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On some fractional order differential equations with weighted conditions

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Abstract. In this paper, we study some Cauchy problems with weighted conditions of a fractional order differential equation . We study by using some fixed point Theorems the existence of at least one solution in the two spaces $C_{1-\kappa}(I)$ and $C(I)$, where $I = [0, \hbar]$.

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Contents

1. Introduction and Background

A Cauchy problem in mathematics asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain. A Cauchy problem can be an initial value problem or a boundary value problem. Also, Cauchy problems are very natural in physics: The typical example is the solution of Newton's equation in classical mechanics, which is a second-order equation for the position of a particle. We know indeed that the motion of a particle is uniquely specified by its initial position and velocity.

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An important result about Cauchy problems for ordinary differential equations is the existence and uniqueness theorem, which states that, under mild assumptions, a Cauchy problem always admits a unique solution in a neighbourhood of the point x_0 where the initial conditions are given.

In this work, we study the two weighted Cauchy-type problems

$$\begin{cases} D^{\kappa}\, u(\varsigma) \;=\; f\Big(\varsigma, u, \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\Big), \;\; \varsigma \;>\; 0, \;\; 0 \;<\; \kappa \;<\; 1, \\ \\ \varsigma^{1-\kappa}\, u(\varsigma)|_{\varsigma=0} \;=\; b, \; b \;\in\; \Re \end{cases} \tag{1.1}$$

and

$$\begin{cases} D^{\kappa}\, u(\varsigma) \;=\; f\Big(\varsigma, u, \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\Big), \;\; \varsigma \;>\; 0, \;\; 0 \;<\; \kappa \;<\; 1, \\ \\ u(0) \;=\; 0. \end{cases} \tag{1.2}$$

The weighted Cauchy-type problems were studied in many papers see [1]-[7].

In [8], the author studied the existence of a solution of the weighted problem

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f(t, u(t)) \;+\; \int_0^t \, g(t, s, u(s))\, ds, \;\; t \;>\; 0, \\ \\ t^{1-\alpha}\, u(t)|_{t=0} \;=\; b, \quad \text{where } 0 \;<\; \alpha \;<\; 1, \; b \;\in\; \Re, \end{cases} \tag{1.3}$$

in the space $C_{1-\alpha}(I)$, where the functions f and g satisfied the following conditions

(1) $t^{1-\alpha} f(t, u)$ is continuous on $R^+ \times C^0_{1-\alpha}(\Re^+)$ and

$$|f(t, u)| \le t^{\mu} \varphi(t) |u|^{m_1}, \; \mu \ge 0, \; m_1 > 1,$$

(2) $s^{1-\alpha} g(t, s, u(s))$ is continuous on $D_{\Re^+} \times C^0_{1-\alpha}(\Re^+)$ where

$$D_{\Re^+} = \{(t, s) \in \Re^+ \times \Re^+, \; 0 \le s \le t\}$$

and

$$|g(t, s, u(s))| \le (t-s)^{\beta-1} s^{\sigma} \psi(s) |u|^{m_2}, \; 0 < \beta < 1, \; \sigma \ge 0, \; m_2 > 1,$$

where $\varphi(t)$ and $\psi(t)$ are such that

(3) $\varphi(t)$ is continuous and $t^{\mu-(1-\alpha)m_1}\varphi(t)$ is continuous in case

$$\mu - (1-\alpha) m_1 < 0,$$

(4) $\psi(t)$ is continuous and $t^{\sigma-(1-\alpha)m_2}\psi(t)$ is continuous in case

$$\sigma - (1-\alpha) m_2 < 0.$$

Problem (1.3) is a special case of our problem (1.1), we will study the existence of at least one solution of problem (1.1) in the space $C_{1-\kappa}(I)$ under similar conditions of paper [8].

2. Preliminaries and Definitions

In this section, we state the definitions and theorems which will be used in our paper.

Let $L_1 = L_1[J]$ be the class of Lebesgue integrable functions on the interval J, $J = [0, \infty)$, with norm defined by

$$||f|| = \int_J | f(\varsigma) | \, d\varsigma, \ f \in L_1,$$

then we have the following definition for the fractional (arbitrary) order integration.

Definition 2.1. *The fractional (arbitrary) order integral of the function $f \in L_1[a, b]$ of order $\beta > 0$ is defined by (see [9] - [11])*

$$I_a^\beta f(\varsigma) = \int_a^\varsigma \frac{(\varsigma - s)^{\beta - 1}}{\Gamma(\beta)} f(s) \, ds,$$

or

$$I_a^\beta f(\varsigma) = \int_0^{\varsigma - a} \frac{u^{\beta - 1}}{\Gamma(\beta)} f(\varsigma - u) \, du.$$

When $a = 0$, we can write $I^\beta f(\varsigma) = I_0^\beta f(\varsigma) = f(\varsigma) \star \phi_\beta(\varsigma)$, where

$$\phi_\beta(\varsigma) := \begin{cases} \frac{\varsigma^{\beta - 1}}{\Gamma(\beta)}, & \varsigma > 0, \\ 0, & \varsigma \leq 0, \end{cases}$$

and ϕ satisfies the property

$$\phi_{\beta_1}(\varsigma) \star \phi_{\beta_2}(\varsigma) = \phi_{\beta_1 + \beta_2}(\varsigma).$$

Also $\phi_\beta(\varsigma) \to \delta(\varsigma)$ as $\beta \to 0$, where $\delta(\varsigma)$ is the Dirac-delta function (see [5]).

For $\kappa, \beta \in R^+$, we have

(a) $I_a^\kappa : L_1 \to L_1$,

(b) $I^\kappa I^\beta f(t) = I^{\kappa+\beta} f(t)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\beta \in (0, 1)$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by (see [9] - [11])*

$$D_a^\beta f(\varsigma) = \frac{d}{d\varsigma} I_a^{1 - \beta} f(\varsigma) = \frac{1}{\Gamma(1 - \beta)} \frac{d}{d\varsigma} \int_a^\varsigma (\varsigma - s)^{-\beta} f(s) \, ds.$$

Theorem 2.3. ***(Schauder fixed point Theorem)****[12]*

Let W be a convex subset of a Banach space X, and $T : W \to W$ is compact, continuous map. Then T has at least one fixed point in W.

3. Main Results

Define the two spaces

$$C(I) := \{u : u(\varsigma) \text{ is continuous on } I = [0, \hbar], ||u|| = \max_{\varsigma \in I} |u(\varsigma)|\}$$

and

$$C_{1-\kappa}(I) = \{u : \varsigma^{1-\kappa}u(\varsigma) \text{ is continuous on } I \text{ with the weighted norm } ||u||_{1-\kappa} = ||\varsigma^{1-\kappa}u||\}.$$

Our paper will be divided into two parts, in the first part we will study the existence of a solution for problem (1.1) in the space $C_{1-\kappa}(I)$. And in the second part we will study the existence of a solution for problem (1.2) in the space $C(I)$.

3.1. Solution in $C_{1-\kappa}(I)$

Suppose that the two functions f and h satisfy the following conditions

(1*) for each $\varsigma \in I,\ f(\varsigma,\cdot,\cdot)$ is continuous,
for each $(u,v) \in \Re \times \Re,\ f(\cdot,u,v)$ is measurable, and

$$|f(\varsigma,u,v)| \leq \varsigma^{\mu}\varphi(\varsigma)|u|^{m_1} + |v|, \mu \geq 0, m_1 > 1,$$

(2*) for each $(\varsigma,s) \in I \times I,\ h(\varsigma,s,\cdot)$ is continuous,
for each $u \in \Re,\ h(\cdot,\cdot,u)$ is measurable, and

$$|h(\varsigma,s,u(s))| \leq (\varsigma - s)^{\beta-1}s^{\sigma}\psi(s)|u|^{m_2}, 0 < \beta < 1, \sigma \geq 0, m_2 > 1,$$

where $\varphi(\varsigma)$ and $\psi(\varsigma)$ are continuous functions.

3.1.1. Integral representation

In ([1]-[2]) the authors proved that the Cauchy problem (1.1) is equivalent to the nonlinear integral equation of fractional order

$$u(\varsigma) = b\,\varsigma^{\kappa-1} + \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))\,d\theta\right) ds. \tag{3.1}$$

Define the operator T by

$$Tu(\varsigma) = b\,\varsigma^{\kappa-1} + \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds.$$

It is clear that the fixed point of the operator T is the solution of the integral equation (3.1).

3.1.2. Existence of solution

Theorem 3.1. *Assume that assumptions (1*)-(2*) and (3)-(4) are satisfied, then the weighted Cauchy-type problems (1.1) has at least one solution $u \in C_{1-\kappa}(I)$.*

Proof. Define the set

$$S_r = \left\{u \in C_{1-\kappa}(I) : ||u - b\,\varsigma^{\kappa-1}||_{1-\kappa} \leq r\right\}.$$

Now,

$$||Tu - b\,\varsigma^{\kappa-1}||_{1-\kappa} = \max_{\varsigma \in I}\left|\varsigma^{1-\kappa}I^{\kappa}f\left(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma,s,u(s))\,ds\right)\right|$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa} I^{\kappa}\left|f\left(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma} |h(\varsigma, s, u(s))|\, ds\right)$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma}(\varsigma-s)^{\beta-1}s^{\sigma}\psi(s)|u(s)|^{m_2}\, ds\right)$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\varphi(s)|u(s)|^{m_1}s^{(1-\kappa)m_1}s^{-(1-\kappa)m_1}ds$$

$$+\max_{\varsigma\in I}\varsigma^{1-\kappa}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\, d\theta\, ds$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa}||\varphi||\;||u||_{1-\kappa}^{m_1}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu-(1-\kappa)m_1}ds$$

$$+\max_{\varsigma\in I}\varsigma^{1-\kappa}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\theta^{(1-\kappa)m_2}\theta^{-(1-\kappa)m_2}\, d\theta\, ds$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa}||\varphi||\;||u||_{1-\kappa}^{m_1}\tfrac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)(m_1+1)+2)}\varsigma^{\mu-(1-\kappa)(m_1+1)+1}$$

$$+\max_{\varsigma\in I}\varsigma^{1-\kappa}||\psi||\;||u||_{1-\kappa}^{m_2}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma-(1-\kappa)m_2}\, d\theta\, ds$$

$$\leq ||\varphi||\;||u||_{1-\kappa}^{m_1}\tfrac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+\max_{\varsigma\in I}\varsigma^{1-\kappa}||\psi||\;||u||_{1-\kappa}^{m_2}\int_0^{\varsigma}\tfrac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\tfrac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}s^{\sigma-(1-\kappa)m_2+\beta}\, ds$$

$$\leq ||\varphi||\;||u||_{1-\kappa}^{m_1}\tfrac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+\max_{\varsigma\in I}\varsigma^{1-\kappa}||\psi||||u||_{1-\kappa}^{m_2}\tfrac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}\tfrac{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\varsigma^{\sigma-(1-\kappa)m_2+\kappa+\beta}$$

$$\leq ||\varphi||\;||u||_{1-\kappa}^{m_1}\tfrac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+||\psi||\;||u||_{1-\kappa}^{m_2}\tfrac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\hbar^{\sigma-(1-\kappa)m_2+\beta+1}.$$

If $u \in S_r$, then

$$||Tu-b\varsigma^{\kappa-1}||_{1-\kappa} \leq \tfrac{\Gamma(\mu-(1-\kappa)m_1+1)||\varphi||\;(r+|b|)^{m_1}}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+\tfrac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)||\psi||\;(r+|b|)^{m_2}}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\hbar^{\sigma-(1-\kappa)m_2+\beta+1}$$

$$\leq C_1\,(r+|b|)^{m_1}\,\hbar^{\gamma}\;+\;C_2\,(r+|b|)^{m_2}\,\hbar^{\delta},$$

where

$$C_1 \;=\; \frac{\Gamma(\mu-(1-\kappa)m_1+1)||\varphi||}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)},$$

$$C_2 \;=\; \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)||\psi||}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)},$$

$$\gamma \;=\; \mu-(1-\kappa)m_1+1 \;>\; 0$$

and

$$\delta \;=\; \sigma-(1-\kappa)m_2+\beta+1 \;>\; 0.$$

If we take $r=|b|$ and $\hbar$ very small, then

$$||Tu-b\varsigma^{\kappa-1}||_{1-\kappa} \;\leq\; r,$$

then $T(S_r)\subset S_r$.

Now, we prove that T is continuous on S_r. Indeed: let $u_1,u_2\in S_r$, then we get

$$||Tu_1-Tu_2||_{1-\kappa}=\max\nolimits_{\varsigma\in I}\left|\varsigma^{1-\kappa}I^{\kappa}\left[f\left(\varsigma,u_1(\varsigma),\textstyle\int_0^{\varsigma}h(\varsigma,s,u_1(s))ds\right)-f\left(\varsigma,u_2(\varsigma),\textstyle\int_0^{\varsigma}h(\varsigma,s,u_2(s))\,ds\right)\right]\right|$$

$$\leq\max\nolimits_{\varsigma\in I}\varsigma^{1-\kappa}\textstyle\int_0^{\varsigma}\frac{(\varsigma\,-\,s)^{\kappa-1}}{\Gamma(\kappa)}\left|f\left(s,u_1(s),\textstyle\int_0^{s}h(s,\theta,u_1(\theta))d\theta\right)-f\left(s,u_2(s),\textstyle\int_0^{s}h(s,\theta,u_2(\theta))\,d\theta\right)\right|ds.$$

From the continuity of f and h, we can deduce that for a given $\epsilon>0$ there exists a $\delta_1>0$ such that for all $(s,u_1,v_1),(s,u_2,v_2)\in I\times C_{1-\kappa}(I)\times C_{1-\kappa}(I)$, we have

$$s^{1-\kappa}\left|f\left(s,u_1(s),\int_0^{s}h(s,\theta,u_1(\theta))d\theta\right)-f\left(s,u_2(s),\int_0^{s}h(s,\theta,u_2(\theta))\,d\theta\right)\right|<\epsilon$$

provieded that $||u_1-u_2||_{1-\kappa}<\delta_1$.

To prove that $T(S_r)$ is equicontinuous, let $\tau_1, \tau_2 \in I, \tau_1 < \tau_2, |\tau_2 - \tau_1| < \delta$, then

$$\tau_2^{1-\kappa} Tu(\tau_2) - \tau_1^{1-\kappa} Tu(\tau_1) = {\tau_2}^{1-\kappa} \int_0^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$-{\tau_1}^{1-\kappa} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$= {\tau_2}^{1-\kappa} \int_0^{\tau_1} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$+{\tau_2}^{1-\kappa} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$-{\tau_1}^{1-\kappa} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$\leq (\tau_2^{1-\kappa} - \tau_1^{1-\kappa}) \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} f(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta) ds$$

$$+{\tau_2}^{1-\kappa} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds,$$

$$|\tau_2^{1-\kappa} Tu(\tau_2) - \tau_1^{1-\kappa} T_1 u(\tau_1)| \leq (\tau_2^{1-\kappa} - \tau_1^{1-\kappa}) \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} |f(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta)| ds$$

$$+{\tau_2}^{1-\kappa} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} \left| f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) \right| ds$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right) \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} \left(s^{\mu} \varphi(s) |u|^{m_1} + \int_0^s |h(s,\theta,u(\theta))| d\theta\right) ds$$

$$+{\tau_2}^{1-\kappa} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} \left(s^{\mu} \varphi(s) |u(s)|^{m_1} + \int_0^s |h(s,\theta,u(\theta))| d\theta\right) ds$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right) \left[\int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1} \varphi(s) s^{(1-\kappa)m_1} |u(s)|^{m_1} ds \right.$$

$$\left. + \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^s (s-\theta)^{\beta-1} \theta^{\sigma-(1-\kappa)m_2} \psi(\theta) \theta^{(1-\kappa)m_2} |u(\theta)|^{m_2} d\theta \, ds \right]$$

$$+{\tau_2}^{1-\kappa} \left[\int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1} \varphi(s) s^{(1-\kappa)m_1} |u(s)|^{m_1} ds \right.$$

$$\left. + \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^s (s-\theta)^{\beta-1} \theta^{\sigma-(1-\kappa)m_2} \psi(\theta) \theta^{(1-\kappa)m_2} |u(\theta)|^{m_2} d\theta \, ds \right]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right) \left[||\varphi|| \, ||u||_{1-\kappa}^{m_1} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1} \, ds \right.$$

$$\left. + ||\psi|| \, ||u||_{1-\kappa}^{m_2} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^s (s-\theta)^{\beta-1} \theta^{\sigma-(1-\kappa)m_2} d\theta \, ds \right]$$

$$+{\tau_2}^{1-\kappa} \left[||\varphi|| \, ||u||_{1-\kappa}^{m_1} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1} \, ds \right.$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \int_0^{\tau_1} \frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\sigma-(1-\kappa)m_2+\beta} ds\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\sigma-(1-\kappa)m_2+\beta} ds\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \frac{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} {\tau_1}^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[||\varphi||||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} (\tau_2-\tau_1)^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} {\tau_1}^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} (\tau_2-\tau_1)^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[C_1\ ||u||_{1-\kappa}^{m_1}\ \tau_1^{\gamma+\kappa-1} + C_2\ ||u||_{1-\kappa}^{m_2} {\tau_1}^{\delta+\kappa-1}\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[C_1\ ||u||_{1-\kappa}^{m_1}\ (\tau_2-\tau_1)^{\gamma+\kappa-1} + C_2\ ||u||_{1-\kappa}^{m_2}\ (\tau_2-\tau_1)^{\delta+\kappa-1}\Bigg].$$

Therefore TS_r is equi-continuous, by Arzela-Ascoli Theorem then TS_r is relatively compact. Therefore the conditions of Schauder fixed point Theorem are hold, which implies that T has a fixed point in S_r. Then the nonlinear integral equation (3.1) has at least one solution $u \in C_{1-\kappa}(I)$ and consequently the weighted Cauchy-type problem (1.1) has at least one solution $u \in C_{1-\kappa}(I)$.

3.2. Solution in $C(I)$

3.2.1. Integral representation

Lemma 3.2. *The Cauchy problem (1.2) is equivalent to the nonlinear integral equation of fractional order*

$$u(\varsigma) = \int_0^\varsigma \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))\, d\theta\right) ds. \tag{3.2}$$

Proof: Let $u(\varsigma)$ be a solution of

$$D^\kappa u(\varsigma) = \frac{d}{d\varsigma} I^{1-\kappa} u(\varsigma) = f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

integrate both sides, we get

$$I^{1-\kappa} u(\varsigma) - I^{1-\kappa} u(\varsigma)|_{\varsigma=0} = I\, f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

operating by I^κ on both sides of the last equation, we get

$$I u(\varsigma) - I^\kappa\, C = I^{1+\kappa} f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

differentiate both sides, we get

$$u(\varsigma) - C_1\, \varsigma^{\kappa-1} = I^\kappa f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

from the initial condition, we find that $C_1 = 0$, then we get (3.2)
Define the operator F by

$$F u(\varsigma) = \int_0^\varsigma \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta)) d\theta\right) ds.$$

It is clear that the fixed point of the operator F is the solution of the integral equation (3.2).

3.2.2. Existence of solution

Theorem 3.3. *Assume that the assumptions (1*)-(2*) are satisfied. Then the weighted Cauchy-type problem (1.2) has at least one solution $u \in C(I)$.*

Proof. Define the set

$$B_r = \left\{ u \in C(I) : ||u|| \le r_1 \right\}.$$

Now,

$$||Fu|| = \max_{\varsigma\in I} \left| I^\kappa f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\bigg|f\bigg(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\bigg)\bigg|$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\bigg(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma} |h(\varsigma, s, u(s))|\, ds\bigg)$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\bigg(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma} (\varsigma - s)^{\beta-1} s^{\sigma}\psi(s)|u(s)|^{m_2}\, ds\bigg)$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \max_{\varsigma\in I} \int_0^{\varsigma} \tfrac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^{s} (s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\, d\theta ds$$

$$\leq \max_{\varsigma\in I} \int_0^{\varsigma} \tfrac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu}\varphi(s)|u(s)|^{m_1} ds$$

$$+ \max_{\varsigma\in I} ||\psi||\, ||u||^{m_2} \tfrac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\beta+\sigma+1)} \int_0^{\varsigma} \tfrac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\beta+\sigma} ds$$

$$\leq \max_{\varsigma\in I} ||\varphi||\, ||u||^{m_1} \tfrac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)} \varsigma^{\kappa+\mu}$$

$$+ \max_{\varsigma\in I} ||\psi||\, ||u||^{m_2} \tfrac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\beta+\sigma+1)} \tfrac{\Gamma(\beta+\sigma+1)}{\Gamma(\kappa+\beta+\sigma+1)} \varsigma^{\kappa+\beta+\sigma}$$

$$\leq ||\varphi||\, ||u||^{m_1} \tfrac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)} \hbar^{\kappa+\mu} + ||\psi||\, ||u||^{m_2} \tfrac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\kappa+\beta+\sigma+1))} \hbar^{\kappa+\beta+\sigma}.$$

If $u \in B_r$, then

$$||Fu|| \leq \frac{\Gamma(\mu+1)||\varphi||\, r_1^{m_1}}{\Gamma(\kappa+\mu+1)} \hbar^{\kappa+\mu} + \frac{\Gamma(\beta)\Gamma(\sigma+1)||\psi||\, r_1^{m_2}}{\Gamma(\kappa+\beta+\sigma+1)} \hbar^{\kappa+\beta+\sigma}.$$

If we take $\hbar$ very small, then

$$||Fu|| \;\leq\; r_1,$$

then $F(B_r) \subset B_r$.
From the continuity of f and h, we obtain that the operator F is continuous.
To prove that $F(B_r)$ is equicontinuous
Let $\tau_1, \tau_2 \in [0, \hbar], \tau_1 < \tau_2, |\tau_2 - \tau_1| < \delta$, then

$$Fu(\tau_2) - Fu(\tau_1) = \int_0^{\tau_2} \tfrac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\bigg(s, u(s), \int_0^{s} h(s, \theta, u(\theta))d\theta\bigg)\, ds$$

$$- \int_0^{\tau_1} \tfrac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\bigg(s, u(s), \int_0^{s} h(s, \theta, u(\theta))d\theta\bigg)\, ds$$

$$= \int_0^{\tau_1} \tfrac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\bigg(s, u(s), \int_0^{s} h(s, \theta, u(\theta))d\theta\bigg)\, ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\,f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)\,ds$$

$$-\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\,f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)\,ds$$

$$\leq\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\,f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)ds,$$

$$-\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\,f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)\,ds$$

$$\Bigg|Fu(\tau_2)-Fu(\tau_1)\Bigg|\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\,\Bigg|f\Bigg(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\Bigg)\Bigg|ds$$

$$\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\Bigg(s^{\mu}\varphi(s)|u(s)|^{m_1}+\int_0^s|h(s,\theta,u(\theta))|d\theta\Bigg)ds$$

$$\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\varphi(s)|u(s)|^{m_1}ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}d\theta ds$$

$$\leq||\varphi||\;||u||^{m_1}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\;ds$$

$$+||\psi||\;||u||^{m_2}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s(s-\theta)^{\beta-1}\theta^{\sigma}d\theta\;ds$$

$$\leq||\varphi||\;||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\kappa+1)}(\tau_2-\tau_1)^{\mu+\kappa}$$

$$+||\psi||\;||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\sigma+\beta}ds$$

$$\leq||\varphi||\;||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\kappa+1)}(\tau_2-\tau_1)^{\mu+\kappa}$$

$$+||\psi||\;||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+\kappa+1)}(\tau_2-\tau_1)^{\sigma+\beta+\kappa}.$$

Therefore FB_r is equi-continuous, by Arzela-Ascoli Theorem then FB_r is relatively compact. Therefore the conditions of Schauder fixed point Theorem are hold, which implies that F has a fixed point in B_r. Then the nonlinear integral equation (3.2) has a solution $u\in C(I)$ and consequently from Lemma 3.2, we get that the weighted Cauchy-type problem (1.2) has a solution $u\in C(I)$.

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References

[1] A. M. A. El-Sayed and Sh. A. Abd El-Salam, Weighted Cauchy-type problem of a functional differ-integral equation, *EJQTDE*, **30**(2007) 1-9.

[2] A. M. A. El-Sayed and Sh. A. Abd El-Salam, L_p- solution of weighted Cauchy-type problem of a differ-integral functional equation, *Inter. J. of Nonlinear Sci.*, **5**(2008), 1–9.

[3] A. M. A. El-Sayed and Sh. A. Abd El-Salam, Coupled system of a fractional order differential equations with weighted initial conditions, *Open Math.*, **17**(2019), 1737-1749.

[4] F. M. Gaafar, Cauchy-type problems of a functional differential equations with advanced arguments, *Journal of Fractional Calculus and Applications*, **5**(2)(2014), 71-77.

[5] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. 1, Fizmatgiz, Moscow, (1959) (in Russian).

[6] K. M. Furati and N. E. Tatar, Long time behavior for a nonlinear fractional model, *J. Math. Anal. Appl.*, **332**(2007), 441-454.

[7] K. M. Furati and N. E. Tatar, Power-type estimates for a nonlinear fractional differential equation, *Nonlinear Analysis*, **62**(2005), 1025-1036.

[8] K. M. Furati and N. E. Tatar, An existence results for a nonlocal fractional differential problem, *Journal of Fractional Calculus*, **26**(2004), 43-51.

[9] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).

[10] I. Podlubny, *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).

[11] S. Samko, A. Kilbas and O. L. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).

[12] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, (1985).