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A common fixed point theorem for four weakly compatible self maps of a S-metric space using (CLR) property

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Abstract. In this paper, by employing a contractive condition of integral type, we obtain a unique common fixed point for four weakly compatible self maps of a S-metric space which satisfy common limit range property.

AMS Subject Classifications: 54H25, 47H10.

Keywords: S-metric space, Fixed point, Weakly compatibility, Common limit range property.

Contents

1. Introduction

Gerald Jungck [7] introduced the concept of compatibility to generalized the notion of commutative property. Further Jungck and Rhoades [8] proposed weakly compatibility of mappings. Also they proved that for a pair of mappings compatibility always implies weakly compatibility but not conversely.

To prove common fixed point theorems, Sintunavarat et al [15] initiated common limit range (CLR) property, which generalized the (E.A) property proposed M. Aamri, D. El Moutawakil [1].

Several authors Dhage, Gahler, Sedghi, Mustafa [3–5, 14, 16] generalized the notion of metric space by introducing 2-metric space, D^*-metric spaces and G-metric spaces.

Shaban Sedghi et al [13] proposed S-metric space as further generalization of metric spaces. This concept of S-metric spaces generated lot of interest among many researches.

In this paper, we prove a common fixed point theorem for four weakly compatible self maps of S-metric space satisfying common limit range property along with an integral type contractive condition [2]. Our result generalizes the results already proved in literature [6]. A suitable example is provided to validate our theorem.

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2. Preliminaries

Definition 2.1. *[13] Let M be non empty set. A function $S : M^3 \longrightarrow [0, \infty)$ is said to be a S-metric on M, if for each $\nu, \omega, \vartheta, \lambda \in M$*

1. $S(\nu, \omega, \vartheta) \geq 0$
2. $S(\nu, \omega, \vartheta) = 0 \Leftrightarrow \nu = \omega = \vartheta$
3. $S(\nu, \omega, \vartheta) \leq S(\nu, \nu, \lambda) + S(\omega, \omega, \lambda) + S(\vartheta, \vartheta, \lambda)$

then (M, S) is called a S-metric space.

Lemma 2.2. *[11] In a S-metric space we have $S(\nu, \nu, \omega) = S(\omega, \omega, \nu)$ for all $\nu, \omega \in M$.*

Definition 2.3. *[12] Let (M, S) be a S- metric space. A sequence (ν_n) in M is said to be convergent if there is a $\nu \in M$ such that $S(\nu_n, \nu_n, \nu) \to 0$ as $n \to \infty$, that is, for each $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(\nu_n, \nu_n, \nu) < \epsilon$ and we denote this by writing $\lim_{n\to\infty} \nu_n = \nu$.*

Definition 2.4. *[12] Let (M, S) be a S- metric space. A sequence (ν_n) in M is said to be Cauchy sequence if for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $S(\nu_n, \nu_n, \nu_m) \to 0$ for each $n, m \geq n_0$.*

Definition 2.5. *[12] A S- metric space (M, S) is said to be complete if for every Cauchy sequence converges to some point in it.*

Lemma 2.6. *[12] In a S-metric space (M, S), if there exist two sequences (ν_n) and (ω_n) such that $\lim_{n\to\infty} \nu_n = \nu$ and $\lim_{n\to\infty} \omega_n = \omega$, then $\lim_{n\to\infty} S(\nu_n, \nu_n, \omega_n) = S(\nu, \nu, \omega)$.*

Definition 2.7. *[8] The self mappings H, J of a S-metric space (M, S) are called weakly compatible if $HJ\nu = JH\nu$ whenever $H\nu = J\nu$ for any ν in M.*

Definition 2.8. *[9] In a S-metric space (M, S), the two pairs of self mappings (H, K) and (J, L) of M are said to satisfy common (E.A) property if there exist two sequences (ν_n) and (ω_n) in M such that*

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma, \text{ where } \gamma \in M.$$

Definition 2.9. *[15] In a S-metric space (M, S), the two pairs of self mappings (H, K) and (J, L) on M are said to satisfy common limit range property with respect to K and L, denoted by (CLR_{KL}) if there exists two sequences (ν_n) and (ω_n) in M such that*

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma, \text{ where } \gamma \in K(M) \cap L(M).$$

Remark 2.10. *Throughout this paper $f : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable function which is summable on compact subset of $[0, \infty)$ with $\int_0^\epsilon f(\gamma)d\gamma > 0$, for any $\epsilon > 0$.*

Remark 2.11. *Throughout this paper $g : [0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing continuous function on $(0, \infty)$ and $g(\gamma) = 0 \Leftrightarrow \gamma = 0$.*

Remark 2.12. *Throughout this paper $h : [0, \infty) \longrightarrow [0, \infty)$ is a upper semicontinuous function on $(0, \infty)$ with $h(0) = 0$ and $h(\gamma) < \gamma$, for any $\gamma > 0$*

Lemma 2.13. *[10] Let $f : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable function which is summable on compact subset of $[0, \infty)$ with $\int_0^\epsilon f(\gamma)d\gamma > 0$, for any $\epsilon > 0$ and $\{\rho_n\}_{n\geq 1}$ be a non negative sequence with $\lim_{n\to\infty} \rho_n = \theta$. Then we have*

$$\lim_{n\to\infty} \int_0^{\rho_n} f(\gamma)d\gamma = \int_0^{\theta} f(\gamma)d\gamma.$$

3. Main Result

Now we state our main theorem.

Theorem 3.1. *In a S-metric space (M, S), suppose H, J, K and L are self mappings of M satisfying the following conditions*

(i) The pairs (H, K) and (J, L) satisfy (CLR_{KL}) property

(ii) The pairs (H, K) and (J, L) are weakly compatible

(iii)

$$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma$$

where

$$\begin{aligned} p(\nu,\nu,\omega) = max\{&S(K\nu,K\nu,L\omega), S(H\nu,H\nu,K\nu), S(J\omega,J\omega,L\omega), \\ &\frac{S(K\nu,K\nu,J\omega)+S(L\omega,L\omega,H\nu)}{2}, \frac{S(H\nu,H\nu,K\nu)S(J\omega,J\omega,L\omega)}{1+S(K\nu,K\nu,L\omega)}, \\ &\frac{S(H\nu,H\nu,L\omega)S(J\omega,J\omega,K\nu)}{1+S(K\nu,K\nu,L\omega)}, \\ &S(H\nu,H\nu,K\nu)(\frac{1+S(K\nu,K\nu,J\omega)+S(L\omega,L\omega,H\nu)}{1+S(H\nu,H\nu,K\nu)+S(L\omega,L\omega,J\omega)})\} \end{aligned}$$

then H, J, K and L have a unique common fixed point in M.

Proof. From the (CLR_{KL}) property of the pairs (H, K) and (J, L), we have two sequences (ν_n) and (ω_n) in M such that

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma, \ where\ \gamma \in K(M) \cap L(M). \tag{3.1}$$

Also there exists a point $\eta \in M$ such that $K\eta = \gamma$, from (3.1), we have

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma = K\eta.$$

We now claim that $H\eta = K\eta$, for if $H\eta \neq K\eta$ then $S(H\eta, H\eta, K\eta) > 0$.
Keeping $\nu = \eta$ and $\omega = \omega_n$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\eta,H\eta,J\omega_n)} f(\gamma)d\gamma) \leq g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma) - \int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma. \tag{3.2}$$

Then

$$\begin{aligned} p(\eta,\eta,\omega_n) = max\{&S(K\eta,K\eta,L\omega_n), S(H\eta,H\eta,K\eta), S(J\omega_n,J\omega_n,L\omega_n), \\ &\frac{S(K\eta,K\eta,J\omega_n)+S(L\omega_n,L\omega_n,H\eta)}{2}, \frac{S(H\eta,H\eta,K\eta)S(J\omega_n,J\omega_n,L\omega_n)}{1+S(K\eta,K\eta,L\omega_n)}, \\ &\frac{S(H\eta,H\eta,L\omega_n)S(J\omega_n,J\omega_n,K\eta)}{1+S(K\eta,K\eta,L\omega_n)}, \\ &S(H\eta,H\eta,K\eta)(\frac{1+S(K\eta,K\eta,J\omega_n)+S(L\omega_n,L\omega_n,H\eta)}{1+S(H\eta,H\eta,K\eta)+S(L\omega_n,L\omega_n,J\omega_n)})\}. \end{aligned}$$

Now

$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = max\{S(\gamma,\gamma,\gamma), S(H\eta,H\eta,\gamma), S(\gamma,\gamma,\gamma), \frac{S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,H\eta)}{2}, \frac{S(H\eta,H\eta,\gamma)S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)}, \frac{S(H\eta,H\eta,\gamma)S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)}, S(H\eta,H\eta,\gamma)(\frac{1+S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,H\eta)}{1+S(H\eta,H\eta,\gamma)+S(\gamma,\gamma,\gamma)})\}$$

$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = max\{0, S(H\eta,H\eta,\gamma), 0, \frac{S(\gamma,\gamma,H\eta)}{2}, 0, 0, S(H\eta,H\eta,\gamma)\}$$

$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = S(H\eta,H\eta,\gamma).$$

On taking the limit in (3.2), we get

$$\begin{aligned} g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) &= \limsup_{n\to\infty} g(\int_0^{S(H\eta,H\eta,J\omega_n)} f(\gamma)d\gamma) \\ &\leq \limsup_{n\to\infty} \{g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma) - \int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma\} \\ &\leq \limsup_{n\to\infty} (g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma)) - \liminf_{n\to\infty} \int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma. \end{aligned}$$

From Lemma 2.13, we get

$$\begin{aligned} g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) &\leq g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) - \int_0^{h(S(H\eta,H\eta,\gamma))} f(\gamma)d\gamma \\ &< g(\int_0^{S(H\eta,H\eta,\gamma)} g(\gamma)d\gamma). \end{aligned}$$

Which is a contradiction and hence $H\eta = K\eta$.
Therefore we get

$$H\eta = K\eta = \gamma. \tag{3.3}$$

Similarly there exists a point $\xi \in M$ such that $L\xi = \gamma$, from (3.1), we have

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma = L\xi.$$

We now claim that $J\xi = L\xi$, for if $J\xi \neq L\xi$ then $S(J\xi, J\xi, L\xi) > 0$.
Keeping $\nu = \nu_n$ and $\omega = \xi$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\nu_n,H\nu_n,J\xi)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma. \tag{3.4}$$

Then

$$p(\nu_n,\nu_n,\xi) = max\{S(K\nu_n,K\nu_n,L\xi), S(H\nu_n,H\nu_n,K\nu_n), S(J\xi,J\xi,L\xi), \frac{S(K\nu_n,K\nu_n,J\xi)+S(L\xi,L\xi,H\nu_n)}{2}, \frac{S(H\nu_n,H\nu_n,K\nu_n)S(J\xi,J\xi,L\xi)}{1+S(K\nu_n,K\nu_n,L\xi)}, \frac{S(H\nu_n,H\nu_n,L\xi)S(J\xi,J\xi,K\nu_n)}{1+S(K\nu_n,K\nu_n,L\xi)}, S(H\nu_n,H\nu_n,K\nu_n)(\frac{1+S(K\nu_n,K\nu_n,J\xi)+S(L\xi,L\xi,H\nu_n)}{1+S(H\nu_n,H\nu_n,K\nu_n)+S(L\xi,L\xi,J\xi)})\}.$$

Now

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = max\{S(\gamma,\gamma,\gamma), S(\gamma,\gamma,\gamma), S(J\xi,J\xi,\gamma), \frac{S(\gamma,\gamma,J\xi)+S(\gamma,\gamma,\gamma)}{2}, \frac{S(\gamma,\gamma,\gamma)S(J\xi,J\xi,\gamma)}{1+S(\gamma,\gamma,\gamma)}, \frac{S(\gamma,\gamma,\gamma)S(J\xi,J\xi,\gamma)}{1+S(\gamma,\gamma,\gamma)}, S(\gamma,\gamma,\gamma)(\frac{1+S(\gamma,\gamma,J\xi)+S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,J\xi)})\}$$

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = max\{0,0,S(J\xi,J\xi,\gamma), \frac{S(\gamma,\gamma,J\xi)}{2}, 0,0,0\}$$

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = S(J\xi,J\xi,\gamma).$$

On taking the limit in (3.4), we get

$$\begin{aligned} g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) &= \limsup_{n\to\infty} g(\int_0^{S(H\nu_n,H\nu_n,J\xi)} f(\gamma)d\gamma) \\ &\leq \limsup_{n\to\infty}\{g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma\} \\ &\leq \limsup_{n\to\infty}(g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma)) - \liminf_{n\to\infty}\int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma. \end{aligned}$$

From Lemma 2.13, we get

$$\begin{aligned} g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) &\leq g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) - \int_0^{h(S(\gamma,\gamma,J\xi))} f(\gamma)d\gamma \\ &< g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma). \end{aligned}$$

Which is a contradiction and hence $J\xi = L\xi$.
Therefore we get

$$J\xi = L\xi = \gamma. \tag{3.5}$$

From (3.3) and (3.5), we get

$$H\eta = K\eta = J\xi = L\xi = \gamma.$$

Now we establish γ is a common fixed point of H, J, L and K.
Clearly $HK\eta = KH\eta$
from which we get

$$H\gamma = K\gamma$$

and
$JL\xi = LJ\xi$ which implies

$$J\gamma = L\gamma.$$

Now we prove that $H\gamma = \gamma$, for if $H\gamma \neq \gamma$ then $S(H\gamma, H\gamma, \gamma) > 0$.
Substituting $\nu = \gamma$ and $\omega = \xi$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\gamma,H\gamma,J\xi)} f(\gamma)d\gamma) \leq g(\int_0^{p(\gamma,\gamma,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\gamma,\gamma,\xi))} f(\gamma)d\gamma. \tag{3.6}$$

Then

$$p(\gamma,\gamma,\xi) = max\{S(K\gamma,K\gamma,L\xi), S(H\gamma,H\gamma,K\gamma), S(J\xi,J\xi,L\xi),$$
$$\frac{S(K\gamma,K\gamma,J\xi)+S(L\xi,L\xi,H\gamma)}{2}, \frac{S(H\gamma,H\gamma,K\gamma)S(J\xi,J\xi,L\xi)}{1+S(K\gamma,K\gamma,L\xi)},$$
$$\frac{S(H\gamma,H\gamma,L\xi)S(J\xi,J\xi,K\gamma)}{1+S(K\gamma,K\gamma,L\xi)},$$
$$S(H\gamma,H\gamma,K\gamma)(\frac{1+S(K\gamma,K\gamma,J\xi)+S(L\xi,L\xi,H\gamma)}{1+S(H\gamma,H\gamma,K\gamma)+S(L\xi,L\xi,J\xi)})\}$$

$$p(\gamma,\gamma,\xi) = max\{S(H\gamma,H\gamma,\gamma), S(H\gamma,H\gamma,H\gamma), S(\gamma,\gamma,\gamma),$$
$$\frac{S(H\gamma,H\gamma,\gamma)+S(\gamma,\gamma,H\gamma)}{2}, \frac{S(H\gamma,H\gamma,H\gamma)S(\gamma,\gamma,\gamma)}{1+S(H\gamma,H\gamma,\gamma)},$$
$$\frac{S(H\gamma,H\gamma,\gamma)S(\gamma,\gamma,H\gamma)}{1+S(H\gamma,H\gamma,\gamma)},$$
$$S(H\gamma,H\gamma,H\gamma)(\frac{1+S(H\gamma,H\gamma,\gamma)+S(\gamma,\gamma,H\gamma)}{1+S(H\gamma,H\gamma,H\gamma)+S(\gamma,\gamma,\gamma)})\}$$

$$p(\gamma,\gamma,\xi) = max\{S(H\gamma,H\gamma,\gamma), 0, 0, S(H\gamma,H\gamma,\gamma), 0, \frac{S(H\gamma,H\gamma,\gamma)S(H\gamma,H\gamma,\gamma)}{1+S(H\gamma,H\gamma,\gamma)}, 0\}$$

$$p(\gamma,\gamma,\xi) = S(H\gamma,H\gamma,\gamma).$$

From (3.6), we get

$$g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma) \leq g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma) - \int_0^{h(S(H\gamma,H\gamma,\gamma))} f(\gamma)d\gamma$$
$$< g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma).$$

Which is a contradiction and hence $H\gamma = \gamma$.
Therefore we get

$$H\gamma = K\gamma = \gamma. \tag{3.7}$$

Similarly we can prove that

$$J\gamma = L\gamma = \gamma. \tag{3.8}$$

From (3.7) and (3.8), we get

$$H\gamma = K\gamma = J\gamma = L\gamma = \gamma.$$

Proving γ is a fixed point of H, J, K and L.
For if $\zeta(\zeta \neq \gamma)$ is in M such that

$$H\zeta = K\zeta = J\zeta = L\zeta = \zeta.$$

On taking $\nu = \gamma$ and $\omega = \zeta$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\gamma,H\gamma,J\zeta)} f(\gamma)d\gamma) \leq g(\int_0^{p(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) - \int_0^{h(p(\gamma,\gamma,\zeta))} f(\gamma)d\gamma. \tag{3.9}$$

Then

$$\begin{aligned} p(\gamma,\gamma,\zeta) = max\{&S(K\gamma,K\gamma,L\zeta),S(H\gamma,H\gamma,K\gamma),S(J\zeta,J\zeta,L\zeta),\\ &\frac{S(K\gamma,K\gamma,J\zeta)+S(L\zeta,L\zeta,H\gamma)}{2},\frac{S(H\gamma,H\gamma,K\gamma)S(J\zeta,J\zeta,L\zeta)}{1+S(K\gamma,K\gamma,L\zeta)},\\ &\frac{S(H\gamma,H\gamma,L\zeta)S(J\zeta,J\zeta,K\gamma)}{1+S(K\gamma,K\gamma,L\zeta)},\\ &S(H\gamma,H\gamma,K\gamma)(\frac{1+S(K\gamma,K\gamma,J\zeta)+S(L\zeta,L\zeta,H\gamma)}{1+S(H\gamma,H\gamma,K\gamma)+S(L\zeta,L\zeta,J\zeta)})\} \end{aligned}$$

$$\begin{aligned} p(\gamma,\gamma,\zeta) = max\{&S(\gamma,\gamma,\zeta),S(\gamma,\gamma,\gamma),S(\zeta,\zeta,\zeta),\\ &\frac{S(\gamma,\gamma,\zeta)+S(\zeta,\zeta,\gamma)}{2},\frac{S(\gamma,\gamma,\gamma)S(\zeta,\zeta,\zeta)}{1+S(\gamma,\gamma,\zeta)},\\ &\frac{S(\gamma,\gamma,\zeta)S(\zeta,\zeta,\gamma)}{1+S(\gamma,\gamma,\zeta)},\\ &S(\gamma,\gamma,\gamma)(\frac{1+S(\gamma,\gamma,\zeta)+S(\zeta,\zeta,\gamma)}{1+S(\gamma,\gamma,\gamma)+S(\zeta,\zeta,\zeta)})\} \end{aligned}$$

$$p(\gamma,\gamma,\zeta) = max\{S(\gamma,\gamma,\zeta),0,0,S(\gamma,\gamma,\zeta),0,\frac{S(\gamma,\gamma,\zeta)S(\gamma,\gamma,\zeta)}{1+S(\gamma,\gamma,\zeta)},0\}$$

$$p(\gamma,\gamma,\zeta) = S(\gamma,\gamma,\zeta).$$

From (3.9), we get

$$\begin{aligned} g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) &\leq g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) - \int_0^{h(S(\gamma,\gamma,\zeta))} f(\gamma)d\gamma\\ &< g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma). \end{aligned}$$

Which is a contradiction and hence $\gamma = \zeta$.
Proving that H, J, K and L have a unique common fixed point in M. ■

As an illustration we have the following example.

Example 3.2. *Let $M = (0,1]$. Define $S(\nu,\omega,\vartheta) = |\nu-\vartheta| + |\omega-\vartheta|$, where $\nu,\omega,\vartheta \in M$, then S is a S-metric on M. Now let H, J, K and L be self maps on M, defined by*

$$H(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}],\\ \frac{1}{5}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases} \qquad J(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}],\\ \frac{1}{3}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases}$$

$$K(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}],\\ \frac{1}{7}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases} \qquad L(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}],\\ \frac{1}{9}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases}$$

Also take $f(\gamma) = 3\gamma$, $g(\gamma) = \dfrac{\gamma}{3}$ and h(γ) as floor function.

Let (ν_n) and (ω_n) be sequences in M with $\nu_n = \dfrac{1}{n+1}$ and $\omega_n = \dfrac{1}{n+3}$, where $n \geq 1$, then

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} H(\frac{1}{n+1}) = \frac{1}{2},$$

$$\lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} K(\frac{1}{n+1}) = \frac{1}{2},$$

$$\lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} J(\frac{1}{n+3}) = \frac{1}{2},$$

$$\lim_{n\to\infty} L\omega_n = \lim_{n\to\infty} L(\frac{1}{n+3}) = \frac{1}{2}.$$

$$Thus \lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \frac{1}{2}\ and\ \frac{1}{2} \in K(M) \cap L(M).$$

Proving (H, K) and (J, L) satisfy (CLR_{KL}) property.

Also $H\nu = K\nu$, for all $\nu \in (0, \frac{1}{2}]$

H(K(ν)) $= \frac{1}{2} = K(H(\nu))$

therefore (H, K) is weakly compatible.

Similarlry (J, L) is also weakly compatible.

Now we verify the condition (iii) of Theorem 3.1 in different cases.

Case(i): Let $\nu, \omega \in (0, \frac{1}{2}]$

then $H\nu = K\nu = J\omega = L\omega = \frac{1}{2}$ and $p(\nu, \nu, \omega) = 0$, $S(H\nu, H\nu, J\omega) = 0$ from condition (iii) of Theorem 3.1, we get

$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = 0$ also $g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = 0.$

Case(ii): Let $\nu, \omega \in (\frac{1}{2}, 1]$

$H\nu = \frac{1}{5}, K\nu = \frac{1}{7}, J\omega = \frac{1}{3}, L\omega = \frac{1}{9}$ and

$$p(\nu, \nu, \omega) = max\{\frac{4}{63}, \frac{4}{35}, \frac{4}{9}, \frac{88}{315}, \frac{16}{335}, \frac{64}{1005}, \frac{4}{35}\} = \frac{4}{9}$$

$S(H\nu, H\nu, J\omega) = \frac{4}{15}$, then condition (iii) of Theorem 3.1, we get

$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = g(\int_0^{\frac{4}{15}} 3\gamma d\gamma) = \frac{8}{225}$ and

$$g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = g(\int_0^{\frac{4}{9}} 3\gamma d\gamma) - \int_0^{h(\frac{4}{9})} 3\gamma d\gamma$$
$$= \frac{8}{81}.$$

Thus $\frac{8}{225} < \frac{8}{81}$.

Case(iii): Let $\nu \in (0, \frac{1}{2}], \omega \in (\frac{1}{2}, 1]$

$H\nu = K\nu = \frac{1}{2}, J\omega = \frac{1}{3}, L\omega = \frac{1}{9}$ and

$$p(\nu,\nu,\omega) = max\{\frac{7}{9}, 0, \frac{4}{9}, \frac{5}{9}, 0, \frac{7}{48}, 0\} = \frac{7}{9}$$

$S(H\nu, H\nu, J\omega) = \frac{1}{3}$, then condition (iii) of Theorem 3.1, we get

$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = g(\int_0^{\frac{1}{3}} 3\gamma d\gamma) = \frac{1}{18}$ and

$$g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = g(\int_0^{\frac{7}{9}} 3\gamma d\gamma) - \int_0^{h(\frac{7}{9})} 3\gamma d\gamma$$
$$= \frac{49}{162}.$$

Thus $\frac{1}{18} < \frac{49}{162}$.
From above cases
$g\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma$.
Similarly we can check condition (iii) of Theorem 3.1 in case if $\nu \in (\frac{1}{2}, 1]$, $\omega \in (0, \frac{1}{2}]$.
Hence condition (iii) of Theorem 3.1 is satisfied in different cases.
Thus all conditions of Theorem 3.1 are satisfied and clearly $\frac{1}{2}$ is the unique common fixed point of H, J, K and L.

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