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Approximation results for local solution of the initial value problems of nonlinear first order ordinary hybrid integrodifferential equations

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Abstract. In this paper, we establish a couple of approximation results for local existence and uniqueness of the solution of a initial value problem of nonlinear first order ordinary hybrid integrodifferential equations by using the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2022) and Dhage *et al.* (2022). An approximation result for Ulam-Hyers stability of the local solution of the considered hybrid differential equation is also established. Finally, our main abstract results are also illustrated with the help of a couple of numerical examples.

AMS Subject Classifications: 34A12, 34A34, 34A45, 47H10.

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Contents

1. Introduction

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$, for some $t_0, a \in \mathbb{R}$ with $a > 0$, we consider the initial value problem (IVP) of nonlinear first order ordinary hybrid integrodifferential equations (in short HIGDEs),

$$\left.\begin{aligned} x'(t) &= f\Big(t, x(t), \int_{t_0}^{t} g(s, x(s))\, ds\Big), \quad t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 1.1. *A function $x \in C(J, \mathbb{R})$ is said to be a* solution *of the HIGDE (1.1 if it satisfies the equations in (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J. If a solution x lies in a neighborhood $\mathcal{N}(x_0)$ of some point $x_0 \in C(J, \mathbb{R})$, then we say it is a* local solution *or* neighborhood solution *(in short* nbhd solution*) of the HIGDE (1.1) defined on J.*

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Remark 1.2. *It is well-known that an open ball $B(x,r)$ in $C(J,\mathbb{R})$ centered at a point x of radius $r>0$ is a neighborhood of the point x, so if a solution x^* of the HIGDE (1.1) lies in a closed ball $\overline{B(x,r)}$ in $C(J,\mathbb{R})$, then it is a* local solution *in view of the fact that $\overline{B(x,r)} \subset B(x,r+\epsilon)$ for every $\epsilon>0$.*

The HIGDE (1.1) is quite familiar to the subject of nonlinear analysis. If $f(t,x,y)=f(t,x)$, then it includes the nonlinear IVPs of nonlinear first order hybrid differential equation (HDE)

$$\left.\begin{aligned} x'(t) &= f(t,x(t)),\ \ t\in J,\\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.2}$$

and on choosing $f(t,x,y)n = f(t,y)$, it yields the onlinear first order hybrid integrodifferential equation (HIGDE)

$$\left.\begin{aligned} \frac{dx}{dt} &= \int_{t_0}^{t} f(s,x(s)),\ \ t\in J,\\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.3}$$

as particular cases. The HDE (1.2) and HIGDE (1.3) have been studied for approximation results in Dhage and Dhage [11] and Dhage and Dhage [12] respectively via hybrid fixed point theory recently developed in Dhage [8] and Dhage *et al.* [10]. Similarly, the HIGDE (1.1) can be studied for a variety of different aspects of the solution by using different nonlinear operator theoretic techniques from nonlinear functional analysis. The existence result for the local solution of the IVP (1.1) can be proved by using the Schauder fixed point principle, see for example, Coddington [2], Lakshmikantham and Leela [17], Granas and Dugundji [14] and references therein. Similarly, the approximation result for the uniqueness of solution can be proved by using well-known Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the topic of nonlinear analysis. But to the knowledge of the present authors, the approximation result for local existence and uniqueness theorems without using the Lipschitz condition is not discussed so far in the theory of nonlinear differential and integral equations. In this paper, we undertake the study of approximation results for local existence and uniqueness of solution under weaker form of one sided or partial Lipschitz condition but via construction of the algorithms based on the Dhage monotone iteration method and a hybrid fixed point theorem of Dhage *et al.* [10] and Dhage [8].

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyer stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5.

2. Auxiliary Results

We place the problem of HIGDE (1.1) in the function space $C(J,\mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|, \tag{2.1}$$

and an order relation $\preceq$ in $C(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \ \forall\ t\in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{2.3}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t),\ \forall\ t \in J.$$

It is known that the Banach space $C(J,\mathbb{R})$ together with the order relations $\preceq$ becomes a partially ordered Banach space which we denote for convenience, by $\big(C(J,\mathbb{R}),K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J,\mathbb{R})$ of radius r, for some $r>0$, by

$$B_r(x_0)=\{x\in C(J,\mathbb{R}) \mid \|x-x_0\|<r\}=B(x,r)$$

and

$$B_r[x_0]=\{x\in C(J,\mathbb{R}) \mid \|x-x_0\|\leq r\}=\overline{B(x,r)}$$

receptively. It is clear that $B_r[x_0]=\overline{B_r(x_0)}$ and $B_r(x_0)\subset B_r[x_0]\subset B_{r+\epsilon}(x_0)$ for every $\epsilon>0$. Let $M>0$ be a given real number. Denote

$$B_r^M[x_0]=\big\{x\in B_r[x_0]\,\big|\,|x(t_1)-x(t_2)|\leq M\,|t_1-t_2| \text{ for } t_1,t_2\in J\big\}. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. *The set $B_r^M[x_0]$ is compact in $C(J,\mathbb{R})$.*

Proof. By definition $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J,\mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J,\mathbb{R})$ in view of the condition (2.1). Now by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J,\mathbb{R})$ and the proof of the lemma is complete. □

It is well-known that the fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations. See Granas and Dugundji [14] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [8] and Dhage *et al.* [10].

Theorem 2.2 (Dhage [8])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E,\|\cdot\|,\preceq,\big)$ with every chain C in S is Janhavi set and let $\mathcal{T}: S\to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0\in S$ such that $x_0\preceq \mathcal{T}x_0$ or $x_0\succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x=x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.3 (Dhage [8])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r>0$, in a regular partially ordered Banach space $\big(E,\|\cdot\|,\preceq,\big)$ and let $\mathcal{T}:E\to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0\in X$ such that $x_0\preceq\mathcal{T}x_0$ or $x_0\succeq\mathcal{T}x_0$ satisfying*

$$\|x_0-\mathcal{T}x_0\|\leq(1-q)r \tag{2.5}$$

for some real number $r>0$, then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to x^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

Remark 2.4. *We note that every pair of elements in a partially ordered set* ($\mathbf{poset}$) $(E,\preceq)$ *has a lower or upper bound if $(E,\preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E,\|\cdot\|,\preceq)$ is called a* **partially lattice ordered Banach space***. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. The details of the lattice structure of the Banach spaces appear in Birkhoff [1].*

As a consequence of Remark 2.4, we obtain

Theorem 2.5. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially lattice ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote it by (X, K). Similarly, an ordered Banach space (X, K), where partial order $\preceq$ defined by the cone K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\big(C(J, \mathbb{R}), K\big)$ of continuous real-valued functions defined on the closed and bounded interval J is lattice ordered Banach space, where the cone K is given by $K = \{x \in CJ, \mathbb{R}) \mid x \succeq 0\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [15]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [3, 4].

Lemma 2.6 (Dhage [3, 4])**.** *Every ordered Banach space (X, K) is regular.*

Lemma 2.7 (Dhage [3, 4])**.** *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.6 and 2.7 we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.8 (Dhage [8] and Dhage *et al.* [10])**.** *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.9 (Dhage [8])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.*

Theorem 2.10 (Dhage [8])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a lattice ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.*

The details of the notions of partial order, Janhavi set, regularity of ordered space, monotonicity of mappings, partial continuity, partial closure, partial completeness, partial compactness and partial contraction etc. and related applications appear in Dhage [3–8], Dhage and Dhage [9], Dhage *et al.* [10, 13] and references therein.

3. Local Approximation Results

We consider the following set of hypotheses in what follows.

(H$_1$) The function f is continuos anf bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(H$_2$) $f(t, x, y)$ is nondecreasing in x and y for each $t \in J$.

(H$_3$) $g(t, x)$ is nondecreasing in x for each $t \in J$.

(H$_4$) $f(t, \alpha_0, y) \geq 0$ for all $t \in J$ and $y \geq 0$.

(H$_5$) $g(t, \alpha_0) \geq 0$ for all $t \in J$.

Then we have the following useful lemma.

Lemma 3.1. *If $h \in L^1(J, \mathbb{R})$, then the IVP of ordinary first order linear differential equation*

$$x'(t) = h(t), \ \ t \in J, \quad x(t_0) = \alpha_0, \tag{3.1}$$

is equivalent to the integral equation

$$x(t) = \alpha_0 + \int_{t_0}^{t} h(s)\, ds, \ \ , t \in J. \tag{3.2}$$

Theorem 3.2. *Sppose that the hypotheses (H$_1$), through (H$_5$) hold. Furthermore, if $M_f\, a \leq r$ and $M_f \leq M$, then the HIGDE (1.1) has a solution x^* in $B_r^M[x_0]$, where, $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= \alpha_0, \ \ t \in J,\\ x_{n+1}(t) &= \alpha_0 + \int_{t_0}^{t} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds, \ \ t \in J, \end{aligned}\right\} \tag{3.3}$$

where $n = 0, 1, \ldots,$ is monotone nondecreasing and converges to x^*.

Proof. Set $X = C(J, \mathbb{R})$. Clearly, (X, K) is a partially ordered Banach space. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (2.4). By Lemma 2.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 3.1, the HIGDE (1.1) is equivalent to the nonlinear hybrid integral equation (HIE)

$$x(t) = \alpha_0 + \int_{t_0}^{t} f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds, \ \ t \in J. \tag{3.4}$$

Now, define an operator $\mathcal{T}$ on $B_r^M[x_0]$ into X by

$$\mathcal{T}x(t) = \alpha_0 + \int_{t_0}^{t} f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds, \ \ t \in J. \tag{3.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.8 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned} |\mathcal{T}x(t) - x_0(t)| &= \left| \int_{t_0}^{t} f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \right| \\ &\leq \int_{t_0}^{t} \left| f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big) \right| ds \\ &= M_f \int_{t_0}^{t_0+a} ds \\ &= M_f\, a \leq r. \end{aligned}$$

Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - x_0\| \leq M_f\, a \leq r,$$

which implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then, we have

$$\begin{aligned}
|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)| &\leq \left| \int_{t_1}^{t_2} \left| f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big) \right| ds \right| \\
&\leq M_f\, |t_1 - t_2| \\
&\leq M\, |t_1 - t_2|.
\end{aligned}$$

Therefore, $\mathcal{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathcal{T}(B_r^M[x_0]) \subset B_r^M[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, since (H$_2$) and (H$_3$) hold, we have that

$$\begin{aligned}
\mathcal{T}x(t) &= \alpha_0 + \int_{t_0}^{t} f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&\geq \alpha_0 + \int_{t_0}^{t} f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}y(t)
\end{aligned}$$

for all $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, by dominated cnonvergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n &= \lim_{n\to\infty} \left[\alpha_0 + \int_{t_0}^{t} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds\right] \\
&= \alpha_0 + \lim_{n\to\infty} \int_{t_0}^{t} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds \\
&= \alpha_0 + \int_{t_0}^{t} \left[\lim_{n\to\infty} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\right]\, ds \\
&= \alpha_0 + \int_{t_0}^{t} f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x(t)
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. As $\{\mathcal{T}x_n\} \subset B_r^M[x_0]$, $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of points in X. As a reult, we have that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly on J. Hence $\mathcal{T}$ is partially continuouus operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the order relation $x_0 \preceq \mathcal{T}x_0$.*

Since (H$_4$) and (H$_5$) hold, one has

$$\begin{aligned}
x_0(t) &= \alpha_0 + \int_{t_0}^{t} f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&\leq x_0(t) + \int_{t_0}^{t} f\Big(s, \alpha_0, \int_{t_0}^{s} g(\tau, \alpha_0)\, d\tau\Big)\, ds \\
&= \alpha_0 + \int_{t_0}^{t} f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x_0(t)
\end{aligned}$$

for all $t \in J$. As a result, we have that $x_0 \preceq \mathcal{T}x_0$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.8, and so $\mathcal{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to $x^* \subset B_{r+\epsilon}(x_0)$ for arbitrary $\epsilon > 0$. Since every open ball is a neighborhood of its center point, we infer that the HIE (3.4) and consequently the HIGDE (1.1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of partial Lipschitz condition. We need the following hypotheses in what follows.

(H_6) There exists a constant $k > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_1) + \ell_2(x_2 - y_2)$$

for all $t \in J$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$, where $\big(\ell_1 a + \ell_2 k\, a^2\big) < 1$.

(H_7) There exists a constant $k > 0$ such that

$$0 \leq g(t, x) - g(t, y) \leq k\,(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \geq y$.

Theorem 3.3. *Suppose that the hypotheses (H_1), (H_4) - (H_5) and (H_6) - (H_7) hold. Furthermore, if*

$$M_f a \leq \big[1 - \big(\ell_1 a + \ell_2 k\, a^2\big)\big] r, \quad \big(\ell_1 a + \ell_2 k\, a^2\big) < 1, \tag{3.6}$$

for some real number $r > 0$, then the HIGDE (1.1) has a unique solution x^* in $B_r[x_0]$ defined on J, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*.

Proof. Set $(X, K) = \big(C(J, \mathbb{R}), \preceq\ \big)$ which is a lattice ordered Banach space w.r.t. to the lattice operations $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$, and so every pair of elements of X has a lower and an upper bound. Let $r > 0$ be a fixed real number and consider the closed ball $B_r[x_0]$ centered at $x_0 \in C(J, \mathbb{R})$ of radius r in the partially ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (3.5). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be any two elements such that $x \succeq y$. Then, by hypotheseis (H_2) and (H_3), we obtain

$$\begin{aligned}
&\mathcal{T}x(t) - \mathcal{T}y(t) \\
&= \int_{t_0}^{t} \bigg[f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big) - f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\, d\tau\Big) \bigg]\, ds \\
&\geq 0
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$ and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypotheses

(H_6) and (H_7), we obtain

$$
\begin{aligned}
|\mathcal{T}x(t)-\mathcal{T}y(t)| &= \bigg|\int_{t_0}^{t} f\Big(s,x(s),\int_{t_0}^{s} g(\tau,x(\tau))\,d\tau\Big)\,ds \\
&\qquad - \int_{t_0}^{t} f\Big(s,y(s),\int_{t_0}^{s} g(\tau,y(\tau))\,d\tau\Big)\,ds\,ds\bigg| \\
&\le \bigg|\int_{t_0}^{t} \Big|f\Big(s,x(s),\int_{t_0}^{s} g(\tau,x(\tau))\,d\tau\Big)\,ds \\
&\qquad - f\Big(s,y(s),\int_{t_0}^{s} g(\tau,y(\tau))\,d\tau\Big)\Big|\,ds\bigg| \\
&\le \bigg|\int_{t_0}^{t} \Big[\ell_1\big(x(s)-y(s)\big)+\ell_2\int_{t_0}^{s} k\,\big(x(\tau)-y(\tau)\big)\,d\tau\Big]\,ds\bigg| \\
&= \ell_1\int_{t_0}^{t} |x(s)-y(s)|\,ds + \ell_2 k\int_{t_0}^{t} (t-s)\big(x(s)-y(s)\big)ds \\
&\le \ell_1\int_{t_0}^{t_0+a} \|x-y\|\,ds + \ell_2\,k\,a\int_{t_0}^{t_0+a} \|x-y\|\,ds \\
&= \big[\ell_1 a+\ell_2 k\,a^2\big]\|x-y\| \\
&= \lambda\|x-y\|
\end{aligned}
$$

for all $t\in J$, where $\lambda=\big[\ell_1 a+\ell_2 k\,a^2\big]<1$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x-\mathcal{T}y\|\le\lambda\,\|x-y\|$$

for all comparable elements $x,y\in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant λ. Furthermore, it can be shown as in the proof of Theorem 3.2 that the element $x_0\in B_r^M[x_0]$ satisfies the relation $x_0\preceq\mathcal{T}x_0$ in view of hypothesis (H_4) and (H_5). Finally, by hypothesis (H_1) and condition (3.6), one has

$$
\begin{aligned}
\|x_0-\mathcal{T}x_0\| &= \sup_{t\in J}\bigg|\int_{t_0}^{t} f\Big(s,x_0(s),\int_{t_0}^{s} g(\tau,x_0(\tau))\,d\tau\Big)\,ds\bigg| \\
&\le \sup_{t\in J}\int_{t_0}^{t}\Big|f\Big(s,\alpha_0,\int_{t_0}^{s} g(\tau,\alpha_0)\,d\tau\Big)\Big|\,ds \\
&\le M_f\,a \\
&\le \big[1-\big(\ell_1 a+\ell_2 k\,a^2\big)\big]r
\end{aligned}
$$

which shows that the condition (2.5) of Theorem 2.10 is satisfied. Hence $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.4) and consequently the HIGDE (1.1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*. This completes the proof. □

Remark 3.4. *The conclusion of Theorems 3.2 and 3.3 also remains true if we replace the hypothesis (H_4) with the following one.*

(H_4') The fuction f satisfiies the relation $f(t,x_0,y)<0$ for all $t\in J$ and $y\ge 0$.

In this case, the HDE (1.1) has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to the solution x^*.

Remark 3.5. *If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.2, the HIGDE (1.1) has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 3.3, the HIGDE (1.1) has a unique local positive solution x^* defined on J and the sequence of successive approximations given by (3.3) $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to the unique positive solution x^*.*

4. Approximation of Local Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [18], Huang *et al.* [16] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the HIGDE (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.10. We need the following definition in what follows.

Definition 4.1. *The HIGDE (1.1) is said to be locally Ulam-Hyers stable if for $\epsilon > 0$ and for each local solution $y \in B_r[x_0]$ of the inequality*

$$\left.\begin{aligned} &\left|\frac{dy}{dt} - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\,ds\Big)\right| \leq \epsilon, \;\; t \in J, \\ &y(t_0) = \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{$*$}$$

there exists a constant $K_f > 0$ such that

$$\big|y(t) - \xi(t)\big| \leq K_f \epsilon \tag{$**$}$$

for all $t \in J$, where $\xi \in B_r[x_0]$ is a local solution of the HIGDE (1.1) defined on J. The solution ξ of the HIGDE (1.1) is called Ulam-Hyers stable local solution on J.

Theorem 4.2. *Assume that all the hypotheses of Theorem 3.3 hold. Then the HIGDE (1.1) has a unique Ulam-Hyers stable local solution $x^* \in B_r[x_0]$, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations given by (3.3) converges monotone nondecreasingly to x^*.*

Proof. Let $\epsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (*) on J, that is, we have

$$\left.\begin{aligned} &\left|\frac{dy}{dt} - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\,ds\Big)\right| \leq \epsilon, \quad t \in J, \\ &y(t_0) = \alpha_0 \in \mathbb{R}_+. \end{aligned}\right\} \tag{4.1}$$

By Theorem 3.3, the HIGDE (1.1) has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 2.1, one has

$$\xi(t) = \alpha_0 + \int_{t_0}^{t} f\Big(s, \xi(s), \int_{t_0}^{s} g(\tau, \xi(\tau))\,ds\Big)\,ds, \quad t \in J. \tag{4.2}$$

Now, by integration of (4.1) yields the estimate:

$$\left|y(t) - \alpha_0 - \int_{t_0}^{t} f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\,ds\right| \leq a\,\epsilon, \tag{4.3}$$

for all $t \in J$.

Next, from (4.2) and (4.3) we obtain

$$
\begin{aligned}
\big|y(t)-\xi(t)\big| &= \left|y(t)-\alpha_0-\int_{t_0}^{t} f\Big(s,\xi(s),\int_{t_0}^{s} g(\tau,\xi(\tau))\,d\tau\Big)\,ds\right| \\
&\leq \left|y(t)-\alpha_0-\int_{t_0}^{t} f\Big(s,y(s),\int_{t_0}^{s} g(\tau,y(\tau))\,d\tau\Big)\,ds\right| \\
&\quad + \left|\int_{t_0}^{t}\Big[f\Big(s,y(s),\int_{t_0}^{s} g(\tau,y(\tau))\,d\tau\Big)-f\Big(s,\xi(s),\int_{t_0}^{s} g(\tau,\xi(\tau))\,d\tau\Big)\Big]\,ds\right| \\
&\leq a\,\epsilon+\left|\int_{t_0}^{t}\Big[\ell_1\big(y(s)-\xi(s)\big)+\ell_2\int_{t_0}^{s} k\,\big(y(\tau)-\xi(\tau)\big)\,d\tau\Big]\,ds\right| \\
&= a\,\epsilon+\ell_1\int_{t_0}^{t}|y(s)-\xi(s)|\,ds+\ell_2 k\int_{t_0}^{t}(t-s)\big(y(s)-\xi(s)\big)\big) \\
&\leq a\,\epsilon++\ell_1\int_{t_0}^{t_0+a}\|y-\xi\|\,ds+\ell_2\,k\,a\int_{t_0}^{t_0+a}\|y-\xi\|\,ds \\
&= a\,\epsilon+\big[\ell_1 a+\ell_2 k\,a^2\big]\|y-\xi\| \\
&= a\,\epsilon+\lambda\|y-\xi\|
\end{aligned}
$$

Taking the supremum over t, we obtain

$$\|y-\xi\|\leq a\,\epsilon+\lambda\|y-\xi\|$$

or

$$\|y-\xi\|\leq\left[\frac{a}{1-\lambda}\right]\epsilon,\quad \big(\because\lambda=\big[\ell_1 a+\ell_2 k\,a^2\big]<1\big).$$

Letting $K_f=\left[\dfrac{a}{1-\lambda}\right]>0$, we obtain

$$\big|y(t)-\xi(t)\big|\leq K_f\epsilon$$

for all $t\in J$. As a result, ξ is a Ulam-Hyers stable local solution of the HIGDE (1.1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to ξ. Consequently the HIGDE (1.1) is a locally Ulam-Hyers stable on J. This completes the proof. □

Remark 4.3. *If the given initial condition in the equation (1.1) is such that $\alpha_0>0$, then under the conditions of Theorem 4.2, the HIGDE (1.1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*.*

Remark 4.4. *The local approximation results, Theorems 3.2, 3.3 and 4.2 of this paper includes the local approximation results for the HDE (1.2) and HIGDE (1.3) on J.*

5. The Examples

Below in the following we present a couple of numerical examples to illustrate the abstract ideas involved in the approximation results of this paper for existence, uniqueness and Ulam-Hyer stability of the solution.

Example 5.1. *Given a closed and bounded interval $J=[0,1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HIGDE,*

$$\frac{dx}{dt}=\tanh x(t)+\int_0^t\tanh x(s)\,ds,\ \ t\in[0,1];\quad x(0)=\frac{1}{4}. \tag{5.1}$$

Here, $\alpha_0 = \frac{1}{4}$, $g(t,x) = \tanh x$, $(t,x) \in [0,1]\times\mathbb{R}$ *and* $f(t,x,y) = \tanh x + y$ *for* $(t,x,y) \in [0,1]\times\mathbb{R}\times\mathbb{R}$. *We show that the functions* g *and* f *satisfy all the conditions of Theorem 3.2. Clearly,* f *is bounded on* $[0,1]\times\mathbb{R}\times\mathbb{R}$ *with bound* $M_f = 2$ *and so the hypothesis (H*$_1$*) is satisfied. Also the function* $f(t,x,y)$ *is nondecreasing in* x *and* y *for each* $t \in [0,1]$*. Therefore, hypothesis (H*$_2$*) is satisfied. Next,* $g(t,x)$ *is nondecreasing in* x *for each* $t \in [0,1]$*, so the hypothesis (H*$_3$*) is satisfied. Moreover, we have*

$$f(t,\alpha_0,y) = f\Big(t,\frac{1}{4},y\Big) = \tanh\left(\frac{1}{4}\right) + y \ge 0$$

for each $t \in [0,1]$ *and* $y \ge 0$*, so the hypothesis (H*$_4$*) holds. Finally,* $g(t,\alpha_0) = \tanh(\frac{1}{4}) \ge 0$ *for all* $t \in [0,1]$ *and hypothesis (H*$_5$*) is satisfied. If we take* $r = 2$ *and* $M = 1$*, all the conditions of Theorem 3.2 are satisfied. Hence, the HIGDE (5.1) has a local solution* x^* *in the closed ball* $B_2^1[\frac{1}{4}]$ *of* $C(J,\mathbb{R})$ *which is positive in view of Remark 3.5. Moreover, the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$x_0(t) = \frac{1}{4},\ \ t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + x_{n+1}(t) = \frac{1}{4} + \int_0^t \tanh x_n(s)\,ds + \int_0^t (t-s)\ \tanh x_n(s)\,ds,\ \ t \in [0,1],$$

converges monotone nondecreasingly to x^*.

Example 5.2. *Given a closed and bounded interval* $J = [0,1]$ *in* $\mathbb{R}$*, consider the IVP of nonlinear first order HIGDE,*

$$\frac{dx}{dt} = \frac{1}{4}\ \tan^{-1} x(t) + \frac{1}{4}\int_0^t \tan^{-1} x(s)\,,\ \ t \in [0,1];\quad x(0) = \frac{1}{4}. \tag{5.2}$$

Here, $\alpha_0 = \dfrac{1}{4}$ *and* $g(t,x) = \tan^{-1} x$ *for* $(t,x) \in [0,1]\times\mathbb{R}$*. Again,* $f(t,x,y) = \dfrac{1}{2}\tan^{-1} x + y$ *for each* $t \in [0,1]$*. We show that* f *satisfies all the conditions of Theorem 3.3. Clearly,* f *is bounded on* $[0,1]\times\mathbb{R}\times\mathbb{R}$ *with bound* $M_f = \frac{11}{14}$ *and so, the hypothesis (H*$_1$*) is satisfied. Next, let* $x, y \in \mathbb{R}$ *be such that* $x \ge y$*. Then there exists a constant* ξ *with* $x < \xi < y$ *satisfying*

$$0 \le g(t,x) - g(t,y) \le \frac{1}{1+\xi^2}(x-y) \le (x-y)$$

for all $t \in [0,1]$*. So the hypothesis (H*$_7$*) holds with* $k = 1$*. Moreover,* $g(t,\alpha_0) = g\left(t,\frac{1}{4}\right) = \tan^{-1}\left(\frac{1}{4}\right) \ge 0$ *for each* $t \in [0,1]$*, and so the hypothesis (H*$_4$*) holds. Similarly,*

$$f(t,\alpha_0,y) = \frac{1}{4}\tan^{-1}\alpha_0 + y = \tan^{-1}\left(\frac{1}{4}\right) + y \ge 0$$

for each $t \in [0,1]$ *and for all positive number* y*, so the hypothesis (H*$_4$*) is satisfied. Next, let* $x_1, y_1, x_2, y_2 \in \mathbb{R}$ *with* $x_1 \ge y_1$*,* $x_2 \ge y_2$*. Then,*

$$0 \le f(t,x_1,x_2) - f(t,y_1,y_2) \le \frac{1}{4}\cdot(x_1-y_2) + \frac{1}{4}(x_2-y_2)$$

for each $t \in [0,1]$*. Therefore, hypothesis (H*$_6$*) holds with* $\ell_1 = \frac{1}{4} = \ell_2$*. If we take* $r = 2$*, then we have*

$$M_f a = \frac{11}{14} \le \Big(1 - \frac{1}{2}\Big)\cdot 2 = \big[1 - \big(\ell_1 a + \ell_2 k\, a^2\big)\big]r$$

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.3 are fulfilled. Hence, the HIGDE (5.2) has a unique local solution x^* *in the closed ball* $B_2[\frac{1}{4}]$ *of* $C(J,\mathbb{R})$ *which is positive in view of Remark 3.5.*

Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$
$$x_{n+1}(t) = \frac{1}{4} + \frac{1}{4}\int_0^t \tan^{-1} x_n(s)\, ds + \int_0^t (t-s)\, \tan^{-1} x_n(s)\, ds, \;\; t \in [0,1],$$

converges monotone nondecreasingly to x^*. *Furthermore, the unique local positive solution* x^* *is Ulam-Hyers stable on* $[0,1]$ *in view of Definition 4.1. Consequently the HIGDE (5.2) is a locally Ulam-Hyers stable on the interval* $[0,1]$.

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