MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(02)**(2023), 167–180.
http://doi.org/10.26637/mjm1102/006

Common fixed point theorem for set of quasi triangular α-orbital admissible mappings in complete metric space with application

Rakesh Tiwari[1] and Shashi Thakur[*2]

[1] *Department of Mathematics, Government V. Y. T. Post-Graduate Autonomous College, Durg 491001, Chhattisgarh, India.*
[2] *Department of Mathematics, C. L. C. Government Arts and Commerce College, Dhamdha, Chhattisgarh, India.*

*R*eceived 28 June 2022; *A*ccepted 22 March 2023

Abstract. The purpose of this paper is to construct a common fixed point theorem for pair of quasi triangular α-orbital admissible with an interpolative (φ, ψ)- Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction mappings with reference to simulation function in complete metric space. We adopt an example to validate our main result. Our result extends the result of M. S. Khan et al. [15]. As an application, we provide the existence of a solution for a nonlinear Fredholm integral equations.

AMS Subject Classifications: 47H10, 54H25.

Keywords: Interpolative (φ, ψ)-type $\mathcal{Z}$-contraction, altering distance function, comparison function, simulation function, quasi triangular α-orbital admissible mappings.

Contents

1. Introduction

The Banach contraction principle is pivotal tools in fixed point theory. Many inventors expanded and generalized the Banach contraction principle to many orientations [3, 5, 24, 27, 28]. Samet et al. [25] found the conception of $\alpha - \psi$ contraction type mapping and take advantage of their new concept to established and found several fixed point theorems. Several inventors used the concept of α-admissible mapping to established new results in many spaces [10, 21, 22, 26, 30]. In 2014, Popescu [20] found the two new concept α-orbital admissible and triangular α-orbital admissible and gave the result each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is an triangular α-orbital admissible mapping. Many inventors gave the fixed point and common fixed point result for α-orbital admissible mapping [1, 7, 9, 18, 19]. In 2015, Khojasteh et al.[17] found the notion of simulation function. In the same year, Argoubi et al. [6] clarified the conception of simulation function. Many inventors found the fixed point and common fixed point result for simulation function in discrete spaces [2, 4, 11, 12, 14, 23, 29].

*Corresponding author.** Email address: **shashithakur89520@gmail.com** (Shashi Thakur)

2. Preliminaries

We recall some useful definitions that will be needed in the sequel.

Definition 2.1. *[25] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is α-admissible if $\alpha(u, v) \geq 1$ implies $\alpha(Qu, Qv) \geq 1$.*

Definition 2.2. *[13] Let $Q : Y \to Y$ be a function and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be triangular α-admissible if Q fulfills the following conditions:*

1. *Q is α-admissible,*
2. *if $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$ implies $\alpha(u, v) \geq 1$.*

Qawagneh et al. [22] introduced the notion of triangular α-admissible for set of self mappings on Y.

Definition 2.3. *[22] Let $H, Q : Y \to Y$ be two mappings and $\alpha : Y \times Y \to [0, \infty)$ be a function such that the following conditions hold:*

1. *if $\alpha(u, v) \geq 1$ then $\alpha(Hu, Qv) \geq 1$ and $\alpha(QHu, HQv) \geq 1$;*
2. *if $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$ implies $\alpha(u, v) \geq 1$.*

Then we say that the pair (H, Q) is triangular α-admissible.

Definition 2.4. *[20] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be α-orbital admissible if $\alpha(u, Qu) \geq 1$ implies $\alpha(Qu, Q^2u) \geq 1$.*

Definition 2.5. *[20] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be triangular α-orbital admissible if Q satisfies the following conditions:*

1. *if Q is α-orbital admissible,*
2. *if $\alpha(u, v) \geq 1$ and $\alpha(v, Qv) \geq 1$ implies $\alpha(u, Qv) \geq 1$.*

Definition 2.6. *[19] Let $H, Q : Y \to Y$ be two mappings and $\alpha_s : Y \times Y \to [0, \infty)$ be a function such that the following condition hold:*

1. *if $\alpha_s(u, Qu) \geq s^2$ and $\alpha_s(u, Hu) \geq s^2$ then $\alpha_s(Qu, HQu) \geq s^2$ and $\alpha_s(Hu, QHu) \geq s^2$.*

Then the set (H, Q) is α_s-orbital admissible.

Definition 2.7. *[19] Let $H, Q : Y \to Y$ be two mappings and $\alpha_s : Y \times Y \to [0, \infty)$ be a function such that the following conditions hold:*

1. *the self mappings H, Q are α_s-orbital admissible,*
2. *if $\alpha_s(u, v) \geq s^2$, $\alpha_s(v, Hv) \geq s^2$ and $\alpha_s(v, Qv) \geq s^2$ implies $\alpha_s(u, Hv) \geq s^2$ and $\alpha_s(u, Qv) \geq s^2$.*

Then the set (H, Q) is triangular α_s-orbital admissible.

M. S. Khan et al. [15] introduced the concept of quasi triangular α-orbital admissible mappings as follows:

Definition 2.8. *[15] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be quasi triangular α-orbital admissible if Q satisfies the following conditions:*

1. *if Q is α-orbital admissible,*
2. *if $\alpha(u, v) \geq 1$ implies $\alpha(u, Qv) \geq 1$.*

Definition 2.9. *[17] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function, if it fulfils the following conditions:*

1. $\zeta(0,0) = 0$;
2. $\zeta(v,u) < u - v$ *for all* $u, v > 0$;
3. *if* $\{v_n\}$, $\{u_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} u_n > 0$, *then* $\lim_{n\to+\infty} sup\, \zeta(v_n, u_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Definition 2.10. *[17] Let (Y,d) be a metric space and $Q : Y \to Y$ be mapping. if there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Qu,Qv), d(u,v)) \geq 0.$$

for all $u, v \in Y$. Then Q is called $\mathcal{Z}$-contraction with respect to ζ.

Definition 2.11. *[16] A continuous function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance if it is non-decreasing and $\varphi(l) = 0$ if and only if $l = 0$.*

Definition 2.12. *[8] A function $\psi : [0,\infty) \to [0,\infty)$ is called comparison function if it is monotonically increasing and $\psi^n(l) \to 0$ as $n \to \infty$ for all $l > 0$.*

M. S. Khan et al.[15] gave (φ,ψ)-type $\mathcal{Z}$-contraction with respect to simulation function ζ using an interpolative (φ,ψ) approach in the setting of metric spaces as follows:

Definition 2.13. *[15] A mapping $Q : Y \to Y$ is called an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ if there exists $\alpha : Y \times Y \to \mathbb{R}, \zeta \in \mathcal{Z}, \varphi \in \Phi, \psi \in \Psi, \theta_1, \theta_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1 + \theta_2 < 1$ fulfilling the inequality*

$$\zeta(\alpha(u,v)\varphi(d(Qu,Qv)), \psi(B(u,v))) \geq 0 \;\; for\ all\ u,\ v \in Y,$$

where

$$B(u,v) = [d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Qu) + d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv) + d(v,Qu))]^{1-\theta_1-\theta_2}$$

In this paper, we construct a common fixed point theorem for set of quasi triangular α-orbital admissible mappings which form an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with reference to simulation function in complete metric space.

3. Main Result

In this section, we introduced the conception of quasi triangular α-orbital admissible mapping for set of self mappings H and Q on Y and discuss (φ,ψ)-type $\mathcal{Z}$-contraction with reference to simulation function.

Definition 3.1. *Let $H, Q : Y \to Y$ be two mappings and $\alpha : Y \times Y \to [0,\infty)$ be a function such that the following conditions hold.*

1. *if* $\alpha(u,Qu) \geq 1$ *and* $\alpha(u,Hu) \geq 1$ *then* $\alpha(Qu,HQu) \geq 1$ *and* $\alpha(Hu,QHu) \geq 1$;
2. *if* $\alpha(u,v) \geq 1$ *implies* $\alpha(u,Qv) \geq 1$ *and* $\alpha(u,Hv) \geq 1$.

Then the pair (H,Q) is called quasi triangular α-orbital admissible.

In the following example shows that the mapping (H, Q) is quasi triangular α-orbital admissible but it is not a triangular α-admissible.

Example 3.2. *Let $Y = \{0, 1, 2\}$ with usual metric $d(u, v) = |u - v|$. Let $H : Y \to Y$, $Q : Y \to Y$ and $\alpha : Y \times Y \to \mathbb{R}$ be mappings defined by*

$$HY = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, QY = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \alpha(u, v) = \begin{cases} 1, & \text{if } (u, v) \in A, \\ 0, & \text{otherwise} \end{cases}$$

where, $A = \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2)\}$. Since $(0, 1), (1, 0) \in A$, then we have
$\alpha(0, Q0) = \alpha(Q0, HQ0) = \alpha(1, 0) = 1$, $\alpha(0, H0) = \alpha(H0, QH0) = \alpha(1, 0) = 1$ and
$\alpha(1, Q1) = \alpha(Q1, HQ1) = \alpha(0, 1) = 1$, $\alpha(1, H1) = \alpha(H1, QH1) = \alpha(0, 1) = 1$. Then (H, Q) is α-orbital admissible mappings. Further, we have

$\alpha(0, 1) = \alpha(0, Q1) = \alpha(0, 0) = 1$ *and* $\alpha(0, 1) = \alpha(0, H1) = \alpha(0, 0) = 1$,
$\alpha(1, 0) = \alpha(1, Q0) = \alpha(1, 1) = 1$ *and* $\alpha(1, 0) = \alpha(1, H0) = \alpha(1, 1) = 1$
$\alpha(1, 2) = \alpha(1, Q2) = \alpha(1, 2) = 1$ *and* $\alpha(1, 2) = \alpha(1, H2) = \alpha(1, 0) = 1$.

Hence, (H, Q) is quasi triangular α-orbital admissible mappings. Since $\alpha(u, v) = \alpha(1, 2) = 1$, $\alpha(v, Qv) = \alpha(2, Q2) = \alpha(2, 2) = 0$ and $\alpha(v, Hv) = \alpha(2, H2) = \alpha(2, 0) = 0$ but $\alpha(1, 2) = \alpha(1, Q2) = \alpha(1, 2) = 1$ and $\alpha(1, 2) = \alpha(1, H2) = \alpha(1, 0) = 1$. This shows that the condition $\alpha(v, Qv)$ and $\alpha(v, Hv)$ for triangular α-orbital admissible are not necessity for quasi triangular α-orbital admissible. On the other hand, we have $\alpha(1, 2) = 1, \alpha(H1, Q2) = \alpha(0, 2) = 0$ and $\alpha(QH1, HQ2) = \alpha(1, 0) = 1$ as $(0, 2) \nexists Y$, so (H, Q) is not α-admissible mapping. Further, we have $\alpha(0, 1) = \alpha(1, 2) = 1$, but $\alpha(0, 2) = 0$, so (H, Q) is not triangular α-admissible mapping.

Lemma 3.3. *Let $H, Q : Y \to Y$ be two mappings and $\alpha : Y \times Y \to [0, \infty)$ such that the set (H, Q) is quasi triangular α-orbital admissible. Assume that there exists $u_0 \in Y$ in this manner $\alpha(u_0, Hu_0) \geq 1$. Define a sequence $\{u_n\}$ in Y by $Hu_{2n} = u_{2n+1}$ and $Qu_{2n+1} = u_{2n+2}$. Then $\alpha(u_n, u_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Proof. Since $\alpha(u_0, Hu_0) = \alpha(u_0, u_1) \geq 1$ and H, Q are α-orbital admissible self mappings,

$$\begin{aligned} \alpha(u_0, Hu_0) &\geq 1 \quad implies \\ \alpha(Hu_0, QHu_0) &= \alpha(u_1, Qu_1) = \alpha(u_1, u_2) \geq 1 \\ and \;\; \alpha(u_1, Qu_1) &\geq 1 \quad implies \\ \alpha(Qu_1, HQu_1) &= \alpha(u_2, Hu_2) = \alpha(u_2, u_3) \geq 1 \\ also \;\; \alpha(u_2, Hu_2) &\geq 1 \quad implies \\ \alpha(Hu_2, QHu_2) &= \alpha(u_3, Qu_3) = \alpha(u_3, u_4) \geq 1 \end{aligned}$$

Applying the above argument repeatedly, we obtain $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since (H, Q) is quasi triangular α-orbital admissible mapping and $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then we get $\alpha(u_n, Qu_{n+1}) = \alpha(u_n, u_{n+2}) \geq 1$ and $\alpha(u_n, Hu_{n+1}) = \alpha(u_n, u_{n+2}) \geq 1$. By continuing the process, we get that $\alpha(u_n, u_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

Definition 3.4. *The mappings $H, Q : Y \to Y$ are called an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ if there exists $\alpha : Y \times Y \to \mathbb{R}, \zeta \in \mathcal{Z}, \varphi \in \Phi, \psi \in \Psi, \theta_1, \theta_2, \theta_3 \in (0, 1)$ in this manner $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1 + \theta_2 + \theta_3 < 1$ fulfilling the inequality*

$$\zeta(\alpha(u, v)\varphi(d(Hu, Qv)), \psi(B(u, v))) \geq 0 \;\; for \; all \; u, v \in Y, \tag{3.1}$$

where

$$B(u,v) = [d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Hu)+d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv)+d(v,Qu))]^{\theta_3}.$$
$$[\frac{1}{2}(d(u,Hv)+d(v,Hu))]^{1-\theta_1-\theta_2-\theta_3}$$

Now, we state and prove our main results as follows:

Theorem 3.5. *Let H and Q be self mappings on a metric space (Y,d) which is complete. Suppose that (H,Q) is a quasi triangular α-orbital admissible and forms an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$ and H and Q are continuous, then the mappings H and Q have a unique common fixed point.*

Proof. Let $u_0 \in Y$ be such that $\alpha(u_0, Hu_0) \geq 1$. Define a sequence $\{u_n\}$ in Y such that $u_{2n+1} = Hu_{2n}$ and $u_{2n+2} = Qu_{2n+1}$ for all $n \in \mathbb{N}$. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then it is very easy to show that H and Q have a common fixed point. Hereof, uniquitously the proof we shall assume that $u_n \neq u_{n+1}$ and hence we have $d(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. Now, since the pair (H,Q) is α-orbital admissible, then

$$\begin{aligned}
\alpha(u_0, Hu_0) &\geq 1 \quad implies \\
\alpha(Hu_0, QHu_0) &= \alpha(u_1, Qu_1) = \alpha(u_1, u_2) \geq 1 \\
and \;\; \alpha(u_1, Qu_1) &\geq 1 \quad implies \\
\alpha(Qu_1, HQu_1) &= \alpha(u_2, Hu_2) = \alpha(u_2, u_3) \geq 1 \\
also \;\; \alpha(u_2, Hu_2) &\geq 1 \quad implies \\
\alpha(Hu_2, QHu_2) &= \alpha(u_3, Qu_3) = \alpha(u_3, u_4) \geq 1
\end{aligned}$$

Applying the above argument repeatedly, we get $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By the definition of quasi triangular α-admissibility, we can find that for any $n, m \in \mathbb{N}$ with $m > n$, we have $\alpha(u_n, u_m) \geq 1$.

Suppose $u_{2n} \neq u_{2n+1}$ for all $n \in \mathbb{N}$, by Lemma 3.3, we have $\alpha(u_{2n}, u_{2n+1}) \geq 1$, for all $n \in \mathbb{N}$. From (3.1), we obtain

$$\begin{aligned}
0 &\leq \zeta\Big(\alpha(u_{2n}, u_{2n+1})\varphi(d(Hu_{2n}, Qu_{2n+1})), \psi(B(u_{2n}, u_{2n+1}))\Big) \\
&= \zeta\Big(\alpha(u_{2n}, u_{2n+1})\varphi(d(u_{2n+1}, u_{2n+2})), \psi(B(u_{2n}, u_{2n+1}))\Big) \\
&< \psi(B(u_{2n}, u_{2n+1})) - \alpha(u_{2n}, u_{2n+1})\varphi(d(u_{2n+1}, u_{2n+2}))
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
B(u_{2n}, u_{2n+1}) &= [d(u_{2n}, u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n}, Hu_{2n}) + d(u_{2n+1}, Qu_{2n+1}))]^{\theta_2}.[\frac{1}{2}(d(u_{2n}, Qu_{2n+1}) \\
&\quad + d(u_{2n+1}, Qu_{2n}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n}, Hu_{2n+1}) + d(u_{2n+1}, Hu_{2n}))]^{1-\theta_1-\theta_2-\theta_3} \\
&= [d(u_{2n}, u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n}, u_{2n+1}) + d(u_{2n+1}, u_{2n+2}))]^{\theta_2} \\
&\quad .[\frac{1}{2}(d(u_{2n}, u_{2n+2})]^{1-\theta_1-\theta_2}
\end{aligned} \tag{3.3}$$

Consequently, we arrive

$$\begin{aligned}\varphi(d(u_{2n+1},u_{2n+2})) &\leq \alpha(u_{2n},u_{2n+1})\varphi(d(u_{2n+1},u_{2n+2}))\\ &< \psi(B(u_{2n},u_{2n+1}))\\ &= \psi\big([d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2}))]^{\theta_2}.\\ &\quad [\frac{1}{2}(d(u_{2n},u_{2n+2})]^{1-\theta_1-\theta_2}\big)\\ &\leq \psi\big([d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2}))]^{1-\theta_1}\big). \end{aligned} \tag{3.4}$$

Suppose $d(u_{2n},u_{2n+1}) < d(u_{2n+1},u_{2n+2})$, for $n \geq 1$, then from (3.4), we obtain

$$\varphi(d(u_{2n+1},u_{2n+2})) \leq \psi(d(u_{2n+1},u_{2n+2})) < \varphi(d(u_{2n+1},u_{2n+2})).$$

This is a contradiction. Accordingly, we obtain

$$d(u_{2n+1},u_{2n+2}) \leq d(u_{2n},u_{2n+1}),\ for\ all\ n \geq 1.$$

Identically, we can show that $d(u_{2n},u_{2n+1}) \leq d(u_{2n-1},u_{2n})$. So, we conclude that $d(u_n,u_{n+1}) \leq d(u_{n-1},u_n)$. Hence $d(u_n,u_{n+1})$ is a monotonic decreasing sequence of positive real numbers. So, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(u_n,u_{n+1}) = l$. Now, we show that $l = 0$. We claim that $l > 0$. Now, we have

$$\begin{aligned}0 &\leq \zeta\Big(\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})),\psi(B(u_{n-1},u_n))\Big)\\ &< \psi(B(u_{n-1},u_n)) - \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})).\end{aligned} \tag{3.5}$$

Consequently, we obtain

$$\begin{aligned}\varphi(d(u_n,u_{n+1}) &\leq \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})) \leq \psi(B(u_{n-1},u_n))\\ &\leq \varphi(B(u_{n-1},u_n))\\ &\leq \varphi(d(u_{n-1},u_n))\end{aligned} \tag{3.6}$$

Letting limit as $n \to +\infty$ in (3.6), we get

$$lim_{n\to+\infty}\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})) = lim_{n\to+\infty}\psi(B(u_{n-1},u_n)) = \varphi(l). \tag{3.7}$$

Setting $s_n = \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1}))$, $t_n = \psi(B(u_{n-1},u_n))$ in (3.5), then by definition of simulation function

$$0 \leq lim_{n\to+\infty}sup\zeta(\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})),\psi(B(u_{n-1},u_n))) < 0.$$

Which is a contradiction and thus we have $\lim_{n\to+\infty} d(u_n,u_{n+1}) = 0$. Now, we show that $\{u_n\}$ is a Cauchy sequence. Suppose not, there exists $\epsilon > 0$ for which we can find two sequences m_k and n_k, for all $k \geq 1$ with $u_{m_k} > u_{n_k} \geq k$ such that $d(u_{n_k},u_{m_k}) \geq \epsilon$. Further, we assume that m_k is the smallest number greater than n_k, then $d(u_{n_k},u_{m_{k-1}}) < \epsilon$.

By triangular inequality, we get

$$\epsilon \leq d(u_{n_k},u_{m_k}) \leq d(u_{n_k},u_{m_{k-1}}) + d(u_{m_{k-1}},u_{m_k}) < \epsilon + d(u_{m_{k-1}},u_{m_k}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(u_{n_k},u_{m_k}) = \epsilon. \tag{3.8}$$

Again by triangular inequality, we obtain

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_{k+1}}) + d(u_{n_{k+1}}, u_{m_{k+1}}) + d(u_{m_{k+1}}, u_{m_k}).$$

Also we obtain

$$d(u_{n_{k+1}}, u_{m_{k+1}}) \leq d(u_{n_{k+1}}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_{k+1}}).$$

By using the above two inequalities and taking limit as $k \to +\infty$ with (3.8), we get

$$\lim_{k\to+\infty} d(u_{n_{k+1}}, u_{m_{k+1}}) = \epsilon. \tag{3.9}$$

Furthermore, we obtain

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_{k+1}}) + d(u_{n_{k+1}}, u_{m_k}) \leq d(u_{n_k}, u_{m_k}) + 2d(u_{m_k}, u_{m_{k+1}}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(u_{n_{k+1}}, u_{m_k}) = \epsilon. \tag{3.10}$$

Similarly, we get

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{m_{k+1}}) + d(u_{m_{k+1}}, u_{m_k}) \leq d(u_{n_k}, u_{m_k}) + 2d(u_{m_k}, u_{m_{k+1}}).$$

Taking limit as $k \to +\infty$, we get

$$\lim_{k\to+\infty} d(u_{n_k}, u_{m_{k+1}}) = \epsilon. \tag{3.11}$$

Since (H, Q) is quasi triangular α-orbital admissible, by lemma 3.3, we get $B(u_{n_k}, u_{m_k}) \geq 1$, for all numbers m_k, n_k such that $m_k > n_k$, where $k \geq 1$. From (3.1), we get

$$\begin{aligned}
0 &\leq \zeta\Big(\alpha(u_{n_k}, u_{m_k})\varphi(d(Hu_{n_k}, Qu_{m_k}), \psi(B(u_{n_k}, u_{m_k}))\Big)\\
&= \zeta\Big(\alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}), \psi(B(u_{n_k}, u_{m_k}))\Big)\\
&< \psi(B(u_{n_k}, u_{m_k})) - \alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}) &\leq \alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}})\\
&\leq \psi(B(u_{n_k}, u_{m_k})) < \varphi(B(u_{n_k}, u_{m_k})),
\end{aligned}$$

where

$$\begin{aligned}
B(u_{n_k}, u_{m_k}) = [d(u_{n_k}, u_{m_k})]^{\theta_1}.[\frac{1}{2}(d(u_{n_k}, Hu_{n_k}) + d(u_{m_k}, Qu_{m_k}))]^{\theta_2}.[\frac{1}{2}(d(u_{n_k}, Qu_{m_k})\\
+ d(u_{m_k}, Qu_{n_k}))]^{\theta_3}.[\frac{1}{2}(d(u_{n_k}, Hu_{m_k}) + d(u_{m_k}, Hu_{n_k}))]^{1-\theta_1-\theta_2-\theta_3}
\end{aligned}$$

Taking limit as $k \to +\infty$ together with (3.8), (3.9), (3.10) and (3.11), we get

$$0 \leq \varphi(\epsilon) < \varphi(0) = 0 \Rightarrow \varphi(\epsilon) = 0 \; if \; and \; only \; if \; \epsilon = 0.$$

Which is a contradiction and hence $\{u_n\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $w \in Y$ such that $\lim_{n\to\infty} u_n = w$. Since H and Q are continuous, we find that $Hw = \lim_{n\to\infty} Hu_n = \lim_{n\to\infty} u_{n+1} = w$ and $Qw = \lim_{n\to\infty} Qu_n = \lim_{n\to\infty} u_{n+1} = w$. Therefore w is the common fixed point of H and Q.

To demonstrate the uniqueness of the common fixed point, we suppose that w^* is another common fixed point of H and Q and $\alpha(w,w^*) \geq 1$. Assume that $w \neq w^*$. From (3.1), we get

$$\begin{aligned}\zeta(\alpha(w,w^*)\varphi(d(Hw,Qw^*)),\psi(B(w,w^*))) &\geq 0\\ \zeta(\alpha(w,w^*)\varphi(d(w,w^*)),\psi(B(w,w^*))) &\geq 0\\ \psi(B(w,w^*))-\alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0\\ -\alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0.\end{aligned}$$

Which is contradiction and therefore the mappings H and Q have a unique common fixed point.

Remark 3.6. *For $H = Q$ in Theorem 3.5, we get the following result of M. S. Khan et al.[15]*

Corollary 3.7. *Let Q be a self mapping on a metric space (Y,d) which is complete. Suppose that Q is quasi triangular α-orbital admissible and forms an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0,Qu_0) \geq 1$ and Q is continuous, then Q has a unique fixed point.*

Remark 3.8. *Setting $\zeta(u,v) = \psi(v) - u$ for all $u,v > 0$ in Theorem 3.5, we get the following result.*

Corollary 3.9. *Let $H,Q : Y \to Y$ be self mappings on a metric space (Y,d) which is complete. If there exists $\alpha : Y \times Y \to \mathbb{R}, \varphi \in \Phi, \psi \in \Psi, \theta_1,\theta_2,\theta_3 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1+\theta_2+\theta_3 < 1$ satisfying the inequality*

$$\alpha(u,v)\varphi(d(Hu,Qv)) \leq \psi(B(u,v)) \;\; for\ all\ u,v \in Y.$$

If there exists $u_0 \in Y$ such that $\alpha(u_0,Hu_0) \geq 1$ and H and Q are continuous. Then the mappings H and Q have a unique common fixed point.

Remark 3.10. *By letting $\alpha(u,v) = 1$ for all $u,v \in Y$ and $\varphi = I_Y$ in Corollary 3.9, we find the following result.*

Corollary 3.11. *Let $H,Q : Y \to Y$ be self mappings on a metric space (Y,d) which is complete. If there exists $\psi \in \Psi, \theta_1,\theta_2,\theta_3 \in (0,1)$ such that $\theta_1+\theta_2+\theta_3 < 1$ satisfying the inequality*

$$d(Hu,Qv) \leq \psi(B(u,v)) \;\; for\ all\ u,v \in Y.$$

Then the mappings H and Q have a unique common fixed point.

Now, we illustrate an example to validate our main Theorem 3.5.

Example 3.12. *Let $Y = (-1,1]$ and $d : Y \times Y \to \mathbb{R}$ defined by $d(u,v) = |u-v|$. Define the mappings $H,Q : Y \to Y$ by*

$$HY = \begin{cases} \frac{u}{3}, \text{ if } u \in (-1,0) \\ \frac{u}{9}, \text{ if } u \in [0,1] \end{cases}, \; QY = \begin{cases} \frac{u}{2}, \text{ if } u \in (-1,0) \\ \frac{u}{3}, \text{ if } u \in [0,1]. \end{cases}$$

Also, we define the function $\alpha : Y \times Y \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1, \text{ if } u,v \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

Taking $\zeta(u,v) = \psi(v) - u$, for all $u,v > 0$ in Theorem 3.5, we get

$$\alpha(u,v)\varphi(d(Hu,Qv)) \leq \psi(B(u,v)),$$

for all $u, v \in Y$. *Let* $\varphi(t) = t$, $\psi(t) = kt$, *where* $k = \frac{1}{\sqrt{3}}, \theta_1 = \frac{1}{2}, \theta_2 = \frac{1}{4}, \theta_3 = \frac{1}{6}$, *then* $\varphi(t) \geq \psi(t)$. *Since* $0 \leq u, v \leq 1$, *then we get*

$$\begin{aligned}
&0 \leq |u-v| \leq 1 \Rightarrow 0 \leq |u-v|^{\frac{1}{2}} \leq 1,\\
&0 \leq \frac{1}{2}[|u-Hu| + |v-Qv|] = [(\frac{1}{9})(4u+3v)]^{\frac{1}{4}} \leq (\frac{7}{9})^{\frac{1}{4}},\\
&0 \leq \frac{1}{2}[|u-Qv| + |v-Qu|] = [\frac{1}{6}(|3u-v| + |3v-u|)]^{\frac{1}{6}} \leq (\frac{2}{3})^{\frac{1}{6}} \ \ and\\
and \ \ &0 \leq \frac{1}{2}[|u-Hv| + |v-Hu] = [\frac{1}{18}(|9u-v| + |9v-u|)]^{\frac{1}{12}} \leq (\frac{8}{9})^{\frac{1}{12}}.
\end{aligned}$$

By simple calculation for all $u, v \in Y$, *we obtain*

$$\begin{aligned}
\alpha(u,v)\varphi(d(Hu,Qv)) &= \alpha(u,v)|Hu-Qv| = \frac{3}{9}|u-3v| = \frac{1}{3}|u-3v|\\
&\leq \frac{1}{\sqrt{3}}|u-v|^{\frac{1}{2}}.[(\frac{1}{9})(4u+3v)]^{\frac{1}{4}}.[\frac{1}{6}(|3u-v| + |3v-u|)]^{\frac{1}{6}}.\\
&\ [\frac{1}{18}(|9u-v| + |9v-u|)]^{\frac{1}{12}}\\
&= \psi(B(u,v)).
\end{aligned}$$

Therefore the set (H,Q) *is an interpolative* (φ,ψ)*-Banach-Kannan-Chatterjea type* $\mathcal{Z}$*-contraction with reference to* ζ. *If* $\{u_n\}$ *is a sequence in* Y *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\{u_n\} \subseteq [0,1]$ *for all* $n \in \mathbb{N}$. *Since* $([0,1], d)$ *is a complete metric space, then the sequence* $\{u_n\}$ *converges to* u *in* $[0,1] \subseteq Y$. *If* $\alpha(u,v) \geq 1$, *then* $u, v \in [0,1]$. *So,* $Hu, Qv, QHu, HQv \in [0,1]$. *Therefore,* $\alpha(u,Qu) = 1$ *and* $\alpha(u,Hu) = 1$ *then* $\alpha(Qu,HQu) = 1$ *and* $\alpha(Hu,QHu) = 1$. *Also if* $\alpha(u,v) = 1$ *implies* $\alpha(u,Qv) = 1$ *and* $\alpha(u,Hv) = 1$. *This implies that the pair* (H,Q) *is a quasi triangular* α*-orbital admissible in* Y.
Let $\{u_n\} \subseteq [0,1]$ *for all* $n \in \mathbb{N}$. *This implies that*

$$\lim_{n\to\infty} Hu_n = \lim_{n\to\infty} \frac{1}{9}u_n = \frac{1}{9}u = Hu,$$

and

$$\lim_{n\to\infty} Qu_n = \lim_{n\to\infty} \frac{1}{3}u_n = \frac{1}{3}u = Qu,$$

This implies that the mappings H *and* Q *are continuous. Thus, all supposition of Theorem 3.5 are fulfilled. Hence* H *and* Q *have a unique common fixed point* $u = 0$.

In the following theorem, we put back the continuity of H and Q with the notion of α-regularity.

Theorem 3.13. *Let* H *and* Q *be self mappings on a metric space* (Y,d) *which is complete. Suppose that* (H,Q) *is a quasi triangular* α*-orbital admissible and forms an interpolative* (φ,ψ)*-Banach-Kannan-Chatterjea type* $\mathcal{Z}$*-contraction with respect to* ζ. *If there exists* $u_0 \in Y$ *such that* $\alpha(u_0, Hu_0) \geq 1$ *and* $\{u_n\}$ *in* Y *is* α*-regular, then the mappings* H *and* Q *have a unique common fixed point.*

Proof. Let $u_0 \in Y$ be such that $\alpha(u_0, Hu_0) \geq 1$. Define a sequence $\{u_n\}$ in Y such that $u_{2n+1} = Hu_{2n}$ and $u_{2n+2} = Qu_{2n+1}$ for all $n \in \mathbb{N}$. Since the pair (H,Q) is α-orbital admissible, we find that $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. We suppose that $u_n \neq u_{n+1}$ and hence we have $d(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. By repeating the process as in the proof of Theorem 3.5, we derived that $\{u_n\}$ converges to w. Since $\{u_n\}$ in Y is α-regular, then there exists a subsequence u_{n_k} of $\{u_n\}$ such that $\alpha(u_{n_k}, w) \geq 1$, for each $k \in \mathbb{N} \cup \{0\}$. From (3.1), we get

$$\begin{aligned}
\zeta(\alpha(u_{2n_k}, w)\varphi(d(Hu_{2n_k}, Qw)), \psi(B(u_{2n_k}, w)) &\geq 0\\
\zeta(\alpha(u_{2n_k}, w)\varphi(d(u_{2n_{k+1}}, Qw)), \psi(B(u_{2n_k}, w)) &\geq 0\\
\psi(B(u_{2n_k}, w)) - \alpha(u_{2n_k}, w)\varphi(d(u_{2n_{k+1}}, Qw) &\geq 0
\end{aligned}$$

Consequently, we arrive

$$\varphi(d(u_{2n_{k+1}}, Qw) \leq \alpha(u_{2n_k}, w)\varphi(d(u_{2n_{k+1}}, Qw)) < \psi(B(u_{2n_k}, w)) < \varphi(B(u_{2n_k}, w))$$

where

$$\begin{aligned} B(u_{2n_k}, w) = [d(u_{2n_k}, w)]^{\theta_1}.[\frac{1}{2}(d(u_{2n_k}, Hu_{2n_k}) + d(w, Qw))]^{\theta_2}.[\frac{1}{2}(d(u_{2n_k}, Qw) \\ + d(w, Qu_{2n_k}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n_k}, Hw) + d(w, Hu_{2n_k}))]^{1-\theta_1-\theta_2-\theta_3} \\ = [d(u_{2n_k}, w)]^{\theta_1}.[\frac{1}{2}(d(u_{2n_k}, u_{2n_{k+1}}) + d(w, Qw))]^{\theta_2}.[\frac{1}{2}(d(u_{2n_k}, Qw) \\ + d(w, u_{2n_{k+1}}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n_k}, Hw) + d(w, u_{2n_{k+1}}))]^{1-\theta_1-\theta_2-\theta_3} \end{aligned}$$

Taking $k \to +\infty$, we get $\varphi(d(w, Qw)) = 0$ which implies $d(w, Qw) = 0$. This shows that w is a fixed point of Q. Similarly, we can show that $(Hw, w) = 0$. Hence the mappings H and Q have a common fixed point.

To demonstrate the uniqueness of the common fixed point, we suppose that w^* is another common fixed point of H and Q and $\alpha(w, w^*) \geq 1$. Assume that $w \neq w^*$. From (3.1), we get

$$\begin{aligned} \zeta(\alpha(w, w^*)\varphi(d(Hw, Qw^*)), \psi(B(w, w^*))) \geq 0 \\ \zeta(\alpha(w, w^*)\varphi(d(w, w^*)), \psi(B(w, w^*))) \geq 0 \\ \psi(B(w, w^*)) - \alpha(w, w^*)\varphi(d(w, w^*)) \geq 0 \\ -\alpha(w, w^*)\varphi(d(w, w^*)) \geq 0. \end{aligned}$$

which is contradiction and hence the mappings H and Q have a unique common fixed point.

Remark 3.14. *For $H = Q$ in Theorem 3.13, we get Theorem 2.2 of M. S. Khan et al. [15]*

Corollary 3.15. *Let Q be a self mapping on a metric space (Y, d) which is complete. Suppose that Q is quasi triangular α-orbital admissible and forms an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0, Qu_0) \geq 1$ and $\{u_n\}$ in Y is α-regular, then Q has a unique fixed point in Y.*

Remark 3.16. *Setting $\zeta(u, v) = \psi(v) - u$ for all $u, v > 0$ in Theorem 3.13, we get the following result.*

Corollary 3.17. *Let $H, Q : Y \to Y$ be self mappings on a metric space (Y, d) which is complete. If there exists $\alpha : Y \times Y \to \mathbb{R}, \varphi \in \Phi, \psi \in \Psi, \theta_1, \theta_2, \theta_3 \in (0, 1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1 + \theta_2 + \theta_3 < 1$ satisfying the inequality*

$$\alpha(u, v)\varphi(d(Hu, Qv)) \leq \psi(B(u, v)) \;\; for\ all\ u, v \in Y.$$

If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$ and $\{u_n\}$ in Y is α-regular. Then the mappings H and Q have a unique common fixed point.

Remark 3.18. *By letting $\alpha(u, v) = 1$ for all $u, v \in Y$ and $\varphi = I_Y$ in Corollary 3.17, we get the following result.*

Corollary 3.19. *Let $H, Q : Y \to Y$ be two self mappings on a complete metric space. If there exists $\psi \in \Psi, \theta_1, \theta_2, \theta_3 \in (0, 1)$ such that $\theta_1 + \theta_2 + \theta_3 < 1$, for $t > 0, \psi > 0$ satisfying the inequality*

$$d(Hu, Qv) \leq \psi(B(u, v)) \;\; for\ all\ u, v \in Y.$$

Then the mappings H and Q have a unique common fixed point.

4. Application

We apply our outcome to find an existence theorem for Fredholm integral equations. Let $Y = \mathrm{C}[a,b]$ be a set of all real continuous functions on $[a,b]$ equipped with metric $d(e,j) = max_{t\in[a,b]}|e(t) - j(t)|$ for all $e, j \in \mathrm{C}[a,b]$. Then (Y,d) is a complete metric space.
Now, we consider Fredholm integral equations

$$u(t) = h(t) + \int_a^b K(t,s,u(s))ds \tag{4.1}$$

$$v(t) = h(t) + \int_a^b K(t,s,v(s))ds \tag{4.2}$$

where $t, s \in [a,b]$. Assume that $K : [a,b] \times [a,b] \times Y \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ continuous.

Theorem 4.1. *Let (Y,d) be a metric space equipped with metric $d(e,j) = max_{t\in[a,b]}|e(t)-j(t)|$ for all $e, j \in Y$ and $H, Q : Y \to Y$ are operator on Y defined by*

$$Hu(t) = h(t) + \int_a^b K(t,s,u(s))ds \tag{4.3}$$

$$Qv(t) = h(t) + \int_a^b K(t,s,v(s))ds \tag{4.4}$$

where $t, s \in [a,b]$. Assume that $K : [a,b] \times [a,b] \times Y \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ is continuous. Further, assume that the following conditions hold:
(i) If there exists a continuous function $q : [a,b] \times [a,b] \to [0,\infty)$, $\theta_1, \theta_2, \theta_3 \in (0,1)$ with $\theta_1 + \theta_2 + \theta_3 < 1$ that for all $u, v \in Y, s, t \in [a,b]$ fulfilling the following inequality

$$|K(t,s,u(s)) - K(t,s,v(s))| \leq q(t,s)M(u(s),v(s)) \tag{4.5}$$

$$\begin{aligned} where \quad M(u(s),v(s)) = {} & [|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)| + |v(s)-Qv(s)|)]^{\theta_2}. \\ & [\frac{1}{2}(|u(s)-Qv(s)| + |v(s)-Qu(s)|)]^{\theta_3}[\frac{1}{2}(|u(s)-Hv(s)| \\ & + |v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3} \end{aligned}$$

(ii) If there exists $k \in [0,1)$ and $\alpha : Y \times Y \to (0,\infty)$ such that for each $u \in Y$, we have

$$max_{t\in[a,b]} \int_a^b q(t,s)ds \leq \frac{k}{\alpha(u,v)}.$$

(iii) If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$.

Then the integral equations have a unique common solution in Y.

Proof. From $(4.3), (4.4)$ and (4.5), we obtain

$$\begin{aligned}|Hu(t)-Qv(t)| &= |\int_a^b K(t,s,u(s))ds - \int_a^b K(t,s,v(s))ds| \\ &= \int_a^b |K(t,s,u(s)) - K(t,s,v(s))|ds \\ &\leq \int_a^b q(t,s)M(u(s),v(s))ds \\ &\leq \int_a^b q(t,s)([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\ &\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}.[\frac{1}{2}(|u(s)-Hv(s)| \\ &\quad +|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3})ds.\end{aligned}$$

Taking maximum on both sides for all $t \in [a,b]$, we get

$$\begin{aligned}d(Hu,Qv) &= max_{t\in[a,b]}|Hu(t)-Qv(t)| \\ &\leq max_{t\in[a,b]}\int_a^b q(t,s)([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\ &\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}. \\ &\quad [\frac{1}{2}(|u(s)-Hv(s)|+|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3})ds \\ &\leq (max_{t\in[a,b]}([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\ &\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}.[\frac{1}{2}(|u(s)-Hv(s)| \\ &\quad +|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3}))\int_a^b q(t,s)ds \\ &\leq [d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Hu)+d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv)+d(v,Qu))]^{\theta_3}. \\ &\quad [\frac{1}{2}(d(u,Hv)+d(v,Hu))]^{1-\theta_1-\theta_2-\theta_3} max_{t\in[a,b]}\int_a^b q(t,s)ds \\ &\leq \frac{k}{\alpha(u,v)}B(u,v) \\ or\ \ \alpha(u,v)d(Hu,Qv) &\leq kB(u,v).\end{aligned}$$

Since $Y = \mathrm{C}[a,b]$ is complete metric space. Hence, all the suppositions of Theorem 3.5 are satisfied by setting $\zeta(v,u) = \psi(u) - v$ with $\psi(l) = kl$ and $\varphi(l) = l$ for all $l > 0$, where $k \in [0,1)$ and hence the integral equations have a unique common solution.

5. Conclusion

From our investigations, we conclude that the existence and uniqueness of common fixed point theorem for pair of quasi triangular α-orbital admissible with an interpolative (φ,ψ)- Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction mappings with reference to simulation function in complete metric space. As an application, we find the existence and uniqueness of common solution for nonlinear Fredholm integral equations. An example is given in support of our main result. Our result provides new path for the researchers in the concerned field.

References

[1] H. Afshari, H. Ayadi and E. Karapinar, On generalized $\alpha-\psi$-Geraghty contractions on b-metric spaces, *Georgian Mathematical Journal*, (2018), https://doi.org/10.1515/gmj-2017-0063.

[2] M. A. Alghamdi, S. Gulyaz-Ozyurt and E. Karapinar, A Note on extended Z-contraction, *Mathematics*, **8(2)**(2020), 1-14, https://doi.org/10.3390/math8020195.

[3] H. Alsamir, M. S. Noorani, W. Shatanawi and F. Shaddad, Generalized Berinde-type contractive mappings in B-metric spaces with an application, *J. Math. Anal.*, **6**(2016), 1-12.

[4] O. Alqahtani and E. Karapinar, A Bilateral contraction via simulation function, *Filomat*, **33(15)**(2019), 4837- 4843, https://doi.org/10.2298/FIL1915837A.

[5] A. H. Ansari, W. Shatanawi, A. kurdi and G. Maniu, Best proximity points in complete metric spaces with (P)-property via C-class functions, *J. Math. Anal.*, **7**(2016), 54-67.

[6] H. Argoubi, B. Samet and C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, **8** (2015), 1082-1094, https://doi.org/10.22436/jnsa.008.06.18.

[7] M. Arshad, E. Ameer and E. Karapinar, Generalized contractions with triangular α-orbital admissible mapping on Branciari metric spaces, *J. of Inequality and Applications*, (2016), 63:2016, https://doi.org/10.1186/s13660-016-1010-7.

[8] V. Berinde, Sequence of operators and fixed points in quasi metric space, *Studia Univ. Babes-Bolyai Math.*, **41(4)**(1996), 23-27.

[9] A. Farajzadeh, C. Noytaptim and A. Kaewcharoen, Some fixed point theorems for generalized $\alpha-\eta-\psi$-Geraghty contractive type mappings in partial b-metric spaces, *J. of informatics and Mathematical Sciences*, **10(3)**(2018), 455-478, https://doi.org/10.26713/jims.v10i3.583.

[10] N. Hussain, E. Karapinar and F. Akbar, α-admissible mappings and related fixed point theorems, *J. of Inequalities and Applications*, (2013), 2013:114, https://doi.org/10.1186/1029-242X-2013-114.

[11] E. Karapinar, Fixed points results via simulation functions, *Filomat*, **30(8)**(2016), 2343-2350, https://doi.org/10.2298/FIL1608343K.

[12] E. Karapinar, Revisiting simulation functions via interpolative contrations, *Appl. Anal. Discrete Math.*, **13**(2019), 859-870, https://doi.org/10.2298/AADM190325038K.

[13] E. Karapinar, P. Kumam and P. Salimi, On $\alpha-\psi$-Meir-Keeler contractive mappings, *Fixed Point Theory Appl.*, **2013**(2013), 12 pages, https://doi.org/10.1186/1687-1812-2013-94.

[14] E. Karapinar and F. Khojasteh, An approach to best proximity points results via simulation functions, *J. Fixed Point Theory Appl.*, **19(3)**(2017), 1983-1995, https://doi.org/10.1007/s11784-016-0380-2.

[15] M. S. Khan, Y. M. Singh and E. Karapinar, On the interpolative (ϕ,ψ)-type $\mathcal{Z}$-contraction, *U. P. B. Sci. Bull., Series A*, **83(2)**(2021), 25-38.

[16] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30**(1984), 1-9, https://doi.org/https://doi.org/10.1017/S0004972700001659.

[17] F. Khojasteh, S. Shukla and S. Radenovic, A new approach to the study of fixed point theorems via simulation functions, *Filomat*, **29(6)**(2015), 1189-1194, https://doi.org/10.2298/FIL1506189K.

[18] C. Lang and H. Guan , Common fixed point and coincidence point results for generalized $\alpha-\phi_E$-Geraghty contraction mappings in b-metric spaces, *AIMS Mathematics*, **7(8)**(2022), 14513-14531, https://doi.org/10.3934/math.2022800.

[19] J. Li and H. Guan, Common fixed point of generalized α_s-ψ-Geraghty contractive mappings on b-metric spaces, *American Journal of Applied Mathematics and Statistics*, **9(2)**(2021), 66-74, https://doi.org/10.12691/ajams-9-2-5.

[20] O. Popescu, Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces, *Fixed Point Theory Appl.*, (2014), 1-12, https://doi.org/10.1186/1687-1812-2014-190.

[21] Y. M. Singh, M. S. Khan and S. M. Kang, F-convex contraction via admissible mapping and related fixed point theorems with an application, *Mathematics*, **6**(2018), 1-15, https://doi.org/10.3390/math6060105.

[22] H. Qawagneh, M. S. MD Noorani, W. Shatanawi and H. Alsamir, Common fixed points for pairs of triangular α-admissible mappings, *J. Nonlinear Sci. Appl.*, **10**(2017), 6192-6204, https://doi.org/10.22436/jnsa.010.12.06.

[23] A. F. Roldan-Leopez-de-Hierro, E. Karapinar, C. Roldan-Lopez-de-Hierro and J. Martinez-Morenoa, Coincidence point theorems on metric spaces via simulation functions, *J. Computational and Appl. Math.*, **275**(2015), 345-355, https://doi.org/10.1016/j.cam.2014.07.011.

[24] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, Common fixed points of almost generalized $(\psi,\varphi)_s$- contractive mappings in ordered b-metric spaces, *Fixed Point Theory Appl.*, **2013**(2013), 23 pages, https://doi.org/10.1186/1687-1812-2013-159.

[25] B. Samet, C. Vetro and P. Vetro, Fixed points theorems for $\alpha-\psi$ contractive type mappings, *Nonlinear Anal.*, **75**(2012), 2154-2165, https://doi.org/10.1016/j.na.2011.10.014.

[26] W. Shatanawi, Common fixed points for mappings under contractive conditions of (α,β,ψ)-admissibility type, *Mathematics*, **6**(2018), 1-11, https://doi.org/10.3390/math6110.

[27] W. Shatanawi, M. S. Noorani, H. Alsamir and A. Bataihah, Fixed and common fixed point theorems in partially ordered quasi metric spaces, *J. Math. Computer. Sci.*, **16**(2016), 516-528, http://dx.doi.org/10.22436/jmcs.016.04.05.

[28] W. Shatanawi and M. Postolache, Common fixed point results for mappings under nonlinear contraction of cyclic form inordered metric spaces, *Fixed Point Theory Appl.*, **2013**(2013), 13 pages, https://doi.org/10.1186/1687-1812-2013-60.

[29] Y. Sun, X. L. Liu, J. Deng and M. Zhou, Some fixed point results for α-admissible extended $\mathcal{Z}$-contraction mappings in extended rectangular b-metric spaces, *AIMS Mathematics*, **7(3)**(2021), 3701-3718, https://doi.org/10.3934/math.2022205.

[30] S. Omran and L. Masmali, α-admissible mapping in $\mathcal{C}^*$ algebra-valued b-metric spaces and fixed point theorems, *AIMS Mathematics*, **6(9)**(2021), 10192-10206, https://doi.org/10.3934/math.2021590.