tag{2.3}$$

We have

$$C^h_{jkl;h} = [(r_{jk;l} - r_{jl;k}) - \frac{1}{2(n-2)}(g_{jk}s_{;l} - g_{jl}s_{;k})]. \tag{2.4}$$

Multiplying (2.4) by g^{jk} and using the condition (**??**) we get

$$0 = \frac{s_{;l}}{n-2}. \tag{2.5}$$

Which on siplification gives $s_{;l} = 0$, that is the scalar curvature is constant. Hence we have the following:

Theorem 2.2. *If the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic, then the scalar curvature of the space is constant.*

Let the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic. Then we have

$$0 = 2\alpha_m C^m_{jkl} + \beta^m C_{mjkl} + \gamma_j C^m_{mkl} + \delta_k C^m_{mjl} + \eta_l C^m_{jkm}. \tag{2.6}$$

Multiplying the above equation by g^{jk} we have

$$\frac{s}{n-2}[2\alpha_l + \beta_l - \delta_l + n\eta_l] = 0. \tag{2.7}$$

If the scalar curvature does not vanishes, then we have

$$2\alpha_l + \beta_l - \delta_l + n\eta_l = 0. \tag{2.8}$$

Thus we can state the following:

Theorem 2.3. *Let the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic. If $2\alpha + \beta - \delta + n\eta \neq 0$, then the scalar curvature of the space vanishes.*

If the space is pseudo confrmally symmetric Riemannian manifold then $\alpha = \beta = \gamma = \delta = \eta = A$ then we have:

Corollary 2.4. *If the confrmal curvature tensor of a pseudo confrmally symmetric Riemannian manifold is harmonic, then the scalar curvature of the space vanishes.*

Definition 2.5. *A Riemannian manifold (M^n, g) is said to be recurrent if its curvature tensor R_{ijkl} of type (0,4) satisfies the condition*

$$R_{ijkl;m} = B_m R_{ijkl} \tag{2.9}$$

where the 1-form B is non zero.

Multiplying the equation (2.9) by g^{il} and then multiplying by g^{jk} we get

$$r_{jk;m} = B_m r_{jk} \tag{2.10}$$

and then

$$s_{;m} = B_m s \tag{2.11}$$

Using (2.10), (2.11) and (2.1) we get

$$g^{il}g^{jk}C_{ijkl;m} = -\frac{n}{n-2}B_m s. \tag{2.12}$$

From (1.6)

$$g^{il}g^{jk}C_{ijkl;m} = -\frac{s}{n-2}[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.13}$$

If $[a + (n-2)b] \neq 0$, then from the above two equations we get

$$B_m = \frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n} \tag{2.14}$$

Thus we can state that:

Theorem 2.6. *If a generalized pseudo conformally symmetric Riemannian manifold is recurrent, then the 1-forms B, α, β, γ, δ, η satisfy the relation $B = \frac{2n\alpha+\beta+\gamma+\delta+\eta}{n}$.*

If a generalized pseudo conformally symmetric Riemannian manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that :

Corollary 2.7. *If a pseudo conformally symmetric Riemannian manifold is recurrent, then the 1-forms B and α are related by $B = \frac{2(n+2)}{n}\alpha$.*

From the Ricci identity and a parallel vector field V, it follows that

$$0 = V^t_{;jk} - V^t_{;kj} = V^m R^t_{mjk}. \tag{2.15}$$

Taking covariant derevative of the above equation we get

$$V^m R^t_{mjk;l} = 0. \tag{2.16}$$

Multiplying by g_{ti} we get

$$V^m R_{imjk;l} = 0 \tag{2.17}$$

Using second Bianchi identity we obtain

$$V^m R_{jkli;m} = 0. \tag{2.18}$$

Multiplying by g^{ji} and then multiplying by g^{kl} we have

$$V^m r_{kli;m} = 0 \tag{2.19}$$

$$V^m s_{;m} = 0. \tag{2.20}$$

Using the above equations it follows that

$$V^m C_{ijkl;m} = 0. \tag{2.21}$$

Or,

$$[2\alpha_m C_{ijkl} + \beta_i C_{mjkl} + \gamma_j C_{imkl} + \delta_k C_{ijml} + \eta_l C_{ijkm}]V^m = 0. \tag{2.22}$$

Or,

$$[\frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n}]V^m = 0. \tag{2.23}$$

Which leads the following:

Theorem 2.8. *If a generalized pseudo conformally symmetric manifold (M^n, g) admits a parallel vector field V, then either s=0 or* $[\frac{2n\alpha_m+\beta_m+\gamma_m+\delta_m+\eta_m}{n}]V^m = 0$.

Let a generalized pseudo conformally symmetric manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that:

Corollary 2.9. *If a pseudo conformally symmetric manifold (M^n, g) admits a parallel vector field V and $[a + (n-2)b] \neq 0$, then either s=0 or $\alpha_m V^m = 0$.*

and then Multiplying (1.6) by g^{il} and then multiplying the relation thus obtained by g^{jk}, we obtain

$$-(\frac{s_{;m}}{n-2})n = -(\frac{s}{n-2})[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.24}$$

Since $[a + (n-2)] \neq 0$, we have

$$s_{;m} = \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]}{n}s. \tag{2.25}$$

Taking covariant derivative of (2.25), we get

$$\begin{aligned} s_{;mt} = &\frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n}s \\ &+ \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)s_{;t}]}{n}. \end{aligned} \tag{2.26}$$

Or,

$$\begin{aligned} s_{;mt} = &\frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n}s \\ &+ \frac{[(2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)(2n\alpha_t + \beta_t + \gamma_t + \delta_t + \eta_t)]s}{n}. \end{aligned} \tag{2.27}$$

Therefore from the above relation we can write

$$\begin{aligned} 0 = s_{;mt} - s_{;tm} = &\frac{s}{n}[2n(\alpha_{m;t} - \alpha_{t;m}) + (\beta_{m;t} - \beta_{t;m}) \\ &+ (\gamma_{m;t} - \gamma_{t;m}) + (\delta_{m;t} - \delta_{t;m}) + (\eta_{m;t} - \eta_{t;m})]. \end{aligned} \tag{2.28}$$

Thus we can state that:

Theorem 2.10. *Let the scalar curvature of a generalized pseudo conformally symmetric manifold does not vanish. Then if four 1-forms are closed then all the 1-forms are closed.*

If the manifold pseudo conformally symmetric manifold then, $\alpha = \beta = \gamma = \delta = \eta = A$. Then we have from (2.28)

$$0 = s_{;mt} - s_{;tm} = \frac{2n+4}{n}s[A_{m;t} - A_{t;m}]. \tag{2.29}$$

If $s \neq 0$ then

$$A_{m;t} - A_{t;m} = 0. \tag{2.30}$$

Thus we have the following

Corollary 2.11. *If the scalar curvature of a generalized pseudo conformally symmetric manifold does not vanish, then the 1-form A is closed.*

3. Conclusions

I have studied a new space namely generalized pseudo conformally symmetric manifold. Some geometric properties of such spaces are obtained. we have studied the harmonic nature of conformal curvature tensor. In future, different properties of these spaces can be obtained by imposing differnt restriction on the Ricci tensor of such spaces.

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References

[1] M. Ali, Q. Khan, M. Vasiulla, On generalized pseudo symmetric manifolds,*Bull. Cal. Math. Soc.*, **113(4)** (2021), 353-362.

[2] É. Cartan, Sur une classe remarquable d'espaces de Riemann, I,*Bull. de la Soc. Math. de France*, **54**, (1926), 214–216.

[3] É. Cartan, Sur une classe remarquable d'espaces de Riemann, II, *Bull. de la Soc. Math. de France*, **55**, (1927), 114–134.

[4] M. C. Chaki , On pseudosymmetric manifolds,*An. Sti. Ale Univ., "AL. I. CUZA" Din Ias*, **33** (1987), 53–58.

[5] M. C. Chaki , On pseudosymmetric Ricci symmetric manifolds, *Bulg. J. Phys.*, **15** (1988), 526–531.

[6] M. C. Chaki , On generalized pseudo-symmetric manifolds, *Publ. Math. Debrecen*, **45** (1994), 305–312.

[7] R. Deszcz, On pseudosymmetric spaces,*Bull. Soc. Math. Belg. Sér. A*, **44** No. 1, (1992), 1–34.

[8] Y. Ishii, On conharmonic transformations, *Tensor, N. S.*, **11**, (1957), 73–80.

[9] J. Kim, A type of conformal curvature tensor,*Far East J. Math. Sci.*, **99** No. 1, (2016), 61-74.

[10] J. Kim, On pseudo semi-conformal symmetric manifolds,*Bull. Korean. Math. Soc.*, **54** ,(2017), 177-186.

[11] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.*, **20** ,(1956), 47–87.

[12] A. A. Shaikh, R. Deszcz, M. Hotloś, J. Jelowicki, H. Kundu, On pseudosymmetric manifolds,*Publ. Math. Debrecen*, **86(3-4)**, (2015), 433–456.

[13] A. A. Shaikh, S. K. Hui, On weakly conharmonically symmetric manifolds,*Tensor, N. S.*, **70**, No. 2, (2008), 119–134.

[14] A. A. Shaikh, H. Kundu, On equivalency of various geometric structures, *J. Geom.*, **105**, (2014), 139–165.

[15] A. A. Shaikh, I. Roy, S. K. Hui, On totally umbilical hypersurfaces of weakly conharmonically symmetric spaces, *Global J. Science Frontier Research*, **10(4)**, (2010), 28–31.

[16] L. Tamássy, T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds,*Coll. Math. Soc., J. Bolyai*, **56**, (1989), 663–670.

[17] Z. I. SZABÓ, Stucture theorems on Riemannian spaces satisfying $R(X,Y).R=0$, I, The local version, *J. Diff. Geom.*, **17**, (1982), 531–582.

[18] A. G. WALKER, On Ruse's space of recurrent curvature ,*Proc. of London Math. Soc.*, **52** , (1950), 36-54.