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# Study on generalized pseudo conharmonically symmetric manifolds

AKSHOY PATRA *[1]

[1] *Government College of Engineering and Textile Technology, Berhampore, 742101, Dist. Murshidabad, West Bengal, India.*

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**Abstract.** In this paper, a type of Riemannian manifold, namely Generalized pseudo conharmonically symmetric manifold is studied. Several geometric properties of such spaces are studied. By imposing different restrictions on the conharmonic curvature tensor, we have obtained several properties. If the conharmonic curvature tensor is harmonic then the form of the scalar curvature is obtained. Also, the relations among the 1-forms under various conditions are obtained.

**AMS Subject Classifications**: 53C20; 53C21; 53C44.

**Keywords**: Pseudo symmetry, Second Bianchi Identity, Conharmonic curvature tensor, Harmonic curvature tensor.

## Contents



## 1. Introduction

The geometry of a space mainly depends on the curvature of the space. One of the most important geometric properties of a space is symmetry. Cartan began the study of local symmetry of Riemannian spaces and studied elaborately ([2], [3]). According to him, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$. During the last sixty years, the notion of locally symmetric manifolds has been generalized by many authors in a weaker sense. They have weakened in different directions with several defining conditions by giving some curvature restrictions. Various weaker symmetries are studied as generalizations or extensions of Cartan's notion, such as recurrent manifolds by Walker [18], semi-symmetric manifolds by Szabó ([17]), pseudosymmetric manifolds in the sense of Deszcz [7], pseudosymmetric manifolds in the sense of Chaki [4], generalized pseudosymmetric manifolds by Chaki [6], weakly symmetric manifolds by Selberg [11] and weakly symmetric manifolds by Támassy and Binh [16].
According to Chaki a Riemannian manifold is said to be pseudo symmetric if

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= 2\alpha(X)R(Y,Z,U,V)\\ &+ \alpha(Y)R(X,Z,U,V) + \alpha(Z)R(Y,X,U,V)\\ &+ \alpha(U)R(Y,Z,X,V) + \alpha(V)R(Y,Z,U,X)\end{aligned} \tag{1.1}$$

where and $\alpha$ is a 1-form, $X,Y,Z,U,V \in \chi(M)$. Pseudo symmetric manifolds are studied by many authors ([5] , [6], [12]). Ishii [8], introduced the notion of conharmonic transformation under which a harmonic function

*Corresponding author. Email address: akshoyp@gmail.com (Akshoy Patra)

 

transforms into a harmonic function. $\overline{C}$, the conharmonic curvature tensor of type (0,4) on an $(M^n, g)$ is defined as follows

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{1.2}$$

in terms of local coordinates

$$\overline{C}_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}r_{il} - g_{ik}r_{jl} + g_{il}r_{jk} - g_{jl}r_{ik}) \tag{1.3}$$

which remains invariant under conharmonic transformation where $R$ and $S$ are the Riemannian curvature and Ricci curvature tensor respectively. $g \wedge S$ is the Kulkarni-Nomizu product [14].

In [13], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. They also studied it elaborately [15]. The conharmonic curvature tensor has many applications in the theory of general relativity.

In 2021, Ali, Khan and Vasiulla [1] introduced generalized pseudo symmetric manifold and studied various properties. Also in 2017, Kim introduced pseudo semiconformally symmetric manifolds [10] and studied various properties. Motivating by the above studies in this paper, I would like to introduce generalized pseudo conharmonically symmetric manifold, which is defined by

$$\begin{aligned}(\nabla_X \overline{C})(Y, Z, U, V) &= 2\alpha(X)\overline{C}(Y, Z, U, V) \\ &+ \beta(Y)\overline{C}(X, Z, U, V) + \gamma(Z)\overline{C}(Y, X, U, V) \\ &+ \delta(U)\overline{C}(Y, Z, X, V) + \eta(V)R(Y, Z, U, X)\end{aligned} \tag{1.4}$$

where and $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ are 1-forms.
In terms of local coordinates

$$\overline{C}_{ijkl;m} = 2\alpha_m\overline{C}_{ijkl} + \beta_i\overline{C}_{mjkl} + \gamma_j\overline{C}_{imkl} + \delta_k\overline{C}_{ijml} + \eta_l\overline{C}_{ijkm} \tag{1.5}$$

## 2. Generalized pseudo conharmonically symmetric manifolds

**Definition 2.1.** *The conharmonic curvature tensor of a Riemannian manifold $(M^n, g)$ is said to be harmonic if the divergence of the curvature tensor $\overline{C}^i_{jkl}$ of type (1,3) vanishes, i.e.,*

$$\overline{C}^h_{jkl;h} = 0. \tag{2.1}$$

By virtue of second Bianchi identity we have

$$R^h_{jkl;h} = r_{jk;l} - r_{jl;k}. \tag{2.2}$$

And then

$$r^k_{l;k} = \frac{1}{2}s;l. \tag{2.3}$$

We have

$$\overline{C}^h_{jkl;h} = [(\frac{n-3}{n-2})(r_{jk;l} - r_{jl;k}) - \frac{1}{2(n-2)}(g_{jk}s_{;l} - g_{jl}s_{;k})] \tag{2.4}$$

Multiplying (2.4) by $g^{jk}$ and using the condition (2.1) we get

$$0 = (\frac{n-3}{n-2})(s_{;l} - r^k_{l;k}) - \frac{1}{2(n-2)}(ns_{;l} - s_{;l}). \tag{2.5}$$

Which on siplification gives $s_{;l} = 0$, that is the scalar curvature is constant. Hence we have the following:

**Theorem 2.2.** *If the conharmonic curvature tensor of a generalized pseudo conharmonically symmetric Riemannian manifold is harmonic, then the scalar curvature of the space is constant.*

Let the conharmonic curvature tensor of a generalized pseudo conharmonically symmetric Riemannian manifold is harmonic. Then we have

$$0 = 2\alpha_m \overline{C}^m_{jkl} + \beta^m \overline{C}_{mjkl} + \gamma_j \overline{C}^m_{mkl} + \delta_k \overline{C}^m_{mjl} + \eta_l \overline{C}^m_{jkm}. \tag{2.6}$$

Multiplying the above equation by $g^{jk}$ we have

$$\frac{s}{n-2}[2\alpha_l + \beta_l - \delta_l + n\eta_l] = 0. \tag{2.7}$$

If the scalar curvature does not vanishes then we have

$$2\alpha_l + \beta_l - \delta_l + n\eta_l = 0. \tag{2.8}$$

Thus we can state the following:

**Theorem 2.3.** *If the conharmonic curvature tensor of a generalized pseudo conharmonically symmetric Riemannian manifold is harmonic, then either the scalar curvature of the space vanishes or the 1-forms are related by (2.8).*

If the space is pseudo conharmonic symmetric Riemannian manifold then $\alpha = \beta = \gamma = \delta = \eta$ then we have

**Corollary 2.4.** *If the conharmonic curvature tensor of a pseudo conharmonically symmetric Riemannian manifold is harmonic, then the scalar curvature of the space vanishes.*

**Definition 2.5.** *A Riemannian manifold $(M^n, g)$ is said to be recurrent if its curvature tensor $R_{ijkl}$ of type (0,4) satisfies the condition*

$$R_{ijkl;m} = B_m R_{ijkl} \tag{2.9}$$

*where the 1-form $B$ is non zero.*

Multiplying the equation (2.9) by $g^{il}$ and then multiplying by $g^{jk}$ we get

$$r_{jk;m} = B_m r_{jk} \tag{2.10}$$

and then

$$s_{;m} = B_m s \tag{2.11}$$

Using (2.10), (2.11) and (1.3) we get

$$g^{il} g^{jk} \overline{C}_{ijkl;m} = -\frac{n}{n-2} B_m s. \tag{2.12}$$

From (1.5)

$$g^{il} g^{jk} \overline{C}_{ijkl;m} = -\frac{s}{n-2}[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.13}$$

From the above two equations we get

$$B_m = \frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n}. \tag{2.14}$$

Thus we can state that:

**Theorem 2.6.** *If a generalized pseudo conharmonically symmetric Riemannian manifold is recurrent, then the 1-forms $B$, $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ satisfy the relation $B = \frac{2n\alpha+\beta+\gamma+\delta+\eta}{n}$.*

If a generalized pseudo conharmonically symmetric Riemannian manifold is pseudo conharmonically symmetric Riemannian manifold then $\alpha = \beta = \gamma = \delta = \eta$, which yields the following:

**Corollary 2.7.** *If a pseudo conharmonically symmetric Riemannian manifold is recurrent, then the 1-forms $B$ and $\alpha$ are related by $B = \frac{2(n+2)}{n}\alpha$.*

From the Ricci identity and a parallel vector field $V$, it follows that

$$0 = V^t_{;jk} - V^t_{;kj} = V^m R^t_{mjk}. \tag{2.15}$$

Taking covariaint derevative of the above equation we get

$$V^m R^t_{mjk;l} = 0. \tag{2.16}$$

Multiplying by $g_{ti}$ we get

$$V^m R_{imjk;l} = 0 \tag{2.17}$$

Using second Bianchi identity we obtain

$$V^m R_{jkli;m} = 0. \tag{2.18}$$

Multiplying by $g^{ji}$ and then multiplying by $g^{kl}$ we have

$$V^m r_{kli;m} = 0 \tag{2.19}$$

and then

$$V^m s_{;m} = 0. \tag{2.20}$$

Using the above equations it follows that

$$V^m \overline{C}_{ijkl;m} = 0. \tag{2.21}$$

Or,

$$[2\alpha_m \overline{C}^m_{jkl} + \beta^m \overline{C}_{mjkl} + \gamma_j \overline{C}^m_{mkl} + \delta_k \overline{C}^m_{mjl} + \eta_l \overline{C}^m_{jkm}]V^m = 0. \tag{2.22}$$

Or,

$$[\frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n}]V^m = 0. \tag{2.23}$$

Which leads the following

**Theorem 2.8.** *If a generalized pseudo conharmonically symmetric manifold $(M^n, g)$ admits a parallel vector field $V$, then either s=0 or $[\frac{2n\alpha_m+\beta_m+\gamma_m+\delta_m+\eta_m}{n}]V^m = 0$.*

Multiplying (1.5) by $g^{il}$ and then multiplying the relation thus obtained by $g^{jk}$, we obtain

$$-(\frac{s_{;m}}{n-2})n = -(\frac{s}{n-2})[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.24}$$

Or,

$$s_{;m} = \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]}{n} s. \tag{2.25}$$

Taking covariant derivative of (2.25), we get

$$\begin{aligned} s_{;mt} = & \frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n} s \\ & + \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m) s_{;t}]}{n}. \end{aligned} \tag{2.26}$$

Or,

$$\begin{aligned} s_{;mt} = & \frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n} s \\ & + \frac{[(2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)(2n\alpha_t + \beta_t + \gamma_t + \delta_t + \eta_t)] s}{n}. \end{aligned} \tag{2.27}$$

Therefore from the above relation we can write

$$\begin{aligned} 0 = s_{;mt} - s_{;tm} = & \frac{s}{n}[2n(\alpha_{m;t} - \alpha_{t;m}) + (\beta_{m;t} - \beta_{t;m}) \\ & + (\gamma_{m;t} - \gamma_{t;m}) + (\delta_{m;t} - \delta_{t;m}) + (\eta_{m;t} - \eta_{t;m})]. \end{aligned} \tag{2.28}$$

Thus we are in a position to state the following:

**Theorem 2.9.** *Let the scalar curvature of a generalized pseudo conharmonically symmetric Riemannian manifold $(M^n, g)$ does not vanish. If the four 1-forms are closed, then all the 1-forms are closed.*

If the manifold is pseudo conharmonically symmetric manifold then $\alpha = \beta = \gamma = \delta = \eta = A$ then we have from (2.28)

$$0 = s_{;mt} - s_{;tm} = \frac{2n+4}{n} s[A_{m;t} - A_{t;m}]. \tag{2.29}$$

If $s \neq 0$ then

$$A_{m;t} - A_{t;m} = 0. \tag{2.30}$$

Thus we have the following:

**Corollary 2.10.** *If the scalar curvature of a pseudo conharmonically symmetric manifold does not vanish, then the 1-form A is closed.*

## 3. Conclusions

I have studied a new space namely generalized pseudo conharmonically symmetric manifold. Some geometric properties of such spaces are obtained. we have studied the harmonic nature of conharmonic curvature tensor. In future, different properties of these spaces can be obtained by imposing differnt restriction on the Ricci tensor of such spaces.

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