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Results using primitive function module

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Abstract. A real or complex valued function defined on the set of all positive integers is called an arithmetic function and an arithmetic function is said to be completely multiplicative function if f is not identically zero and $f(mn) = f(m)f(n)$ for all m, n. The objective of this paper is to present a result of completely multiplicative function of two variables using primitive function module.

AMS Subject Classifications: 11A07, 11A25.

Keywords: Arithmetic function, Multiplicative function, Primitive function module.

Contents

1. Introduction

A real or complex valued function defined on the set of all positive integers is called an arithmetic function. An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and $\mathrm{f(mn)} = \mathrm{f(m)f(n)}$ whenever $(\mathrm{m}, \mathrm{n}) = 1$. The function $\mathrm{f(m, n)}$ of two variables defined for pairs of positive integers m and n is said to be multiplicative in both the arguments m and n if $\mathrm{f}(1, 1) = 1$ and $\mathrm{f}(\mathrm{m}_1\ \mathrm{m}_2, \mathrm{n}_1\mathrm{n}_2) = \mathrm{f}(\mathrm{m}_1,\ \mathrm{m}_2)\,\mathrm{f}(\mathrm{m}_2, \mathrm{n}_2)$ where $(\mathrm{m}_1\mathrm{n}_1,\ \mathrm{m}_2\mathrm{n}_2) = 1$. Many identities have been established by various researchers discussed in [3, 7, 9].

Definition 1.1. *An arithmetic function is said to be completely multiplicative function if f is not identically zero and $f(mn) = f(m)f(n)$ for all m, n.*

Definition 1.2. *Strongly Multiplicative function: A multiplicative arithmetic function* f *is said to be strongly multiplicative function if for every prime* P, *we have*

$$f(p) = f\left(p^2\right) = f\left(p^3\right) = \cdots\ldots\ldots\ldots\ldots$$

Definition 1.3. *An arithmetic function $f(n, r)$ is said to be primitive function module r if $f(n, r) = f(\gamma(n, r), r)$ for all $\gamma(n, r) = \gamma((n, r))$.*

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Definition 1.4. *An arithmetic function $f(n,r)$ is said to be completely primitive function module r if $f(n,r)=f\left(n',r'\right)$ for all n,n^1 and all positive r,r' Such that*

$$\frac{\gamma(r)}{\gamma(n,r)}=\frac{\gamma\left(r'\right)}{\gamma\left(n',r'\right)}$$

Let $g(r)$ and $h(r)$ be two arithmetic functions.
Define

$$f(n,r)=\sum_{d\mid(n,r)} h(d)g\left(\frac{r}{d}\right)\mu\left(\frac{r}{d}\right) \tag{1.1}$$

and

$$F(r)=f(0,r)=\sum_{d\mid r} h(d)g\left(\frac{r}{d}\right)\mu\left(\frac{r}{d}\right). \tag{1.2}$$

2. Preliminaries

We use the following lemma proved by E.Cohen ([4], P404).

Lemma 2.1. *Suppose $f(n,r)$ is completely primitive module r. Then*

$$f(n,r)=\sum_{\substack{d\mid\gamma(r)\\(d,n)=r_1}} G(d)\Leftrightarrow G\left(r_1\right)=\sum_{d\mid r_1} f\left(\frac{r_1}{d},d\right)\mu\left(\frac{r_1}{d}\right)$$

for any square free r_1.

Lemma 2.2. *Let $h(v)$ is completely multiplicative function. Then $F(v)=h\left(\frac{v}{\gamma(v)}F(\gamma(v))\right)$.*

Proof. From (1.2), We have

$$\begin{aligned}
F(v) &= \sum_{d\mid n} h(d)g\left(\frac{v}{d}\right)\mu\left(\frac{v}{d}\right)\\
&= \sum_{d\delta=v} h\left(\frac{v}{\delta}\right)g(\delta)\mu(\delta)\\
&= \sum_{d\delta=v} h\left(\frac{v}{\gamma(v)}\cdot\frac{\gamma(n)}{\delta}\right)g(\delta)\mu(\delta)\\
&= h\left(\frac{v}{\gamma(v)}\right)\sum_{d\delta=v} h\left(\frac{\gamma(v)}{\delta}\right)g(\delta)\mu(\delta)\\
&= h\left(\frac{v}{\gamma(v)}\right)F(\gamma(v)).
\end{aligned}$$

Because of factor $\mu(\delta)$, since $\mu(\delta)=0$ for square number. Hence ranging d over divisor of v or over divisor of $\gamma(v)$ is same. This completes the lemma.

■

Also we need the following result:

Lemma 2.3. *Let $g(r)$ be multiplicative, $h(r)$ is completely multiplicative and for all primes $P, h(p)\neq 0, h(p)\neq g(P)$. Then $F(r)\neq 0$ for all r.*

Proof. Since $F(1)=1$ We may assume that $r>1$. Note that $F(r)=(h*\mu g)(r)$.

$F(r)$ is multiplicative, since $h(r), g(r)$ and $\mu(r)$ are multiplicative functions. To prove $F(P^\alpha)\neq 0$ for all primes P and $\alpha>1$.

Consider

$$\begin{aligned} F\left(P^\alpha\right) &= \sum_{k=0}^{\alpha} h\left(P^k\right) g\left(P^{\alpha-k}\right) \mu\left(P^{\alpha-k}\right) \\ &= h\left(P^\alpha\right)-h\left(P^{\alpha-1}\right) g(P) \\ &= h(P)^{\alpha-1}[h(P)-g(P)] \\ &\neq 0. \end{aligned}$$

Since $h(P)-g(P)\neq 0, h(P)\neq 0$.

■

3. Main Results

Theorem 3.1. *If $g(r)$ is multiplicative, $h(r)$ is completely multiplicative and for all prime $P, h(P)\neq 0, h(P)\neq g(P)$, then*

$$\sum_{\substack{d\mid r\\(d,n)=1}} \frac{g(d)}{F(d)}\mu^2(d)=\frac{h(r)}{F(r)}\frac{F((n,r))}{h((n,r))}.$$

Proof. Denote

$$J(n,r)=\frac{h(r)}{F(r)}\frac{F((n,r))}{h((n,r))}. \tag{3.1}$$

$J(n,r)$ is properly defined since $F(r)\neq 0, h(n,r)\neq 0$.

By Lemma 2.2, we get,

$$\begin{aligned} J(n,r) &= \frac{h(r)h\left(\frac{n,r}{\gamma(n,r)}\right)F(\gamma(n,r))}{h\left(\frac{r}{\gamma(r)}\right)F(\gamma(r))h((n,r))} \\ &= \frac{h(r)h((n,r))F(\gamma(n,r))h(\gamma(r))}{h(\gamma(n,r))h(r)F(\gamma(r))h((n,r))} \\ &= \frac{h(\gamma(r))F(\gamma(n,r))}{h(\gamma(n,r))F(\gamma(r))} \\ &= \frac{h\left(\frac{\gamma(r)}{\gamma(n,r)}\right)}{F\left(\frac{\gamma(r)}{\gamma(n,r)}\right)} \\ &= \frac{h(m)}{F(m)}, \end{aligned}$$

where $m=\frac{\gamma(r)}{\gamma(n,r)}$.

Thus

$$J(n,r)=\frac{h(m)}{F(m)}, \quad \text{where } m=\frac{\gamma(r)}{\gamma(n,r)}. \tag{3.2}$$

Now we prove that

$$\mathrm{J}(n, \mathrm{r}) \quad \text{is completely primitive} \quad (\bmod r). \tag{3.3}$$

That is, we have to show that $J(n, r)=J\left(n^1, r^1\right)$ for all n, n^1, r, r^1 with

$$\frac{\gamma(r)}{\gamma(n, r)}=\frac{\gamma\left(r^1\right)}{\gamma\left(n^1, r^1\right)}$$

By (3.2), we have

$$\begin{aligned} J(n, r) & =\frac{h\left(\frac{\gamma(r)}{\gamma(n, r)}\right)}{F\left(\frac{\gamma(r)}{\gamma(n, r)}\right)} \\ & =\frac{h\left(\frac{\gamma\left(r^{\prime}\right)}{\gamma\left(n^{\prime}, r^{\prime}\right)}\right)}{F\left(\frac{\gamma\left(r^{\prime}\right)}{\gamma\left(n^{\prime}, r^{\prime}\right)}\right)} \\ & =J\left(n^{\prime}, r^{\prime}\right) . \end{aligned}$$

Therefore by Lemma 2.1, we have

$$J(n, r)=\sum_{\substack{d \mid \gamma(r) \\ (d, n)=1}} G(d) \Leftrightarrow G\left(r_1\right)=\sum_{d \mid r_1} J\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right) .$$

Consider

$$\begin{aligned} G\left(r_1\right) & =\sum_{d \mid r_1} J\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right) . \\ & =\sum_{d \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right) \quad \text{by} \quad (3.2) \end{aligned}$$

which by multiplicativity of $\mu(r)$ and $F(r)$ gives

$$\begin{aligned} & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} \sum_{d \mid r_1} h(d) \mu(d) F\left(\frac{r_1}{d}\right) \\ & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} \sum_{d \mid r_1} h(d) \mu(d) \sum_{D \delta=\frac{r_1}{d}} h(D) g(\delta) \mu(\delta), \end{aligned}$$

where $\mathrm{E}=\mathrm{Dd}$. But

$$\sum_{d \mid E} \mu(d)= \begin{cases}1 & \text{if } E=1 \\ 0 & \text{if } E>1 .\end{cases}$$

Therefore,

$$\begin{aligned} G\left(r_1\right) & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} g\left(r_1\right) \mu\left(r_1\right) \\ & =\frac{\mu^2\left(r_1\right) g\left(r_1\right)}{F\left(r_1\right)} . \end{aligned}$$

Now we have

$$\begin{aligned} J(n, r) &= \sum_{\substack{d \mid \gamma(r) \\ (d, n)=1}} G(d) \\ &= \sum_{\substack{d \mid \gamma(r) \\ (d, n)=1}} \frac{\mu^2(d) g(d)}{F(d)}. \end{aligned}$$

■

Theorem 3.2.

$$F(r) \sum_{\substack{d \mid r \\ (d, n)=1}} \frac{h(d)}{F(d)} \cdot \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d \mid (n, r)} h(d) f\left(\frac{r}{d}\right),$$

where $f(n) = g(n)\mu(n)$.

Proof. Let

$$Q(n, r) = F(r) \sum_{\substack{d \mid r \\ (d, n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right).$$

Let

$$Q(n, r) = F(r) \sum_{\substack{d \mid r \\ (d, n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right).$$

Let r_1 and r_2 be the uniquely determined positive integers such that $r = r_1 r_2$ where $(r_1, r_2) = 1, \gamma(r_2) = \gamma(n, r)$.

Then

$$\begin{aligned} Q(n, r) &= F(r) \mu(r_2) \sum_{d_1 \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right) \\ &= F(r) \mu(r_2) G(r_1) \\ &= F(r_1) F(r_2) \mu(r_2) \frac{\mu^2(r_1) g(r_1)}{F(r_1)} \\ &= \mu(r) \mu(r_1) g(r_1) \sum_{d \mid (n, r)} h(d) g\left(\frac{r_2}{d}\right) \mu\left(\frac{r_2}{d}\right). \end{aligned}$$

In view of the presence of $\mu(r)$ and the fact that $\gamma(r_2) = \gamma(n, r)$, we have

$$Q(n, r) = \mu(r) \sum_{d \mid (n, r)} h(d) g\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right).$$

That is,

$$F(r) \sum_{\substack{d \mid r \\ (d, n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d \mid (n, r)} h(d) f\left(\frac{r}{d}\right),$$

where $f(n) = \mu(n) g(n)$. ■

Remark 3.3. *Substituting* $h(n) = n^k, f(n) = \mu(n)$ *and* $F(r) = J_k(r)$ *in Theorem 3.2, we get a well known identity known as Brauer - Rademacher identity.*

$$J_k(r) \sum_{d|r} \frac{d^k}{J_k(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d|(n,r)} d^k \mu\left(\frac{r}{d}\right).$$

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