MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(03)**(2022), 216–223.
http://doi.org/10.26637/mjm1003/004

Interval-valued intuitionistic fuzzy linear transformation

R. SANTHI [1] AND N. UDHAYARANI [2] *

[1,2] *PG and Research Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi-642001, Tamil Nadu, India.*

Received 12 November 2021; Accepted 17 April 2022

Abstract. In this paper we introduce the concept of interval-valued intuitionistic fuzzy relations(in biefly IVIFR) and composition of IVIFR-equations. Then we continued it to interval-valued intuitionistic fuzzy linear transformation(in brief IVIFL-transformation) and discussed its properties. Also introduced the concept of composition of IVIFL-transformations.

AMS Subject Classifications: 03E72, 15A03.

Keywords: IVIF-Relations, Composition of IVIF-Relations, IVIFL-transformations, $IVIF_0L$-transformations, $IVIF_IL$-transformations.

Contents

1. Introduction and Background

Zadeh [19] introduced fuzzy set and properties of fuzzy sets. At [1], Atanassov introduced the intuitionistic fuzzy set which was broadened to interval-valued intuitionistic fuzzy set by Atanassov and Gargov [2] whose membership and nonmembership functions are intervals. In 2011, Lin and Huang [5, 8] introduced the basic concepts of (T, S)-composition matrix and (T, S)- interval-valued intuitionistic fuzzy equivalence matrix. Initially, Shyamal and Pal [16] introduced interval-valued fuzzy matrix. Then Intuitionistic fuzzy matrices introduced by Madhumangal Pal et al. [9]. Pal and Susanta K. Khan [10] introduced some basic operators in interval-valued intuitionistic fuzzy matrices. Xu and Yager [18] introduced intuitionistic and interval-valued intuitionistic fuzzy preference relations and their measures of similarity in decision making methods. Also

*Corresponding author. Email addresses: **santhifuzzy@yahoo.co.in** (R.Santhi), **udhayaranin@gmail.com** (N. Udhayarani)

Ze-shui Xu and Jian chen [20], approaches the decision making methods using interval-valued intuitionistic judgement matrices.

In 1977, Katsaras and Liu [6] introduced fuzzy vector and fuzzy topological vector spaces. In 1994, Terao and Kitsunezaki [17] introduced fuzzy sets and linear mappings on vector spaces. Kim and Roush [7] presents some basic concepts of generalized fuzzy matrices. Bhowmik and Pal [4] described and studied the concept of generalized intuitionistic fuzzy matrices.

Narayanan et al. [13] introduced the notion of intuitionistic fuzzy continuous mappings and intuitionistic fuzzy bounded linear operators from one intuitionistic fuzzy n-normed linear space to another. Recently Moumitha Chiney and Samanta studied and introduced the concept of intuitionistic fuzzy vector spaces [12]. And Santhi and Udhayarani introduced and studied the concept of [15] interval-valued intuitionistic fuzzy vector spaces. Then intuitionistic fuzzy linear transformations described by Meenakshi and Gandhimathi [11] and Rajkumar Pradhan and Madhumangal Pal [14].

In this paper, we introduced the concept of interval-valued intuitionistic fuzzy linear transformations and scrutinized some of its properties. In section 2, some basic concepts and properties are reviewed. In section 3, we introduced the concept of $IVIFR$-equations and its composition. Also introduced the concept of $IVIFL$-transformations.

2. Preliminaries

This section briefly discussed about the basic concepts of IVIFS which were used in the following sections.

Definition 2.1. ***Interval-valued fuzzy vector:*** *An interval valued fuzzy vector is an n-tuple of elements from an interval -valued fuzzy algebra. That is, an IVFV is of the form* $(x_1, x_2, ..., x_n)$*, where each element* $x_i \in F$*,* $i = 1, 2, ..., n$.

Definition 2.2. ***Interval-valued fuzzy vector space:*** *An interval-valued fuzzy vector space(IVFV Space) is a pair* $(E, A(x))$*, if E is a vector space in crisp sense and* $A : E \to D[0,1]$ *with the property, that for all* $a, b \in F$ *and* $x, y \in E$*, then,*
$\underline{A}(ax + by) \geq \underline{A}(x) \wedge \underline{A}(y)$ *and*
$\overline{A}(ax + by) \geq \overline{A}(x) \wedge \overline{A}(y)$.

Definition 2.3. ***Interval-valued intuitionistic fuzzy set:*** *Let* $D[0,1]$ *be the set of closed subintervals of the interval* $[0,1]$ *and* $X(\neq \phi)$ *be a given set. An interval-valued intuitionistic fuzzy set in X is described as,* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$*, where* $\mu_A(x) : X \to D[0,1]$*,* $\nu_A(x) : X \to D[0,1]$ *with the condition* $0 \leq sup(\mu_A(x)) + sup(\nu_A(x)) \leq 1$ *for any* $x \in X$*. The intervals* $\mu_A(x)$ *and* $\nu_A(x)$ *denotes the degree of belongingness and the degree of nonbelongingness of the element* x *to the set* A*. Thus for each* $x \in X$*,* $\mu_A(x)$ *and* $\nu_A(x)$ *are closed intervals and their lower and upper end points are denoted by* $\mu_{A_L}(x), \mu_{A_U}(x), \nu_{A_L}(x)$ *and* $\nu_{A_U}(x)$*. We can denote it by:*

$$A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]\rangle / x \in X\},$$

where $0 \leq \mu_{A_U}(x) + \nu_{A_U}(x) \leq 1$*,* $\mu_{A_L}(x) \geq 0$*,* $\nu_{A_L}(x) \geq 0$*. For each element x, we can compute the unknown degree(hesitancy degree) of an intuitionistic fuzzy interval of* $x \in X$ *in A defined as follows:*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \mu_{A_U}(x) - \nu_{A_U}(x), 1 - \mu_{A_L}(x) - \nu_{A_L}(x)]$$

Especially, if $\mu_A(x) = \mu_{A_U}(x) = \mu_{A_L}(x)$ *and* $\nu_A(x) = \nu_{A_U}(x) = \nu_{A_L}(x)$*, then the given IVIFS A is reduced to an ordinary intuitionistic fuzzy set.*

Definition 2.4. *For any two IVIFSs* $A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]\rangle / x \in X\}$ *and* $B = \{\langle x, [\mu_{B_L}(x), \mu_{B_U}(x)], [\nu_{B_L}(x), \nu_{B_U}(x)]\rangle / x \in X\}$ *then the following two relations are explained as:*

1. $A \subseteq B$ *if and only if*
   (a) $\mu_{A_U}(x) \leq \mu_{B_U}(x)$,
   (b) $\mu_{A_L}(x) \leq \mu_{B_L}(x)$,
   (c) $\nu_{A_U}(x) \geq \nu_{B_U}(x)$,
   (d) $\nu_{A_L}(x) \geq \nu_{B_L}(x)$, *for any* $x \in X$.
2. $A = B$ *if and only if*
   (a) $\mu_{A_U}(x) = \mu_{B_U}(x)$,
   (b) $\mu_{A_L}(x) = \mu_{B_L}(x)$,
   (c) $\nu_{A_U}(x) = \nu_{B_U}(x)$,
   (d) $\nu_{A_L}(x) = \nu_{B_L}(x)$, *for any* $x \in X$.

Definition 2.5. ***Interval-valued Intuitionistic Fuzzy Vector Space:*** *The Mathematical system of interval-valued intuitionistic fuzzy algebra is defined with two binary operations '+' and '.' on the set* $\widetilde{V}$ *satisfying the following properties:*
Let $[x_L, x_U]$, $[y_L, y_U]$ *and* $[z_L, z_U]$ *be the elements of* $\widetilde{V}$.

1. $Idempotent : [x_L, x_U] + [x_L, x_U] = max\{[x_L, x_U], [x_L, x_U]\} = [x_L, x_U]$
2. $Commutative : [x_L, x_U] + [y_L, y_U] = [y_L, y_U] + [x_L, x_U]$
3. $Associativity : [x_L, x_U] + ([y_L, y_U] + [z_L, z_U]) = ([x_L, x_U] + [y_L, y_U]) + [z_L, z_U]$
4. *Absorption:*
   (a) $[x_L, x_U] + ([x_L, x_U].[y_L, y_U]) = [x_L, x_U]$
   (b) $[x_L, x_U].([x_L, x_U] + [y_L, y_U]) = [x_L, x_U]$
5. *Universal Bounds:*
   (a) $[x_L, x_U] + \phi = [x_L, x_U]$
   (b) $[x_L, x_U] + I = I$
   (c) $[x_L, x_U].\phi = \phi$
   (d) $[x_L, x_U].I = [x_L, x_U]$

where $\phi = \langle [0,0], [1,1]\rangle$ *is the zero element and* $I = \langle [1,1], [0,0]\rangle$, *is the identity element.*

Definition 2.6. *The pair* $(V, \langle [\mu_L(x), \mu_U(x)], [\nu_L(x), \nu_U(x)]\rangle) = \widetilde{V}$ *is said to be an interval-valued intuitionistic fuzzy vector space, if* $\alpha_{\mu_U} : \widetilde{V} \to D[0,1]$, $\alpha_{\nu_U} : \widetilde{V} \to D[0,1]$, $\alpha_{\mu_L} \geq 0$ *and* $\alpha_{\nu_L} \geq 0$ *with the property that for all* $\alpha, \beta \in \widetilde{V}$ *and* $x, y \in F$, *then*

1. $\left\langle [(\alpha_{\mu_{L_1}} + \beta_{\mu_{L_1}}), (\alpha_{\mu_{U_1}} + \beta_{\mu_{U_1}})][(\alpha_{\nu_{L_1}} + \beta_{\nu_{L_1}}), (\alpha_{\nu_{U_1}} + \beta_{\nu_{U_1}})]\right\rangle \in \widetilde{V}$
2. $\{\langle [((\alpha_L \wedge \mu_L), (\alpha_U \wedge \mu_U)), (((1-\alpha_L) \vee \nu_L, (1-\alpha_U) \vee \nu_U))]\rangle\} \in \widetilde{V}$,

where $\alpha_{\mu_{L_1}} + \beta_{\mu_{L_1}} = \alpha_{\mu_{L_1}} \vee \beta_{\mu_{L_1}}$,
$\alpha_{\mu_{U_1}} + \beta_{\mu_{U_1}} = \alpha_{\mu_{U_1}} \vee \beta_{\mu_{U_1}}$,
$\alpha_{\nu_{L_1}} + \beta_{\nu_{L_1}} = \alpha_{\nu_{L_1}} \wedge \beta_{\nu_{L_1}}$,
$\alpha_{\nu_{U_1}} + \beta_{\nu_{U_1}} = \alpha_{\nu_{U_1}} \wedge \beta_{\nu_{U_1}}$.

3. Main Results

In this section, the concept of interval-valued intuitionistic fuzzy linear realtional equations(in briefly IVIFLR-equation) was defined and its basic properties were discussed. Also, this section defines interval-valued intuitionistic fuzzy linear transformations(in brief IVIFL- transformations) on IVIFV- space and discuss its properties.

Definition 3.1. *For interval-valued intuitionistic fuzzy relation $R(x,y)$ we register the lower and upper end points of membership and non-membership value of x in relation with y under R defined by $[\mu_{RL},\mu_{RU}] = [\alpha_{\mu L},\alpha_{\mu U}]$ and $[\nu_{RL},\nu_{RU}] = [\alpha_{\nu L},\alpha_{\nu U}]$ and represented as $\langle[\alpha_{\mu L},\alpha_{\mu U}],[\alpha_{\nu L},\alpha_{\nu U}]\rangle$.*

Note 3.2. *An interval-valued intuitionistic fuzzy binary relation R (in brief IVIF-binary relation) can be represented as an interval-valued intuitionistic fuzzy matrix M_R (in brief IVIF-Matrix).*

Example 3.3. *If R is an $IVIF$-Relation with $X = \{X_1,X_2,X_3\}$ and $Y = \{Y_1,Y_2\}$ that indicates the relational concept 'that the element of set X are different kind of air-conditioning systems to the physical structure of the computer lab of the set Y'. The upper and lower end points of the membership and nonmembership values can be represented by the following IVIF-matrix,*

$$[M_{R_1}] = \begin{bmatrix} \langle[0.3,0.5],[0.2,0.4]\rangle & \langle[0.2,0.5],[0.1,0.5]\rangle & \langle[0.3,0.6],[0.1,0.2]\rangle \\ \langle[0.1,0.3],[0.2,0.5]\rangle & \langle[0.2,0.4],[0.1,0.3]\rangle & \langle[0.1,0.3],[0.2,0.5]\rangle \end{bmatrix}$$

Definition 3.4. *If $R_1(X,Y)$ and $R_2(Y,Z)$ are two IVIF-relations then its IVIF-composition is described by the max-min operation and signified as $R_1 \circ R_2$ with respect to IVIF-matrices of R_1 and R_2. That is, if there is two IVIF-binary relations $R_1(X,Y)$, $R_2(Y,Z)$ then $R(X,Z)$ defined on the sets $X = \{[x_i]/i \in N_s\}, i = 1,2,...,s$, $Y = \{[y_j]/j \in N_m\}, j = 1,2,...,m$, and $Z = \{[z_k]/k \in N_n\}, k = 1,2,...,n$, where N is the set of all positive integers. Let the corresponding IVIF-matrices be denoted by,*

$R_1 = [a_{ij}] = \left\langle \left[\mu_{a_{ijL}},\mu_{a_{ijU}}\right],\left[\nu_{a_{ijL}},\nu_{a_{ijU}}\right]\right\rangle$

$R_2 = [b_{jk}] = \left\langle \left[\mu_{b_{jkL}},\mu_{b_{jkU}}\right],\left[\nu_{b_{jkL}},\nu_{b_{jkU}}\right]\right\rangle$

$R = [r_{ik}] = \langle[\mu_{r_{ikL}},\mu_{r_{ikU}}],[\nu_{r_{ikL}},\nu_{r_{ikU}}]\rangle$ *then the IVIF-composition $R(X,Z)$ of $R_1(X,Y)$ and $R_2(Y,Z)$ is given by*

$$R_1 \circ R_2 = R \tag{3.1}$$

That is,

$$\left\langle \max_j\{\min\left[\left(\mu_{a_{ijL}},\mu_{b_{jkL}}\right),\left(\mu_{a_{ijU}},\mu_{b_{jkU}}\right)\right]\}, \min_j\{\max\left[\left(\nu_{a_{ijL}},\nu_{b_{jkL}}\right),\left(\nu_{a_{ijU}},\nu_{b_{jkU}}\right)\right]\}\right\rangle = \langle[\mu_{r_{ikL}},\mu_{r_{ikU}}],[\nu_{r_{ikL}},\nu_{r_{ikU}}]\rangle$$

where $i \in N_s, j \in N_m$ and $k \in N_n$.

The above equations renders IVIF-relational equations and we get it from performing the max-min operations on R_1 and R_2.

Example 3.5. *Let R_1 and R_2 be two IVIF-matrices and*

$$[M_{R_1}] = \begin{bmatrix} \langle[0.3,0.5],[0.2,0.4]\rangle & \langle[0.2,0.5],[0.1,0.5]\rangle & \langle[0.1,0.5],[0.1,0.2]\rangle \\ \langle[0.1,0.3],[0.2,0.5]\rangle & \langle[0.2,0.6],[0.1,0.3]\rangle & \langle[0.1,0.3],[0.2,0.5]\rangle \end{bmatrix}$$

$$[M_{R_2}] = \begin{bmatrix} \langle [0.2, 0.5], [0.1, 0.2] \rangle & \langle [0.3, 0.6], [0.1, 0.3] \rangle \\ \langle [0.2, 0.6], [0.1, 0.4] \rangle & \langle [0.1, 0.5], [0.1, 0.3] \rangle \\ \langle [0.2, 0.7], [0.1, 0.3] \rangle & \langle [0.1, 0.6], [0.1, 0.2] \rangle \end{bmatrix}$$

Then the composition of these two IVIF-matrices is,

$$[M_{R_1}] \circ [M_{R_2}] = [M]$$

$$[M] = \begin{bmatrix} \langle [0.2, 0.5], [0.1, 0.3] \rangle & \langle [0.3, 0.5], [0.1, 0.2] \rangle \\ \langle [0.2, 0.6], [0.1, 0.4] \rangle & \langle [0.1, 0.5], [0.1, 0.3] \rangle \end{bmatrix}$$

Definition 3.6. *If M_2 and M are given IVIF-matrices in*

$$M_1 \circ M_2 = M \tag{3.2}$$

then we can determine particular IVIF-matrices for M_1 which will be satisfy (3.2). Each of this particular IVIF-matrix for M_1 that satisfies (3.2) is called its IVIF-solution and the set

$$\mathscr{M}(M_2, M) = \{M_1 / M_1 \circ M_2 = M\} \tag{3.3}$$

denotes the IVIF-solution set.

Definition 3.7. *An element p of $\mathscr{M}(A, b)$ is called an IVIF-solution of the equation $Ay = b$ if $p = \left[\left\langle \left[x_{j_{\mu L}}, x_{j_{\mu U}}\right], \left[x_{j_{\nu L}}, x_{j_{\nu U}}\right]\right\rangle / j \in N_m\right]^T$ be defined as,*

$$p = \min \sigma(a_{jk}, b_k) \tag{3.4}$$

where

$$\sigma(a_{jk}, b_k) = \begin{cases} b_k & \text{if } a_{jk} > b_k \\ I & \text{otherwise} \end{cases}$$

$I = \langle [1,1], [0,0] \rangle$, $a_{jk} = \left\langle \left[a_{jk_{\mu L}}, a_{jk_{\mu U}}\right], \left[a_{jk_{\nu L}}, a_{jk_{\nu U}}\right]\right\rangle$ *and* $b_k = \left\langle \left[b_{k_{\mu L}}, b_{k_{\mu U}}\right], \left[b_{k_{\nu L}}, b_{k_{\nu U}}\right]\right\rangle$.

Example 3.8. *Let A and b be two IVIF-matrices,*

$$[A] = \begin{bmatrix} \langle [0.2, 0.7], [0.1, 0.2] \rangle & \langle [0.3, 0.6], [0.1, 0.3] \rangle \\ \langle [0.2, 0.6], [0.1, 0.4] \rangle & \langle [0.1, 0.5], [0.1, 0.3] \rangle \end{bmatrix}$$

and

$$[b] = \left[\langle [0.2, 0.7], [0.1, 0.3] \rangle \ \langle [0.1, 0.6], [0.1, 0.2] \rangle \right]^T$$

Using the above definition of IVIF-solution set, we get,
$p_1 = \min\{\sigma(a_{11}, b_{11}), \sigma(a_{12}, b_{12})\}$
$p_1 = \langle [0.1, 0.6], [0.1, 0.2] \rangle$
$p_2 = \min\{\sigma(a_{21}, b_{11}), \sigma(a_{22}, b_{12})\}$
$p_2 = \langle [1, 1], [0, 0] \rangle$
Then one of the IVIF-solution is $p = [\langle [0.1, 0.6], [0.1, 0.2] \rangle, \langle [1, 1], [0, 0] \rangle]$

Definition 3.9. *An IVIF-transformation $\widetilde{T}$ of $\widetilde{U}$ into $\widetilde{V}$ is called IVIFL-transformations if for every $\alpha, \beta \in \widetilde{V}$ and $x \in F$ then it satisfies the following conditions:*

1. $\widetilde{T}(\alpha + \beta) = \widetilde{T}(\alpha) + \widetilde{T}(\beta)$,
2. $\widetilde{T}(x\alpha) = x.\widetilde{T}(\alpha)$

Where $\alpha = \langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle$ *and* $\beta = \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle$

Example 3.10. *Let* $\widetilde{V}^3$ *and* $\widetilde{V}^2$ *be an IVIFV-spaces over* F. *The IVIF-transformation* $\widetilde{T} : \widetilde{V}^3 \to \widetilde{V}^2$ *defined as* $\widetilde{T}(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2)$ *is an IVIFL- transformation.*

Proposition 3.11. *Let* $\widetilde{T}_1$ *and* $\widetilde{T}_2$ *be two IVIFL- transformations in an IVIFV-space* $\widetilde{V}$ *over* F, *and* $L(\widetilde{V})$ *be the set of all IVIFL-transformations defined on* $\widetilde{V}$, *then* $L(\widetilde{V})$ *is closed under addition and multiplication defined by,*

1. $(\widetilde{T}_1 + \widetilde{T}_2)(\alpha) = \widetilde{T}_1(\alpha) + \widetilde{T}_2(\alpha)$
2. $(x\widetilde{T}_1)(\alpha) = x\widetilde{T}_1(\alpha), \forall \widetilde{T}_1, \widetilde{T}_2 \in \widetilde{V}$ *and* $x \in F$.

Proof. Let $\widetilde{T}_1$ and $\widetilde{T}_2$ be two IVIFL-transformations in an IVIFV- space $\widetilde{V}$ over F.
To prove (1): $\widetilde{T}_1, \widetilde{T}_2 \in \widetilde{V}$ and $x \in F$. Now consider,

$$\begin{aligned}
(\widetilde{T}_1 + \widetilde{T}_2)(\alpha + \beta) &= \widetilde{T}_1(\alpha + \beta) + \widetilde{T}_2(\alpha + \beta) \\
&= \widetilde{T}_1(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle + \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&\quad + \widetilde{T}_2(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle + \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= \widetilde{T}_1(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + \widetilde{T}_1(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&\quad + \widetilde{T}_2(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + \widetilde{T}_2(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= (\widetilde{T}_1 + \widetilde{T}_2)(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) \\
&\quad + (\widetilde{T}_1 + \widetilde{T}_2)(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= (\widetilde{T}_1 + \widetilde{T}_2)(\alpha) + (\widetilde{T}_1 + \widetilde{T}_2)(\beta)
\end{aligned}$$

$\forall \widetilde{T}_1, \widetilde{T}_2 \in L(\widetilde{V})$.

Now consider,

$$\begin{aligned}
(\widetilde{T}_1 + \widetilde{T}_2)(x\alpha) &= \widetilde{T}_1(x\alpha) + \widetilde{T}_2(x\alpha) \\
&= x\widetilde{T}_1(\alpha) + x\widetilde{T}_2(\alpha) \\
&= x(\widetilde{T}_1(\alpha) + \widetilde{T}_2(\alpha)) \\
&= x(\widetilde{T}_1 + \widetilde{T}_2)(\alpha)
\end{aligned}$$

for every $\widetilde{T}_1, \widetilde{T}_2 \in L(\widetilde{V})$ and $x \in F$
To prove (2): For $\alpha \in \widetilde{V}$ and $\widetilde{T} \in L(\widetilde{V})$,

$$(x\widetilde{T})(\alpha + \beta) = x\widetilde{T}(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + (\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle)$$

$$= (\langle[x_{\mu_L}, x_{\mu_U}], [x_{\nu_L}, x_{\nu_U}]\rangle)(\widetilde{T}\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle)$$

$$+\widetilde{T}(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle)$$

$$= x\widetilde{T}(\alpha) + x\widetilde{T}(\beta)$$

Thus $L(\widetilde{V})$ is closed under addition and multiplication. ■

Definition 3.12. *If* $\widetilde{U}$ *and* $\widetilde{V}$ *are IVIFV- spaces then the IVIF-transformation* $\widetilde{T}$ *is defiened by* $\widetilde{T} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T}(\alpha) = \Phi$, *for all* $\alpha \in \widetilde{V}$ *then* $\widetilde{T}$ *is said to be an interval-valued intuitionistic fuzzy zero linear(in briefly* $IVIF_0L$ *)- transformation.*

Definition 3.13. *If $\widetilde{V}$ is an IVIFV-space, then IVIF-transformation $\widetilde{T}$ is defined as,*

$$\widetilde{T}(\alpha) = \alpha, \forall \alpha \in \widetilde{V}$$

then $\widetilde{T}$ is called as an interval-valued intuitionistic fuzzy identity linear(in briefly, $IVIF_I L$)- transformation.

Definition 3.14. *If $\widetilde{U}, \widetilde{V}, \widetilde{W}$ be three IVIFV-spaces over the IVIF-field F such that $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T_2} : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations, then the composition of two IVIFL-transformations, $\widetilde{T_1 T_2}$ is defined by,*

$$(\widetilde{T_1 T_2})\alpha = \widetilde{T}_1(\widetilde{T}_2(\alpha)), \forall \alpha \in \widetilde{W}$$

Remark 3.15. *if Range $(\widetilde{T}_2)$ = Domain $(\widetilde{T}_1)$, then we can define $\widetilde{T_1 T_2}$. Also, $\widetilde{T_1 T_2} \neq \widetilde{T_2 T_1}$.*

Proposition 3.16. *Let $\widetilde{U}, \widetilde{V}$ and $\widetilde{W}$ be an IVIFV-spaces over the IVIF-field F and $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T}_2 : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations, then $\widetilde{T_1 T_2}$ is an IVIFL- transformations from $\widetilde{U} \to \widetilde{W}$.*
Proof: *Let $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$ and $\widetilde{T}_2 : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations. Now define the IVIF-transformation $\widetilde{T_1 T_2} : \widetilde{U} \to \widetilde{W}$ as, $(\widetilde{T_1 T_2})\alpha = \widetilde{T}_1(\widetilde{T}_2(\alpha)), \forall \alpha \in \widetilde{W}$. Let $\alpha, \beta \in \widetilde{W}$ and $x, y \in F$ then,*

$$\begin{aligned}
\widetilde{T_1 T_2}(\alpha + \beta) &= \widetilde{T}_1(\widetilde{T}_2(\alpha + \beta)) \\
&= \widetilde{T}_1(\widetilde{T}_2\alpha + \widetilde{T}_2\beta) \\
&= \widetilde{T}_1(\widetilde{T}_2(\alpha)) + \widetilde{T}_1(\widetilde{T}_2(\beta)) \\
&= \widetilde{T_1 T_2}(\alpha) + \widetilde{T_1 T_2}(\beta)
\end{aligned}$$

Hence $\widetilde{T_1 T_2}$ is an IVIFL- transformations.

Proposition 3.17. *Product of $IVIF_0 L$-transformation with any other IVIFL- transformation is again an $IVIF_0 L$ - transformation.*
Proof: *Let $\widetilde{T_1}$ be any IVIFL-transformation and Φ be an $IVIF_0 L$-transformation. Given $\alpha \in \widetilde{V}$,*
$(\widetilde{T}_1 \Phi)(\alpha) = \widetilde{T}_1(\Phi(\alpha)) = \widetilde{T}_1(\Phi) = \Phi = \Phi(\alpha)$
Similarly, $(\Phi \widetilde{T}_1)(\alpha) = \Phi(\widetilde{T}_1(\alpha) = \Phi$.
Hence $(\widetilde{T}_1 \Phi)(\alpha) = (\Phi \widetilde{T}_1(\alpha)) = \Phi$.

Proposition 3.18. *Product of $IVIF_I L$-transformation with any other IVIFL-transformation is an IVIFL-transformation.*
Proof: *Let $\widetilde{T_1}$ be an IVIFL-transformation and $\mathscr{I}$ be an $IVIF_I L$-transformation. Given $\alpha \in \widetilde{V}$,*
$(\widetilde{T}_1 \mathscr{I})(\alpha) = \widetilde{T}_1(\mathscr{I}(\alpha)) = \widetilde{T}_1(\alpha)$
Like this manner, we can prove, $(\mathscr{I} \widetilde{T}_1)(\alpha) = \widetilde{T}_1(\alpha)$.
Hence, $(\widetilde{T}_1 \mathscr{I})(\alpha) = (\mathscr{I} \widetilde{T}_1)(\alpha) = \widetilde{T}_1(\alpha)$.

4. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20(1)**(1986), 87–96.

[2] K.T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **31**(1989), 343–349.

[3] K.T. ATANASSOV, Operations over interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **64**(1994), 159–174.

[4] M. BHOWMIK AND M. PAL, Intuitionistic fuzzy linear transformations, *Ann. Pure and Appl. Math.*, **1(1)**(2012), 57–68.

[5] H.L. HUANG, (T, S) - based interval-valued intuitionistic fuzzy composition matrix and its application For clustering, *Iranian J Fuzzy Systems*, **9(5)**(2012), 7–19.

[6] A.K. KATSARAS AND D.B. LIU, Fuzzy vector spaces and Fuzzy topological vector vpaces, *Far-East J Math. Sci.*, **58**(1977), 135–146.

[7] K.H. KIM AND F.W. ROUSH, Generalized fuzzy matrices, *Fuzzy Sets and Systems*, **4**(1980), 293–315.

[8] M.L. LIN AND H.L. HUANG, (T, S) -based intuitionistic fuzzy composite matrix and its application, *Int. J Appl. Math. Stat.*, **23**(2011), 54–63.

[9] M. PAL, S.K. KHAN AND A.K. SHYAMAL, Intuitionistic Fuzzy Matrices, *Notes on Intuitionistic Fuzzy sets*, **8(2)**(2002), 51–62.

[10] MADHUMANGAL PAL AND SUSANTA K.KHAN, Interval-Valued Intuitionistic Fuzzy Matrices, *Notes on Intuitionistic Fuzzy Sets*, **11(1)**(2005), 16–27.

[11] A. R. MEENAKSHI AND T. GANDHIMATHI, Intuitionistic fuzzy linear Transformations, *Int. J Compu. Sci. Math.*, **3(1)**(2011), 99–108.

[12] MOUMITA CHINEY AND S.K. SAMANTA, Intuitionistic Fuzzy Basis of an Intuitionistic Fuzzy Vector Space *Notes on Intuitionistic Fuzzy Sets*, **23(4)**(2017),62–74.

[13] A. NARAYANAN, S. VIJAYABALAJI AND N. THILLAIGOVINDHAN, Intuitionistic Fuzzy Linear Operators, *Iranian J. Fuzzy Sys.*, **4**(2007), 89–101.

[14] RAJKUMAR PRADHAN AND MADHUMANGAL PAL, Intuitionistic fuzzy linear Transformations *Ann. Pure Appl. Math.*, **1(1)**(2012), 57–68.

[15] R. SANTHI AND N. UDHAYARANI, Properties of Interval-Valued Intuitionistic Fuzzy Vector Space, *Notes on Intuitionistic Fuzzy Sets*, **25(1)**(2019),12–20.

[16] S.K. SHYAMAL AND M. PAL, Interval-Valued Fuzzy Matrices, *J Fuzzy Math.*, **14(3)**(2006),583–604.

[17] Y. TERAO AND N. KITSUNEZAKI, Fuzzy Sets and Linear Mappings on Vector Spaces, *Mathematica Japonica*, **39(1)**(1994), 61–68.

[18] Z.S. XU AND R.R. YAGER, Intuitionistic and Interval-Valued Intuitionistic Fuzzy Preference Relations and Their Measures of Similarity for the Evaluation of Agreeement within a Group, *Fuzzy Opti. Decision Making*, **8**(2009), 123–139.

[19] L.A. ZADEH, Fuzzy Sets, *Info. Cont.*, **8**(1965), 338–353.

[20] ZE-SHUI XU AND JIAN CHEN, Approach to Group Decision Making Based on Interval-Valued Intuitionistic Judgement Matrices, *Sys. Engi.: Theory and Practices*, **27**(2007), 126–133.