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Nevanlinna theory for the upper half disc-I

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Abstract. In this paper, we prove the Poisson Integral theorem and Poisson-Jenson formula for the upper half disc and consequently introduce the proximity function, the counting function and the characteristic function of a meromorphic function in the upper half disc which are basic functions of Nevanlinna theory.

AMS Subject Classifications: 30D35

Keywords: Nevanlinna theory, Upper half disc, Meromorphic functions, Poisson integral formula, Characteristic functions, etc.

Contents

1. Preamble

Nevanlinna theory for meromorphic functions in the complex plane is about a century old and still is an emerging active area of research. This theory has wide range of applications including complex differential equations and value sharing of meromorphic functions etc. Nevanlinna theory for an angular domain is also developed by some authors like (see [4],[5],[6]) using Carleman's formula.
In this paper, we propose a similar theory for the upper-half plane. Our main tool here is the conformal self map of the upper half disc as defined in (2.1).

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2. Preliminaries

Let $\mathbb{C}$ denote the set of all complex numbers, $\overline{\mathbb{C}}$, the extended complex plane, $\mathbb{C}_+ = \{z : Imz > 0\}$ and $\overline{\mathbb{C}}_+ = \{z : Imz \geq 0\}$. Throughout, let $\mathbf{D} = \{|\xi| < R : Im\xi > 0\}$ and $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$ be open and closed upper-half discs respectively.
In what follows, we see that conformal self map of upper-half disc

$$\Phi_z(\xi) = \frac{R(\xi - z)}{R^2 - \xi\overline{z}} \frac{R^2 - \xi z}{R(\xi - \overline{z})} \tag{2.1}$$

where $z, \xi \in \mathbf{D}$ plays a cardinal role in the development of the theory.

The analogous notations of Nevanlinna theory for meromorphic functions in the upper-half disc shall be introduced as and when required.

3. Main results

We now present the core result namely Poisson Integral formula for the upper-half disc.

Theorem 3.1. *Let $f(z)$ be analytic on the closed upper-half disc $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$. For $z = re^{i\phi}$ $(0 < r < R)$ in $\mathbf{D}$, we have*

$$f(z) = \frac{1}{2\pi}\int_0^\pi f(Re^{i\theta})\left\{\frac{R^2 - |z|^2}{|\xi - z|^2} - \frac{R^2 - |z|^2}{|\xi - \overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R} f(t)\left\{\frac{2r\sin\phi}{|z - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - zt|^2}\right\}dt \tag{3.1}$$

Proof. By the Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - z} d\xi \tag{3.2}$$

If $z_1 = \frac{R^2}{\overline{z}} = \frac{R^2}{r}e^{i\phi}$, then $|z_1| = \frac{R^2}{r} > R$ and hence z_1 lies outside the upper-half circle $\{|\xi| = R : Im\xi \geq 0\}$.
Thus $\frac{f(\xi)}{\xi - z_1} = \frac{f(\xi)}{\xi - \frac{R^2}{\overline{z}}}$ is analytic on the closed upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$.
In view of this and by the Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \frac{R^2}{\overline{z}}} d\xi = 0 \tag{3.3}$$

For $z \in \mathbf{D}$, $\overline{z}$ lies outside the upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$ and $\overline{z}_1 = \frac{R^2}{z}$ is also outside the upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$.
In view of the above observations and by using the Cauchy's theorem again we have,

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \overline{z}} d\xi = 0 \tag{3.4}$$

and

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \frac{R^2}{z}} d\xi = 0. \tag{3.5}$$

Combining the above equations $(3.2)-(3.5)$ we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi-\overline{z}}d\xi \\ &+ \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi - \frac{R^2}{z}}d\xi - \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi-\frac{R^2}{\overline{z}}}d\xi \\ &= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{1}{\xi-z} - \frac{1}{\xi-\overline{z}} + \frac{1}{\xi-\frac{R^2}{z}} - \frac{1}{\xi - \frac{R^2}{\overline{z}}}\right]d\xi \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{1}{\xi-z} - \frac{1}{\xi-\overline{z}} + \frac{z}{R^2-\xi z} - \frac{\overline{z}}{R^2-\xi\overline{z}}\right]d\xi \tag{3.6}$$

$$= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{R^2-|z|^2}{(\xi-z)(R^2-\xi\overline{z})} - \frac{R^2-|z|^2}{(\xi-\overline{z})(R^2-\xi z)}\right]d\xi \tag{3.7}$$

Equation (3.6) can also be rewritten as

$$f(z) = \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{(z-\overline{z})}{(\xi-z)(\xi-\overline{z})} - \frac{R^2(z-\overline{z})}{(R^2-\xi z)(R^2-\xi\overline{z})}\right]d\xi \tag{3.8}$$

For $\xi = \xi_1 + \xi_2$, where $\xi_1 = Re^{i\theta}$, $0 < \theta < \pi$ and $\xi_2 = t,\ -R < t < R$ and in view of (3.7) and $(3,8)$, we have

$$f(z) = \frac{1}{2\pi}\int_0^{\pi} f(Re^{i\theta})\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R} f(t)\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

■

The above Poisson Integral formula for the upper-half disc leads to the following result. This result plays a cardinal role in the development of the Nevanlinna theory for the upper-half disc.

Theorem 3.2. Poisson Jensen formula for upper-half disc
Let $f(z)$ be analytic in the closed upper-half disc $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$ except for the poles $b_1, b_2, ..., b_n$ in $\mathbf{D}$ and $a_1, a_2, ..., a_m$ be zeros of $f(z)$ in $\mathbf{D}$. Then for any $z \neq a_m, b_n$ in $\mathbf{D}$, we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{\pi}\log\left|f(Re^{i\theta})\right|\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R}\log|f(t)|\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

$$- \sum_{|a_m|<R, Ima_m>0}\log\left|\frac{(R^2-\overline{a}_m z)}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{(R^2-a_m z)}\right| + \sum_{|b_n|<R, Imb_n>0}\log\left|\frac{(R^2-\overline{b}_n z)}{R(z-b_n)}\frac{R(z-\overline{b}_n)}{(R^2-b_n z)}\right| \tag{3.9}$$

Proof. Set

$$g(z) = \frac{\prod_{|b_n|<R, Imb_n>0}\frac{(z-b_n)}{(z-\overline{b}_n)}}{\prod_{|a_m|<R, Ima_m>0}\frac{(z-a_m)}{(z-\overline{a}_m)}} f(z) \tag{3.10}$$

Then $g(z)$ is analytic in $\mathbf{D}$ having no zeros and poles in $\mathbf{D}$ and hence there exists an analytic branch $\log g(z)$ in $\mathbf{D}$. In view of the Poisson Integral formula for the upper-half disc i. e. by (3.1), we have

$$\log g(z) = \frac{1}{2\pi}\int_0^{\pi}\log g(Re^{i\theta})\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R}\log g(t)\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log g(Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log g(t)dt \tag{3.11}$$

where

$$\beta_1 = \frac{R^2 - |z|^2}{|\xi - z|^2} - \frac{R^2 - |z|^2}{|\xi - \overline{z}|^2}$$

and

$$\beta_2 = \frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - zt|^2}$$

Taking real part on both sides of equation (3.11), we get

$$\log|g(z)| = \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|g(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|g(t)|\, dt \tag{3.12}$$

By (3.10), we get

$$\log|g(z)| = \log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right| \tag{3.13}$$

and

$$\log|g(t)| = \log|f(t)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{t-b_n}{t-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{t-a_m}{t-\overline{a}_m}\right| \tag{3.14}$$

Since $|t-a_m| = |t-\overline{a}_m|$ and $|t-b_n| = \left|t-\overline{b}_n\right|$, (3.14) leads to

$$\log|g(t)| = \log|f(t)| \tag{3.15}$$

Using (3.13) and (3.15) in (3.12), we get

$$\begin{aligned}
&\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right| \\
&= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \left[\log\left|f(Re^{i\theta})\right| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right|\right] d\theta \\
&+ \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt + \sum_{|b_n|<R, Imb_n>0} \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| d\theta \\
&- \sum_{|a_m|<R, Ima_m>0} \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right| d\theta
\end{aligned} \tag{3.16}$$

Since

$$\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right| = \left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| \tag{3.17}$$

and

$$\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| = \left|\frac{R-\overline{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| \tag{3.18}$$

In view of (3.17) and (3.18), (3.16) becomes.

$$\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\bar{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right|$$
$$= \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$
$$+ \sum_{|b_n|<R, Imb_n>0} \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\bar{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| d\theta - \sum_{|a_m|<R, Ima_m>0} \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| d\theta \tag{3.19}$$

Since a_m' s are zeros of $f(z)$ in $\mathbf{D}$, we have $\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}} \neq 0$ in $\mathbf{D}$ and hence there exists an analytic branch of $\log\left(\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}}\right)$ in $\mathbf{D}$. By the Poisson Integral formula for the upper-half disc, we get

$$\log\left(\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}}\right) = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left(\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right) d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log\left(\frac{R-\frac{\overline{a}_m t}{R}}{R-\frac{a_m t}{R}}\right) dt$$

Using $\left|R^2-\overline{a}_m t\right| = \left|R^2-a_m t\right|$ and taking real part part on both sides of above equation, we get

$$\log\left|\frac{R^2-\overline{a}_m z}{R}\frac{R}{R^2-a_m z}\right| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| d\theta \tag{3.20}$$

Similarly,

$$\log\left|\frac{R^2-\bar{b}_n z}{R}\frac{R}{R^2-b_n z}\right| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\bar{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| d\theta \tag{3.21}$$

From equations (3.19), (3.20) and (3.21), we have

$$\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\bar{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right|$$
$$= \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$
$$+ \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{R^2-\bar{b}_n z}{R}\frac{R}{R^2-b_n z}\right|$$
$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{R^2-\overline{a}_m z}{R}\frac{R}{R^2-a_m z}\right|$$

Hence

$$\log|f(z)| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$
$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{R^2-\overline{a}_m z}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{R^2-a_m z}\right| + \sum_{|b_m|<R, Imb_m>0} \log\left|\frac{R^2-\bar{b}_m z}{R(z-b_m)}\frac{R(z-\bar{b}_m)}{R^2-b_m z}\right|$$

Substituting the values of β_1 and β_2 in the above equation, we obtain the desired equality

$$\log|f(z)| = \frac{1}{2\pi}\int_0^\pi \log\left|f(Re^{i\theta})\right| \left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\} d\theta + \frac{1}{2\pi}\int_{-R}^{R} \log|f(t)| \left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\} dt$$
$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{(R^2-\overline{a}_m z)}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{(R^2-a_m z)}\right| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{(R^2-\bar{b}_n z)}{R(z-b_n)}\frac{R(z-\bar{b}_n)}{(R^2-b_n z)}\right|$$

■

4. Nevanlinna functions in the upper-half disc $\mathbb{D}$

Poisson-Jensen formula for upper-half disc enables us to define Nevanlinna functions in the upper-half disc $\mathbf{D}$: For a meromorphic function f in $\mathbf{D}$ and $a \in \mathbf{D}$, we define

Definition 4.1. ***Proximate function of*** $f - a$ ***in D***

$$m(\mathbf{D}, a, f) := \frac{1}{2\pi}\int_0^{\pi} \log^+ \left|f(Re^{i\theta})\right| \left\{\frac{R^2 - |a|^2}{|\xi - a|^2} - \frac{R^2 - |a|^2}{|\xi - \overline{a}|^2}\right\} d\theta + \frac{1}{2\pi}\int_{-R}^{R} \log^+ |f(t)| \left\{\frac{2r\sin\phi}{|a - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - at|^2}\right\} dt \tag{4.1}$$

where, log^+ is the positive logarithmic function.
Briefly we write this as

$$m(\mathbf{D}, a, f) = m_1(\mathbf{D}, a, f) + m_2(\mathbf{D}, a, f)$$

where,

$$m_1(\mathbf{D}, a, f) = \frac{1}{2\pi}\int_0^{\pi} \log^+ \left|f(Re^{i\theta})\right| \left\{\frac{R^2 - |a|^2}{|\xi - a|^2} - \frac{R^2 - |a|^2}{|\xi - \overline{a}|^2}\right\} d\theta,$$

$$m_2(\mathbf{D}, a, f) = \frac{1}{2\pi}\int_{-R}^{R} \log^+ |f(t)| \left\{\frac{2r\sin\phi}{|a - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - at|^2}\right\} dt$$

Definition 4.2. ***Counting function of*** f ***in D***

$$N(\mathbf{D}, f) = N(\mathbf{D}, \infty, f) = \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{(R^2 - \overline{b}_n a)}{R(a - b_n)}\frac{R(a - \overline{b}_n)}{(R^2 - b_n a)}\right| \tag{4.2}$$

where b_n's are poles of f in $\mathbf{D}$, appearing according to their multiplicities.
and

$$N(\mathbf{D}, a, f) = N(\mathbf{D}, \frac{1}{f - a}) = \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{(R^2 - \overline{a}_m a)}{R(a - a_m)}\frac{R(a - \overline{a}_m)}{(R^2 - a_m a)}\right| \tag{4.3}$$

where a_m's are zeros of $f - a$ in $\mathbf{D}$, appearing according to their multiplicities.
$\overline{N}(\mathbf{D}, f)$, $\overline{N}(\mathbf{D}, a, f)$ denotes the distinct poles f and zeros of $f - a$ in $\mathbf{D}$, respectively.

Definition 4.3. ***Characteristic function of*** $f - a$ ***in D***

$$T(\mathbf{D}, a, f) := m(\mathbf{D}, a, f) + N(\mathbf{D}, a, f) \tag{4.4}$$

5. Properties of Nevanlinna functions in D

As in the Nevanlinna theory for the whole complex plane, we have the following basic results in $(\mathbf{D})$
Let $f_i (i = 1, 2, ..., p)$ be p meromorphic functions in $\overline{\mathbf{D}}$, we have

$$m\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \le \sum_{i=1}^{p} m(\mathbf{D}, a, f_i) + \log p \tag{5.1}$$

$$m\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} m\left(\mathbf{D}, a, f_i\right) \tag{5.2}$$

$$N\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \leq \sum_{i=1}^{p} N(\mathbf{D}, a, f_i) \tag{5.3}$$

$$N\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} N(\mathbf{D}, a, f_i) \tag{5.4}$$

$$T\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \leq \sum_{i=1}^{p} T(\mathbf{D}, a, f_i) + \log p \tag{5.5}$$

$$T\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} T(\mathbf{D}, a, f_i) \tag{5.6}$$

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References

[1] W. K. HAYMAN, *Meromorphic Functions,* Clarendon Press, Oxford, 1964.

[2] C. C. YANG AND H. X. YI, *Uniqueness Theory of Meromorphic Functions,* Kluwer, Dordrecht, 2004.

[3] W. CHERRY, Z. YE, *Nevanlinna's Theory of Value Distribution,* Springer, 2001.

[4] A. A. GOLDBER, I. V. OSTROVSKII, *Value Distribution of Meromorphic Functions,* AMS Translations of Mathematical Monographs, V.236 (2008).

[5] M. A. FEDOROV AND A. F. GRISHIN, Some Questions of the Nevanlinna Theory for the Complex Half-Plane, *Mathematical Physics, Analysis and Geometry*, **1**(1998), 223–271.

[6] J. ZHENG, *Value Distribution of Meromorphic Functions,* Tsinghua University Press, Beijing(Springer), (2010).