MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(04)**(2023), 470–476.
http://doi.org/10.26637/mjm1104/011

Bounds of regular analogue of Lehmer problem

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*R*eceived 10 July 2023; *A*ccepted 09 September 2023

Abstract. In this paper we consider a new problem analogous to Lehmer's problem concerning n for which $\phi(\mathrm{n}) \mid \mathrm{n}-1$, where Φ be the Euler's totient function. The aim of this paper is to improve the bounds for $\omega(\mathrm{n})$ and n, where $n \in S^A$ and $S^A = \cup_{M>1} S_M^A$.

AMS Subject Classifications: 11A25

Keywords: Lehmer problem, Euler totient function, regular analogue of Lehmer problem.

Contents

1. Introduction

Recall that a real or a complex valued function defined on the set of all natural numbers is called an arithmetic function. The set of all arithmetic functions is denoted by A. For $f, g \in A$, their Dirichlet product $f * g$ and the unitary product $f \circ g$ are the arithmetic functions defined by

$$(f * g)(n) = \sum_{d\delta = n} f(d) g(\delta)$$

and

$$(f \circ g)(n) = \sum_{(d,\delta)=1}^{d\delta = n} f(d) g(\delta)$$

while their addition is defined in the usual way, by

$$(f + g)(n) = f(n) + g(n) \text{ for any } n \geq 1$$

It is well known that $(A, +, *)$ is a commutative ring with unity in which the multiplicativity is preserved under the product. Eckford Cohen [2] has proved that $(A, +, \circ)$ also has the above properties. This has motivated Narkiewicz [5] to introduce a class of binary operations on A as follows:

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For any positive integer n, let A_n denote a non-empty set of positive divisors of n. If $f, g \in A$, their A convolution $f\bar{A}g$, as the arithmetic function defined by

$$(f\bar{A}g)(n) = \sum_{d \in A_n} f(d) g\left(\frac{n}{d}\right)$$

where the summation is over those elements $d \in A_n$ for every n.

For example, for any $n \geq 1$, if D_n is the set of all positive divisors of n and U_n is the set of all positive unitary divisors of n, then for any $f, g \in A$, we have $f\bar{D}g = f * g$ (the Dirichlet convolution of f and g) and $f\bar{U}g = f \circ g$ (the unitary convolution of f and g).

Among the several arithmetic convolutions Narkiewicz [5] has studied those for which $(A, +, A)$ is a commutative ring with unity, which preserves the multiplicativity and also having an inverse relative to A, for the function $u \in A$. (given by $u(n) = 1$ for all n) such convolutions are called regular convolutions. As already noted both the Dirichlet convolution and the unitary convolution are regular.

Many researchers carried out research work in different directions. In [1] Ahmad tried to give an analytical proof for lehmer's totient conjecture using Mertens theorem. More recently Zhaoying Liu and Di Han in [9] studied on Generalization of the Lehmer problem and have given upper bound estimations

In this paper we shall use regular analogue of lehmer problem and give some results. The following preliminaries, will be required in our research.

Narkiewicz [5] has characterized the regular convolution on A as follows:

Lemma 1.1. *([5], p. 82-85) An arithmetic convolution A is regular if and only if the following conditions are satisfied:*

(a) $d \in A_m, m \in A_n \Leftrightarrow d \in A_n, \frac{m}{d} \in A_n/d$.

(b) $d \in A_n \Rightarrow \frac{n}{d} \in A_n$.

(c) $\{1, n\} \subseteq A_n$ *for every* n.

(d) $A_{mn} = A_m \times A_n = \{rs : r \in A_m, s \in A_n\}$ *whenever* $(m, n) = 1$.

(e) For every prime power p^k *we have* $A_{p^\alpha} = \left\{1, p^t, p^{2t}, \cdots, p^{kt}\right\}, rt = k$ *for some positive integer* t *and* $p^t \in A_{p^{2t}}, p^{2t} \in A_{p^{3t}}, \cdots$.

In the rest of the paper, A denotes a regular convolution on A.
A positive integer n is called A-primitive, if $A_n = \{1, n\}$. Clearly, in view of (d) of Lemma 1.5, it is easy to see that every primitive number is the power of a prime number. Also it may be noted that the primitive numbers with respect to the Dirichlet convolutions are the primes while the primitive numbers with respect to the unitary convolution are precisely the prime powers.

For a prime power p^k, the least positive integer t such that $p^t \in A_{p^k}$ is called the type of p^k relative to A and is denoted by $\tau_A\left(p^k\right)$. It is easy to see that if $t = \tau_A\left(p^k\right)$. then t divides k and that p^t is A-primitive.

The inverse of the function $u \in A$ defined by $u(n) = 1$ for all n relative to A is denoted by μ_A. That is, $\mu_A \in A$ is such that $\left(u\bar{A}\mu_A\right) = \left(\mu_A\bar{A}u\right) = \varepsilon$ where $\varepsilon \in A$ is given by $\varepsilon(n) = 1$ or 0 according as $n = 1$ or $n > 1$. In other words

$$\sum_{d \in A_n} \mu_A(d) = \varepsilon(n) \text{ for every } n \geq 1.$$

McCarthy [4] has introduced the regular analogue $\phi_A(n)$ of the Euler totient function by

$$\varphi_A(n) = \sum_{d \in A_n} d\mu_A\left(\frac{n}{d}\right).$$

Clearly $\phi_A(n)$ is multiplicative and

$$\phi_A\left(p^k\right) = p^k - p^{k-t} \quad \text{where } t = \tau_A\left(p^k\right).$$

If n is A-primitive then it can be seen easily that $\phi_A(n)$ divides $n-1$. What about the converse of this ? That is, does $\phi_A(n)$ divides $n-1$ imply n is A-primitive. In other words, if

$$S_M^A = \{n : M\phi_A(n) = n-1\}$$

then, is there a non-A-primitive integer in S_M^A, for $M > 1$? Siva Rama Prasad and Subbarao [7] have considered this problem and proved among other things, the following:

S_1^A is the set of all A-primitive numbers.

$$S^A = \cup_{M>1} S_M^A$$

The regular analogue of the Lehmer problem is only to search for non-A-primitive elements in the set S^A. It has been proved that

"If $n \in S^A$ then n is odd and is a product of A-primitive numbers, the type of at least one of which is unity. (1.1)

Suppose $n \in S^A$ has the representation " $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $p_1 < p_2 < \cdots < p_r$ and $t_i = \tau_A\left(p_i^{\alpha_i}\right)$ for $i = 1, 2, 3, \ldots, r$".

Then,

$$p_i^{\alpha_i} \not\equiv 1\,(\text{mod} p_j) \text{ for} i \neq j \text{ and in particular} p_i \not\equiv 1\,(\text{mod} p_j). \tag{1.2}$$

$$If\ n \in S^A \quad and\ 3 \mid n\ then\ \omega(n) \geq 1850. \tag{1.3}$$

$$If\ n \in S^A, 3 \{ n\ and\ 5 \mid n\ then\ \omega(n) \geq 11. \tag{1.4}$$

$$If\ \ n \in S^A, 3\}' n\ and\ 5 \mid n\, then\, \omega(n) \geq 17. \tag{1.5}$$

As a consequence of results proved in ([7], Theorem 3.7) we have

$$If\ n \in S^A, 2 < \omega(n) \leq 16\ then M = 2, 3 \nmid n, 5 \mid n\ and\, 7 \mid n. \tag{1.6}$$

Summarizing the results we have that

$$\omega(n) \geq 11 \text{ for } n \in S^A \tag{1.7}$$

and

$$n < (r-1)^{2^r - 1} \quad \text{for} n \in S^A \quad \text{and} \quad \omega(n) = r. \tag{1.8}$$

The purpose of this paper is to improve (1.7) and (1.8) partially by establishing that, $\omega(n) \geq 17$ and $n < \left(r - \frac{23}{10}\right)^{2^r - 1}$ whenever $n \in S^A$ and 455 is not a unitary divisor of n.

2. Preliminaries

In this section we present the preliminary results that are needed to prove the main theorems. Suppose $n \in S^A$, so that $n \in S_M^A$ for some $M > 1$. Then $M\phi_A(n) = n - 1$ which gives $(n, M) = 1$ and $(n, \phi_A(n)) = 1$. Also $\frac{n}{\varphi_A(n)} > M \geq 2$ so that

$$2 < \frac{n}{\varphi_A(n)} \text{ for all } n \in S^A. \tag{2.1}$$

Throughout this paper, unless otherwise mentioned, n denotes a natural number in S^A with canonical representation

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \text{ where we can assume } p_1 < p_2 < \dots p_r. \tag{2.2}$$

Then by (2.1) we have

$$2 < \prod_{i=1}^{r} \frac{p_i^{k_i}}{p_i^{k_i} - 1}. \tag{2.3}$$

and also by (1.1), at least one $k_i = 1$. Further if $p_1 = 5$ and $p_2 = 7$ then in view of (1.2), we have that $p_i \equiv 1(\text{mod} 5)$ and $p_i \equiv 1(\text{mod} 7)$.

Suppose $\boldsymbol{B}$ is the set of primes containing 5 and 7 and those primes p with $p \not\equiv 1(\bmod 5)$ and $p \not\equiv 1(\bmod 7)$. That is,

$$B = \begin{cases} 5, 7, 13, 17, 19, 23, 37, 47, 53, 59 \\ \quad 67, 73, 79, 83, 89, 97, 103, 107, 109, \dots\} \end{cases} \tag{2.4}$$

The t^{th} element in increasing order of this set is denoted by b_i That is, $b_1 = 5, b_2 = 7, b_3 = 13, \dots$.

The following lemmas proved in [6] are used in the later parts of this paper.

Lemma 2.1. *([6], Lemma 4.3): If $n \in S_M^A$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}$, then*

$$p_i^{\alpha_i} < (r - i + 1) \left(\prod_{j=1}^{i-1} p_j^{\alpha_j} \right)$$

for $i = 2, 3, \dots, r$.

Lemma 2.2. *([6], Lemma 4.4): If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}$ is such that $\frac{n}{\varphi_A(n)} > 2$, then $p_1^{\alpha_1} < 2 + 2\left(\frac{r}{3}\right)$.*

Also we use the fact that

$$\frac{x}{x-1} \text{ is a decreasing function for } x > 1. \tag{2.5}$$

3. Main Results

In this section we first prove the following theorem which improves (1.8) partially.

Theorem 3.1. *If $n \in S^A$ and 455 is not a unitary divisor of n, then $\omega(n) \geq 17$.*

Proof. If $3 \mid n$, there is nothing to prove, in view of (1.3). If $3 \nmid n$ and $5 \nmid n$, (1.5) proves the theorem.

Therefore we may assume that $3 \nmid n$ and $5 \mid n$. If possible, $\omega(n) \leq 16$. Then by (1.6), $n \in S_2^A, 31n, 5 \mid n$ and $7 \mid n$, so that $p_1 = 5$ and $p_2 = 7$ in (2.2). Also in view of (1.2), $p_i(3 \leq i \leq r)$ should be such that $p_i \not\equiv 1(\bmod 5)$ and $p_i \not\equiv 1(\bmod 7)$ so that each such p_i is from the set $\boldsymbol{B}$, given as in (2.4). Then n is of the form

$$n = 5^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}, \text{ where } p_i \in B \text{ for } i \geq 3.$$

First assume that $r = 16$.
Case (i): Let $p_3 \neq b_3$ so that $13 \nmid n$ and hence $p_3 \geq b_4, p_4 \geq b_5, \ldots, p_i \geq b_{i+1}$ for $i = 3, 4, \ldots$. Then we have by (2.5) that

$$\begin{aligned} \frac{n}{\varphi_A(n)} &= \frac{5^{\alpha_1}}{5^{\alpha_1} - 1} \cdot \frac{7^{\alpha_2}}{7^{\alpha_2} - 1} \prod_{i=3}^{16} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1} \\ &< \frac{5}{4} \frac{7}{6} \prod_{i=3}^{16} \frac{b_{i+1}}{b_{i+1} - 1} < 2 \end{aligned}$$

contradicting (2.1).
Case (ii): Suppose $p_3 = b_3$, so that $13 \mid n$ and (3.2) can be written as

$$n = 5^{\alpha_1} 7^{\alpha_2} 13^{\alpha_3} p_4^{\alpha_4} \cdots p_{16}^{\alpha_{16}}.$$

Then, since $13 \mid n$ and $13^2 \equiv 1(\bmod 7)$ we get, by (1.14), that $2 \nmid \alpha_3$ and hence α_3 is odd. Also by (1.14), we have $p_{10} \geq b_{11}, p_{11} \geq b_{12}, p_{12} \geq b_{13}$, $p_{13} \geq b_{15}, p_{14} \geq b_{16}, p_{15} \geq b_{17}$ and $p_{16} \geq b_{18}$. Further since 455 is not unitary divisor of n, we have $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 > 1$. In the case $\alpha_1 = \alpha_2 = 1$, we must have $\alpha_3 \geq 3$. So that $\frac{b_3^{\alpha_3}}{b_3^{\alpha_3 - 1}} < \frac{13^3}{13^3 - 1} = \frac{2197}{2196}$ and therefore

$$\frac{n}{\varphi_A(n)} < \frac{5}{4} \frac{7}{6} \frac{2197}{2196} \prod_{i=4}^{16} \frac{b_i}{b_i - 1} < 2 \tag{3.1}$$

contradicting (2.1).

If one or both of α_1 and α_2 exceed 1 and $\alpha_3 = 1$, then

$$\frac{n}{\varphi_A(n)} < \frac{5}{4} \frac{7}{6} \frac{13}{12} \prod_{i=4}^{16} \frac{b_i}{b_i - 1} < 2, \tag{3.2}$$

and the right of this inequality does not exceed the product on the right of (3.1). Hence $\frac{n}{\varphi_A(n)} < 2$ in this case also, contradicting (2.1). The case of $\alpha_1 > 1, \alpha_2 > 1$ and $\alpha_3 > 1$ can be handled similarly.

Thus $r = 16$ cannot hold. If $r < 16$ then $\frac{n}{\varphi_A(n)}$ should not exceed the right of (3.1) or of (3.2) and in any case $\frac{n}{\varphi_A(n)} < 2$. Hence $\omega(n) > 16$, proving the theorem.

■

Remark 3.2. *Theorem 3.1 is clearly a significant improvement of* (1.7).

Theorem 3.3. *If $n \in S^A$ with $\omega(n) = r$ and 455 does not divide n unitarily then $n < \left(r - \frac{23}{10}\right)^{2^r - 1}$.*

Proof. Suppose $n \in S^A$ with $\omega(n) = r$ and 455 does not divide n unitarily. Then we can write n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where we may assume that $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}$. By Theorem 3.1, we have $r \geq 17$ while Lemma 2.6 gives $p_1^{\alpha_1} < \frac{2}{3}r + 2$. Therefore

$$p_1^{\alpha_1} < r - \frac{18}{5}, \text{ for } r \geq 17. \tag{3.3}$$

Now by Lemma 2.1 and (3.3), we successively have

$$\begin{aligned}
p_1^{\alpha_1} &< r - \frac{18}{5} < \left(r - \frac{23}{10}\right) \\
p_2^{\alpha_2} &< (r-1)p_1^{\alpha 1} < (r-1)\left(r - \frac{18}{5}\right) < \left(r - \frac{23}{10}\right)^2 \\
p_3^{\alpha_3} &< (r-2)p_1^{\alpha 1} p_2^{\alpha_2} < \left(r - \frac{23}{10}\right)^{2^2} \\
&\cdots
\end{aligned}$$

$$p_r^{\alpha_r} < \left(r - \frac{23}{10}\right)^{2^{r-1}}.$$

Multiplying all these inequalities, we get

$$n < \left(r - \frac{23}{10}\right)^{2^r - 1}.$$

proving the theorem.

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Remark 3.4. *We observe that Theorem 3.8 significantly improves* (1.8).

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