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On distinguishing labelling of groups for the conjugation action

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Abstract. In this paper, the conjugation action of various classes of groups on themselves is studied to obtain their distinguishing numbers along with a distinguishing labelling for the said action. An equivalent condition concerning the existence of a 2-distinguishing labelling for the action of a group $\mathcal{G}$ on a $\mathcal{G}$-set $\mathcal{X}$ and a partition of $\mathcal{X}$ into two subsets is established. Also, the distinguishing number for the conjugation action of a group acting on itself is completely characterized.
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1. Introduction

The concept of distinguishing number originated from an elementary problem known as the Frank Rubin's Key problem [5], which states that:

Professor X has n keys on a circular key ring, but he can not see them. Now, the question arises: How many shapes does Professor X need to use in order to keep n keys on the ring and still be able to select the proper key by feel?

Albertson and Collins [1] popularized the aforementioned problem and took it in the realm of graphs by connecting it with the action of symmetries of a graph on its set of vertices. They define the distinguishing number for a graph G to be the minimum k for which the vertices of G can be labeled from 1 to k such that no non-trivial symmetry (automorphism) of the graph G preserves all of the vertex labels.

The surprise answer to the above problem is that only two different handle shapes are required if six or more keys are there in the key ring; but minimum three different handle shapes are required to distinguish the keys for three, four and five keys in the key ring. This motivated us to study and understand the concept of distinguishing labelling and distinguishing number for the conjugation action of a group on itself.

Further, in [6], Tymoczko generalized the notion of the distinguishing number for an arbitrary group action that is, the distinguishing number for the action of an arbitrary group $\mathcal{G}$ on a $\mathcal{G}$-set $\mathcal{X}$ which is not necessarily the action of the automorphism group of a graph on the set of vertices of that graph.

The main objective of this paper is to compute the distinguishing number and a distinguishing labelling for the conjugation action of some well known classes of groups such as Q_{4n} (dicyclic group), V_{8n}, U_{6n} and SD_{8n} (semi-dihedral group) acting on itself. It is observed that for each of these classes of groups the distinguishing number is 2 and this concurrence of the distinguishing number for the conjugation action for all aforementioned groups arises a natural question as follows:

Does there exist any group $\mathcal{G}$ with distinguishing number more than 2 for its conjugation action on itself?

This question stimulates us to completely characterize the distinguishing number for the conjugation of a group on itself. We answer the above question in the sixth section of this paper. In order to achieve this aim, an equivalent condition for the existence of a 2-distinguishing labelling for the action of a group on a $\mathcal{G}$-set $\mathcal{X}$ is established. In addition, by using this condition and some other results, we completely characterize the distinguishing number for the group action of a group $\mathcal{G}$, acting on itself by the conjugation action.

The main results proved in the present paper are:

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(1) The distinguishing numbers and a corresponding distinguishing labelling for the conjugation action of some well known classes of groups such as Q_{4n} (Dicyclic group), V_{8n}, U_{6n} and SD_{8n} (semi-dihedral group) are computed.

(2) A relation between a distinguishing labelling for the action of a group $\mathcal{G}$ on a $\mathcal{G}$-set $\mathcal{X}$ and a partition of the underlying set $\mathcal{X}$ has been identified.

(3) The distinguishing number for the conjugation action of a group on itself is completely characterized.

Our notations are as follows: $\mathcal{G}$ is a group and $\mathcal{X}$ is a $\mathcal{G}$-set. The stabilizer of a subset $\mathcal{A} \subseteq \mathcal{X}$ is $Stab_{\mathcal{G}}(\mathcal{A}) = \{g \in \mathcal{G} : ga = a \text{ for all } a \in \mathcal{A}\}$. We begin this article with the following definitions:

Definition 1.1. *[6] Let $\mathcal{G}$ be a group acting on a nonempty set $\mathcal{X}$. A map $\phi : \mathcal{X} \longrightarrow \{1, 2, ..., k\}$ is said to be a k-distinguishing labelling of the action of $\mathcal{G}$ on the set $\mathcal{X}$ if the only group elements that preserve the labelling are in $Stab_{\mathcal{G}}(\mathcal{X})$. Equivalently, the map ϕ is a k-distinguishing labelling if $\{g : \phi \circ g\ (x) = \phi(x) \text{ for all } x \in \mathcal{X}\} = Stab_{\mathcal{G}}(\mathcal{X})$.*

Definition 1.2. *[6] The distinguishing number $\mathcal{D}_{\mathcal{G}}(\mathcal{X})$ of the set $\mathcal{X}$ with a given group action of $\mathcal{G}$ on $\mathcal{X}$ is the minimum k for which there is a k-distinguishing labelling.*

It is also pertinent to notice that if the $\mathcal{G}$-set $\mathcal{X}$ is equal to $\mathcal{G}$, then under the conjugation action of the group $\mathcal{G}$ on $\mathcal{X}$, the set $Stab_{\mathcal{G}}(\mathcal{X}) = \{g \in \mathcal{G} : ghg^{-1} = h \text{ for all } h \in \mathcal{X}\} = Z(\mathcal{G})$, is the center of the group $\mathcal{G}$. In this case, a map $\phi : \mathcal{X} \longrightarrow \{1, 2, ..., k\}$ is said to be a k-distinguishing labelling for the conjugation action of $\mathcal{G}$ on the set $\mathcal{X}$ if the only group elements that preserve the labelling are in the center of the group $\mathcal{G}$. More precisely, $\{g : \phi(ghg^{-1}) = \phi(h) \text{ for all } h \in \mathcal{G}\} = Z(\mathcal{G})$. Also, the minimum k for which there exists a k-distinguishing labelling ϕ satisfying the above equality is the distinguishing number for the conjugation action of a group on itself.

Moreover, it is not difficult to see that a non-Abelian group $\mathcal{G}$, under the conjugation action, cannot act on itself by fixing each of its elements.

The above observation, in view of Proposition 2.1 [6], leads to the following theorem.

Theorem 1.3. *The distinguishing number for the conjugation action of a non-Abelian group $\mathcal{G}$ acting on itself, is at least* 2.

Throughout this paper, the action of a group $\mathcal{G}$ represents the conjugation action of the group $\mathcal{G}$ on itself, unless stated otherwise.

2. Distinguishing Labelling for the Dicyclic group Q_{4n}

In group theory, a dicyclic group, $\mathcal{G} = Q_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, ab = ba^{-1}\rangle$ is a non-Abelian group of order $4n$ for $n > 1$, which can be viewed as an extension of the cyclic group of order 2 by a cyclic group of order 2n, described by an exact sequence as follows:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Q_{4n} \longrightarrow \mathbb{Z}_{2n} \longrightarrow 1.$$

The dicyclic group is a subgroup of the unit quaternions generated by the elements a and j; where $a = e^{\frac{i\pi}{n}}$ is an n^{th} root of unity. Further, the number of conjugacy classes in Q_{4n} is $n + 3$ namely;

$\{1\}$,
$\{a^n\}$,
$\{a^r, a^{-r}\}; (1 \leq n \leq n-1)$,
$\{a^{2j}b : 0 \leq j \leq n-1\}$,
$\{a^{2j+1}b : 0 \leq j \leq n-1\}$.

Note that one can realize the set of elements of the group Q_{4n} as $Q_{4n} = \{a^k, a^kb : k = 0, 1, 2, ..., 2n-1\}$ and we begin this section with the following lemma.

Lemma 2.1. *For the dicyclic group Q_{4n}, we have*

(1) $ba^j = a^{-j}b$, for any integer j.

(2) For $n > 1$, $Z(Q_{4n}) = \{1, a^n\}$.

As the group Q_{4n} is non-Abelian for $n \geq 2$, while it is Abelian for $n = 1$, therefore in consideration of Proposition 2.1 [6], we have

Proposition 2.2. *If Q_4 acts on itself by conjugation, then $\mathcal{D}_{Q_4}(Q_4) = 1$.*

Before proceeding further, for convenience sake, we partition the set of elements of Q_{4n} for $n \geq 2$, as follows:

$$Q_{4n} = \mathcal{X}_n^{'} \cup \mathcal{X}_n, \text{ where}$$

$\mathcal{X}_n^{'} = \{a^{n+1}, a^{n+2}, ..., a^{2n-1}, a^2b, a^3b\}$ and $\mathcal{X}_n = Q_{4n} \backslash \mathcal{X}_n^{'} = \{1, a, ..., a^n, b, a^4b, a^6b, ..., a^{2n-2}b, ab, a^5b, a^7b, ..., a^{2n-1}b\}$.

Proposition 2.3. *The stabilizer of the subset $\mathcal{X}_n^{'} = \{a^{n+1}, a^{n+2}, ..., a^{2n-1}, a^2b, a^3b\}$ in Q_{4n} is $Stab_{Q_{4n}}(\mathcal{X}_n^{'}) = \{1, a^n\}$.*

Proof. By Lemma 2.1(2), it is enough to show that $Stab_{Q_{4n}}(\mathcal{X}_n^{'}) \subseteq \{1, a^n\}$. For this, let $g \in Stab_{Q_{4n}}(\mathcal{X}_n^{'}) \cap Q_{4n}$. Then $ghg^{-1} = h$, for all $h \in \mathcal{X}_n^{'}$ and we know that an element in Q_{4n} is of the form a^ib^j, where $0 \leq i \leq 2n-1$, $j = 0, 1$, so is g. We claim that if $g = a^ib^j \in Stab_{Q_{4n}}(\mathcal{X}_n^{'})$, then $j = 0$. If possible, let $j \neq 0$, this implies that $g = a^ib$, in this case there exists an element $h = a^{n+1} \in \mathcal{X}_n^{'}$ satisfying $ghg^{-1} = (a^ib)a^{n+1}(b^{-1}a^{-i}) = a^{-n-1} = a^{n-1} \neq a^{n+1} = h$, as $n \geq 2$. This forces j to be 0. Thus, an element $g \in Stab_{Q_{4n}}(\mathcal{X}_n^{'})$ can be of the form $a^i, 0 \leq i \leq 2n-1$. Now, for $h = a^2b \in \mathcal{X}_n^{'}$, we have $ghg^{-1} = h$ if and only if $(a^i)a^2b(a^{-i}) = a^2b$ if and only if $a^{2i+2}b = a^2b$ if and only if $i = 0$ or n. Thus $Stab_{Q_{4n}}(\mathcal{X}_n^{'}) \subseteq \{1, a^n\}$. ■

The next theorem provides a 2-distinguishing labelling for the conjugation action of Q_{4n} on itself.

Theorem 2.4. *Let Q_{4n} with $n \geq 2$, acts on itself by conjugation. Then the map $\phi : Q_{4n} \longrightarrow \{1, 2\}$ defined by $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{X}_n \\ 2\,; & \text{otherwise} \end{cases}$, is a 2-distinguishing labelling of Q_{4n}.*

Proof. Define a labelling, $\phi : Q_{4n} \longrightarrow \{1, 2\}$ for the conjugation action of Q_{4n} on itself by $\phi(x) = 1$, for all $x \in \mathcal{X}_n$ and $\phi(x) = 2$ otherwise. Now, for ϕ to be a 2-distinguishing labelling, it is sufficient to prove that $\{g \in Q_{4n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in Q_{4n}\} = \{1, a^n\}$. Further, by Proposition 2.3, it is enough to show that if $g \in Q_{4n} \setminus \{1, a^n\}$, there exists an element $h \in \mathcal{X}_n^{'}$ such that $ghg^{-1} \notin \mathcal{X}_n^{'}$, that is $ghg^{-1} \in \mathcal{X}_n$. For this assume that $g \in Q_{4n} \setminus \{1, a^n\}$, if $g = a^i$; $i \in \{0, 1, 2, ..., 2n-1\} \setminus \{0, n\}$, choose $h = a^2b \in \mathcal{X}_n^{'}$, then $ghg^{-1} = a^ia^2ba^{-i} = a^{2i+2}b$ and clearly, for $i \in \{0, 1, 2, ..., 2n-1\} \setminus \{0, n\}$, $a^{2i+2}b \in \mathcal{X}_n$. On the other hand, if $g = a^ib$; $i \in \{0, 1, 2, ..., 2n-1\}$ and if $i \neq 2$ or $n+2$, take $h = a^2b \in \mathcal{X}_n^{'}$, then $ghg^{-1} = (a^ib)a^2b(b^{-1}a^{-i}) = a^{i-2}ba^{-i} = a^{2i-2}b \in \mathcal{X}_n$. Otherwise, when $i = 2$ or $n+2$, choose $h = a^3b \in \mathcal{X}_n^{'}$ and we have $ghg^{-1} = (a^ib)a^3b(b^{-1}a^{-i}) = a^{i-3}ba^{-i} = a^{2i-3}b = ab \in \mathcal{X}_n$. Thus $\{g \in Q_{4n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in Q_{4n}\} \subseteq \{1, a^n\}$ and the result follows from Lemma 2.1. ■

By combining Theorem 1.3 and Theorem 2.4 we have

Theorem 2.5. *If Q_{4n} $(n > 1)$ acts on itself by conjugation, then $\mathcal{D}_{Q_{4n}}(Q_{4n}) = 2$.*

3. Distinguishing Labelling for V_{8n} action

The group V_{8n} is introduced by James and Liebeck [4], for an odd positive integer n as follows:

$$V_{8n} = \langle a, b : a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b\rangle.$$

Later, Darafsheh and Poursalavati [2], observed that with the above presentation, the group V_{8n} can also be defined for an arbitrary n. However, the conjugacy classes of the group V_{8n} differ, depending upon whether n is an even or an odd positive integer. When n is odd, the group V_{8n} has $2n+3$ conjugacy classes precisely;
$\{1\}$,
$\{b^2\}$,
$\{a^{2r+1}, a^{-2r-1}b^2\}; (r = 0, 1, ..., n-1)$,
$\{a^{2s}, a^{-2s}\}$,
$\{a^{2s}b^2, a^{-2s}b^2\}; (s = 1, 2, ..., \frac{n-1}{2})$,
$\{a^jb^k : j \text{ even}, k = 1 \text{ or } 3\}$ and
$\{a^jb^k : j \text{ odd}, k = 1 \text{ or } 3\}$.
While when n is even, the group V_{8n} has $2n+6$ conjugacy classes namely;
$\{1\}$,
$\{b^2\}$,
$\{a^n\}$,
$\{a^nb^2\}$,
$\{a^{2r+1}, a^{-2r-1}b^2\}; (r = 0, 1, ..., n-1)$,
$\{a^{2s}, a^{-2s}\}$,
$\{a^{2s}b^2, a^{-2s}b^2\}; (s = 1, 2, ..., \frac{n}{2}-1)$,
$\{a^{2k}b^{(-1)^k} : 0 \le k \le n-1\}$,
$\{a^{2k}b^{(-1)^{k+1}} : 0 \le k \le n-1\}$,
$\{a^{2k+1}b^{(-1)^k} : 0 \le k \le n-1\}$ and
$\{a^{2k+1}b^{(-1)^{k+1}} : 0 \le k \le n-1\}$.

Clearly, V_{8n} is a non-Abelian group of order $8n$ and its elements are of the form $a^r, a^rb, a^rb^2, a^rb^3$; where $r = 0, 1, ..., 2n-1$.

Lemma 3.1. *Let $\mathcal{G} = V_{8n} = \langle a, b : a^{2n} = 1 = b^4, aba = b^{-1}, ab^{-1}a = b\rangle$. Then we have*

(1) $ba^j = a^{-j}b^{(-1)^j}$.

(2) $b^2a^j = a^jb^2$.

(3) $b^3a^j = a^{-j}b^{(-1)^{j+1}}$.

(4) $Z(V_{8n}) = \begin{cases} \langle a^n, b^2\rangle; & \text{if } 2 \mid n \\ \langle b^2\rangle; & \text{if } 2 \nmid n \end{cases}$.

Proof. (1) We will prove the result by induction on j. By hypothesis, $ba = a^{-1}b^{-1}$. Assume that the result holds for all positive integers up to $j-1$, so we have, $ba^{j-1} = a^{1-j}b^{(-1)^{j-1}}$. Further, $ba^j = (ba^{j-1})a = (a^{1-j}b^{(-1)^{j-1}})a = a^{1-j}(b^{(-1)^{j-1}}a) = a^{-j}b^{(-1)^j}$, as $b^{-1}a = a^{-1}b$ and $b^{-1}a^{-1} = ab$. Thus, $ba^j = a^{-j}b^{(-1)^j}$ holds for every non negative integer j. Similarly, we can prove by induction that $ba^{-j} = a^jb^{(-1)^j}$ holds for any non negative integer j. Therefore, (1) holds for any $j \in \mathbb{Z}$.

(2) and (3) follow by using (1), repeatedly.

(4) Straightforward. ■

Next, we partition the set of elements in the group V_{8n}, n odd, as follows:

$$V_{8n} = \mathcal{X}_n^{'} \cup \mathcal{X}_n, \text{ where}$$

$\mathcal{X}_n^{'} = \{a^{-1}b^2, a^{-3}b^2, ..., a^{1-2n}b^2, a^{-2}, a^{-4}, ..., a^{1-n}, a^{-2}b^2, a^{-4}b^2, ..., a^{1-n}b^2, b^3, ab^3\}$ and $\mathcal{X}_n = V_{8n} \setminus \mathcal{X}_n^{'} = \{1, a, ..., a^n, b, a^4b, a^6b, ..., a^{2n-2}b, ab, a^5b, a^7b, ..., a^{2n-1}b\}$.

Proposition 3.2. *The stabilizer $Stab_{V_{8n}}(\mathcal{X}_n^{'})$ of the subset $\mathcal{X}_n^{'}$in V_{8n}, n odd, is the set $\{1, b^2\}$.*

Proof. Let $g \in Stab_{V_{8n}}(\mathcal{X}_n^{'}) \cap V_{8n}$. Then, by definition $ghg^{-1} = h$, for all $h \in \mathcal{X}_n^{'}$. Also, an element $g \in V_{8n}$ will be of the form a^ib^j, where $-n < i \leq n$ and $j = 0, 1, 2, 3$. We claim that if $g \in Stab_{V_{8n}}(\mathcal{X}_n^{'})$, then $j \neq 1, 3$. Otherwise, $g = a^ib$ or a^ib^3, where $-n < i < n$. In this case, there exists an element $h = ab^3 \in \mathcal{X}_n^{'}$ satisfying $ghg^{-1} = \begin{cases} a^{2i-1}b; & \text{if } 2 \mid i \\ a^{2i-1}b^3; & \text{if } 2 \nmid i \end{cases}$. Clearly, $ghg^{-1} = h = ab^3$ if and only if $i = 1$. Furthermore, if $i \neq 1$, then $ghg^{-1} \neq h$, for $h = ab^3$ and in the case when $i = 1$, replace $h = ab^3$ with $b^3 \in \mathcal{X}_n^{'}$. Then $ghg^{-1} = a^2b \neq b^3 = h$. Therefore, an element $g \in Stab_{V_{8n}}(\mathcal{X}_n^{'})$ must be of the form a^ib^j, where $-n < i \leq n$ and $j = 0, 2$.

Next, we shall show that if $g \in Stab_{V_{8n}}(\mathcal{X}_n^{'})$, then $i = 0$. For this, set $h = ab^3 \in \mathcal{X}_n^{'}$. Then $ghg^{-1} = \begin{cases} a^{2i+1}b^3; & \text{if } 2 \mid i \\ a^{2i+1}b; & \text{if } 2 \nmid i \end{cases}$. Again, $ghg^{-1} = h = ab^3$ if and only if $i = 0$. Thus, $Stab_{V_{8n}}(\mathcal{X}_n^{'}) \subseteq \{1, b^2\}$. Further, by Lemma 3.1(4), for any subset S of V_{8n}, $\langle b^2 \rangle \subset Stab_{V_{8n}}(S)$. Hence $Stab_{V_{8n}}(\mathcal{X}_n^{'}) = \{1, b^2\}$. ■

In the forthcoming theorem, we establish a 2-distinguishing labelling for the conjugation action of V_{8n}, n odd, acting on itself.

Theorem 3.3. *If V_{8n}, n odd, acts on itself by conjugation, then the map $\phi : V_{8n} \longrightarrow \{1, 2\}$ defined by $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{X}_n \\ 2\,; & \text{otherwise} \end{cases}$, is a 2-distinguishing labelling.*

Proof. Suppose V_{8n}, n odd, acts on itself by conjugation. Define a labelling, $\phi : V_{8n} \longrightarrow \{1, 2\}$ by $\phi(x) = 1$ for all $x \in \mathcal{X}_n$ and $\phi(x) = 2$ otherwise. Note that for ϕ to be a 2-distinguishing labelling, it is enough to show that

$$\{g \in V_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in V_{8n}\} = \{1, b^2\}.$$

Clearly, by Proposition 3.2, it is equivalent to prove that if $g \in V_{8n} \setminus \{1, b^2\}$, then there exists an element $h \in \mathcal{X}_n^{'}$ such that $ghg^{-1} \notin \mathcal{X}_n^{'}$, that is $ghg^{-1} \in \mathcal{X}_n$. For this, assume that $g \in V_{8n} \setminus \{1, b^2\}$. If the element $g = a^i$ or a^ib^2; $-n < i \leq n, i \neq 0$, then we can choose $h = ab^3 \in \mathcal{X}_n^{'}$, then $ghg^{-1} = \begin{cases} a^{2i+1}b^3; & \text{if } 2 \mid i \\ a^{2i+1}b; & \text{if } 2 \nmid i \end{cases}$. Clearly, $a^{2i+1}b \notin \mathcal{X}_n^{'}$, for $2 \nmid i$. Moreover, if $2 \mid i$, then $a^{2i+1}b^3 \in \mathcal{X}_n^{'}$ if and only if $i = 0$, which is not possible.

On the other hand, the element g should only be of the form $g = a^ib^j$; $-n < i \leq n$, $j = 1$ or 3. Take $h = ab^3 \in \mathcal{X}_n^{'}$, then $ghg^{-1} = \begin{cases} a^{2i-1}b; & \text{if } 2 \mid i \\ a^{2i-1}b^3; & \text{if } 2 \nmid i \end{cases}$. Again, if $2 \mid i$, then $a^{2i-1}b \notin \mathcal{X}_n^{'}$ and observe that in case $2 \nmid i$, then $a^{2i-1}b^3 \in \mathcal{X}_n^{'}$ if and only if $i = 1$ or $i = 1-n$. Note that $i = 1-n$ is not possible, as n is an odd positive integer and $2 \nmid i$. Thus, assume that $i \neq 1$, this implies that $ghg^{-1} \notin \mathcal{X}_n^{'}$. Otherwise, when $i = 1$, the element g will be either ab or ab^3. In any of these cases, for $h = b^3$, we have $ghg^{-1} = abb^3b^{-1}a^{-1} = ab^3a^{-1} = a^2b \notin \mathcal{X}_n^{'}$. This infers that, whenever $g \in V_{8n} \setminus \{1, b^2\}$, there always exists an element $h \in \mathcal{X}_n^{'}$ such that $ghg^{-1} \in \mathcal{X}_n$. Therefore, we conclude that $\{g \in V_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in V_{8n}\} \subseteq \{1, b^2\}$ and hence the result. ■

Now, we turn our attention to examine the distinguishing labelling and distinguishing number for the conjugation action of the group V_{8n}, when n is even. Observe that in this case

$Z(V_{8n}) = Stab_{V_{8n}}(V_{8n}) = \{1, a^n, b^2, a^n b^2\}$.

Assume that $n \in 2\mathbb{Z}^+$ and we partition the set of elements in the group V_{8n} as follows:

$$V_{8n} = \mathcal{Y}_n^{'} \cup \mathcal{Y}_n, \text{ where}$$

$\mathcal{Y}_n^{'} = \{a^{-1}b^2, a^{-3}b^2, ..., a^{1-2n}b^2, a^{-2}, a^{-4}, ..., a^{2-n}, a^{-2}b^2, a^{-4}b^2, ..., a^{2-n}b^2, a^2b, a^3b, a^2b^3, a^3b^3\}$ and $\mathcal{Y}_n = V_{8n} \setminus \mathcal{Y}_n^{'}$. This partition will be used to define a 2-distinguishing labelling for the conjugation action of the group V_{8n}, for even n.

Proposition 3.4. *The stabilizer $Stab_{V_{8n}}(\mathcal{Y}_n^{'})$ of the subset $\mathcal{Y}_n^{'}$ is $\{1, a^n, b^2, a^n b^2\}$.*

Proof. Let $g \in Stab_{V_{8n}}(\mathcal{Y}_n^{'}) \cap V_{8n}$. By hypothesis $ghg^{-1} = h$, for all $h \in \mathcal{Y}_n^{'}$. Also, an element $g \in V_{8n}$ will be of the type $a^i b^j$, where $-n < i \leq n$ and $j = 0, 1, 2, 3$. We claim that if $g \in Stab_{V_{8n}}(\mathcal{Y}_n^{'})$, then $j \neq 1, 3$. Otherwise, either $g = a^i b$ or $a^i b^3$, where $-n < i \leq n$ and there exists an element $h = a^2b^3 \in \mathcal{Y}_n^{'}$ satisfying $ghg^{-1} = \begin{cases} a^{2i-2}b^3; & \text{if } 2 \mid i \\ a^{2i-2}b; & \text{if } 2 \nmid i \end{cases}$. Note that $ghg^{-1} = h = a^2b^3$ if and only if $2 \mid i$ and $i = 2$ or $2 - n$. Thus, if $i \neq 2$ or $2 - n$, then $ghg^{-1} \neq h$ for $h = a^2b^3$. Now, if $i = 2$ or $2 - n$, then replace $h = a^2b^3$ with $a^3b^3 \in \mathcal{X}_n^{'}$ and we have $ghg^{-1} = ab \neq a^3b^3 = h$. Therefore, an element $g \in Stab_{V_{8n}}(\mathcal{Y}_n^{'})$ will be of the form $a^i b^j$, where $-n < i \leq n$ and $j = 0, 2$.

Next, we further show that if $g \in Stab_{V_{8n}}(\mathcal{Y}_n^{'})$, then either $i = 0$ or $i = n$. For this, set $h = a^2b^3 \in \mathcal{Y}_n^{'}$, then $ghg^{-1} = \begin{cases} a^{2i+2}b^3; & \text{if } 2 \mid i \\ a^{2i+2}b; & \text{if } 2 \nmid i \end{cases}$. Clearly, $ghg^{-1} = h = a^2b^3$ if and only if $i = 0$ or n. Hence, an element $g \in Stab_{V_{8n}}(\mathcal{Y}_n^{'})$ should be of the form $a^i b^j$; where $i = 0, n$ and $j = 0, 2$, so $Stab_{V_{8n}}(\mathcal{Y}_n^{'}) \subseteq \{1, a^n, b^2, a^n b^2\}$. Further, since $Z(V_{8n}) = \{1, a^n, b^2, a^n b^2\}$, so $\{1, a^n, b^2, a^n b^2\} \subseteq Stab_{V_{8n}}(\mathcal{Y}_n^{'})$. Hence, $Stab_{V_{8n}}(\mathcal{Y}_n^{'}) = \{1, a^n, b^2, a^n b^2\}$. ■

Theorem 3.5. *Let V_{8n}, n even, acts on itself by conjugation. Then there exists a map $\phi : V_{8n} \longrightarrow \{1, 2\}$ defined by $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{Y}_n \\ 2\,; & \text{otherwise} \end{cases}$, is a 2-distinguishing labelling.*

Proof. Suppose that the group V_{8n}, n even, acts on itself by conjugation. Define a labelling, $\phi : V_{8n} \longrightarrow \{1, 2\}$ for this action by $\phi(x) = 1$ for all $x \in \mathcal{Y}_n$ and $\phi(x) = 2$ otherwise. On imitating the same process as in Theorem 3.3, it suffices to show that

$$\{g \in V_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in V_{8n}\} = \{1, a^n, b^2, a^n b^2\}.$$

Moreover, by Proposition 3.4, it is equivalent to prove that if $g \in V_{8n} \setminus \{1, a^n, b^2, a^n b^2\}$, then there exists an element $h \in \mathcal{Y}_n^{'}$ such that $ghg^{-1} \notin \mathcal{Y}_n^{'}$. For this, assume that $g \in V_{8n} \setminus \{1, a^n, b^2, a^n b^2\}$ and in case the element $g = a^i$ or $a^i b^2$; $-n < i \leq n$, then it is trivial to see that, i can not take the values 0 and n. Thus, set $h = a^2b^3 \in \mathcal{Y}_n^{'}$ and we have $ghg^{-1} = \begin{cases} a^{2i+2}b^3; & \text{if } 2 \mid i \\ a^{2i+2}b; & \text{if } 2 \nmid i \end{cases}$. Also, note that if $2 \nmid i$, then $a^{2i+2}b \in \mathcal{Y}_n^{'}$, if and only if $i = 0$ or n. Again, if $2 \mid i$, then $a^{2i+2}b^3 \in \mathcal{Y}_n^{'}$ if and only if $i = 0, n$. Thus, $ghg^{-1} \notin \mathcal{Y}_n^{'}$, because i cannot take the values 0 and n.

On the other hand, if $g = a^i b$ or $a^i b^3$; $-n < i \leq n$. In this case, take $h = a^2b^3 \in \mathcal{Y}_n^{'}$, then $ghg^{-1} = \begin{cases} a^{2i-2}b; & \text{if } 2 \nmid i \\ a^{2i-2}b^3; & \text{if } 2 \mid i \end{cases}$. Note that $a^{2i-2}b \in \mathcal{Y}_n^{'}$ if and only if $2 \nmid i$ and $i = 2$ or $2 - n$, which is not possible, as n is even. Further, if $2 \mid i$, then $a^{2i-2}b^3 \in \mathcal{Y}_n^{'}$ if and only if $i = 2$ or $-n + 2$. Hence, if $i \neq 2, -n + 2$, then $ghg^{-1} \notin \mathcal{Y}_n^{'}$. Moreover, if we choose $h = a^3b^3$ and if $i = 2$ or $-n + 2$, then $ghg^{-1} = ab \notin \mathcal{Y}_n^{'}$. Thus, we have proved that if $g \in V_{8n} \setminus \{1, a^n, b^2, a^n b^2\}$, then there exists an element $h \in \mathcal{Y}_n'$ such that $ghg^{-1} \notin \mathcal{Y}_n'$. Therefore, we conclude that $\{g \in V_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in V_{8n}\} \subseteq \{1, a^n, b^2, a^n b^2\}$ and by using Lemma 3.1 the reverse inclusion is immediate. ■

By combining Theorem 1.3, Theorem 3.3 and Theorem 3.5, we have Theorem 3.6, which provides the distinguishing number for the conjugation action of the group V_{8n} on itself, for an arbitrary n.

Theorem 3.6. *If V_{8n} acts on itself by conjugation action, then $D_{V_{8n}}(V_{8n}) = 2$.*

4. Distinguishing Labelling for action of the group U_{6n}

Recall from [4] that $\mathcal{G} = U_{6n}$ is a non-Abelian group of order $6n$ and it is generated by two elements a and b such that $a^{2n} = 1, b^3 = 1, ba = ab^{-1}$. The group U_{6n} can be represented as follows:

$$U_{6n} = \langle a, b : a^{2n} = 1, b^3 = 1, ba = ab^{-1} \rangle.$$

However, group U_{6n} can be viewed as a group isomorphic to the semi direct product of a cyclic group of order 3 by a cyclic group of order $2n$. Clearly, the subgroup generated by the generator b is a normal subgroup of order 3 and the subgroup generated by a is a cyclic subgroup of order $2n$. In addition, the group U_{6n} has $3n$ conjugacy classes namely:

$\{a^{2r}\}$,
$\{a^{2r}b, a^{2r}b^2\}$,
$\{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}; (r = 0, 1, 2, ..., n-1)$.

Lemma 4.1. *Let $\mathcal{G} = U_{6n} = \langle a, b : a^{2n} = 1, b^3 = 1, ba = ab^{-1} \rangle$ and j be an arbitrary integer. Then we have*

(1) $ba^j = \begin{cases} a^j b^2; & \text{if } 2 \nmid j \\ a^j b; & \text{if } 2 \mid j \end{cases}$

(2) $b^2 a^j = \begin{cases} a^j b; & \text{if } 2 \nmid j \\ a^j b^2; & \text{if } 2 \mid j \end{cases}$

(3) $Z(U_{6n}) = \langle a^2 \rangle$.

Proof. (1) First we shall show by induction that for any non-negative integer j, $ba^j = \begin{cases} a^j b^2; & \text{if } 2 \nmid j \\ a^j b; & \text{if } 2 \mid j \end{cases}$. Clearly, by group relation $ba = a^{-1}b$ and suppose this holds for all positive integers up to $m-1$. Now, consider $ba^m = (ba^{m-1})a = \begin{cases} (a^{m-1}b^2)a; & \text{if } 2 \nmid m-1 \\ (a^{m-1}b)a; & \text{if } 2 \mid m-1 \end{cases} = \begin{cases} a^m b^2; & \text{if } 2 \nmid m \\ a^m b; & \text{if } 2 \mid m \end{cases}$. Hence, the result holds for all non negative integers. Moreover, by imitating the same process we can similarly prove by induction that $ba^j = \begin{cases} a^j b^2; & \text{if } 2 \nmid j \\ a^j b; & \text{if } 2 \mid j \end{cases}$ holds for any negative integer j and this completes the proof.

Results (2) and (3) are straightforward.

■

Moreover, we can write U_{6n} as a disjoint union of two subsets $\mathcal{X}_n'$ and $\mathcal{X}_n$, where $\mathcal{X}_n' = \{a^i b^2 : i = 1, 2, ..., 2n-1\}$ and $\mathcal{X}_n = Q_{4n} \setminus \mathcal{X}_n'$.

Proposition 4.2. *The stabilizer of the subset $\mathcal{X}_n' = \{a^i b^2 : i = 1, 2, ..., 2n-1\}$ is $Stab_{U_{6n}}(\mathcal{X}_n') = \langle a^2 \rangle$.*

Proof. By Lemma 4.1(3), it is sufficient to show that $Stab_{U_{6n}}(\mathcal{X}_n^{'}) \subseteq \langle a^2 \rangle$. For this, let $g \in Stab_{U_{6n}}(\mathcal{X}_n^{'}) \cap U_{6n}$. Then by hypothesis, $ghg^{-1} = h$, for all $h \in \mathcal{X}_n^{'}$. Also, an element in U_{6n} will be of the form $a^i b^j$, where $0 \leq i \leq 2n-1$ and $j = 0, 1, 2$, and so is g. We claim that if $g = a^i b^j \in Stab_{U_{6n}}(\mathcal{X}_n^{'})$, then $j = 0$. If possible, let $j \neq 0$, then the element g will be of the form $a^i b$ or $a^i b^2$. Now, in case $g = a^i b$ and $2 \nmid i$, then choose $h = b^2$, else in the other case $2 \mid i$, choose $h = ab^2$. Thus by using Lemma 4.1, it follows that $ghg^{-1} \neq h$. On the other hand, when $g = a^i b^2$ and $2 \nmid i$, choose $h = ab^2$ or in case $2 \mid i$, take $h = b^2$. Further, by Lemma 4.1, we observe that in either of these cases $ghg^{-1} \neq h$. Thus, the element g must be of the form a^i, where $0 \leq i \leq 2n-1$. Further, for $h = b^2 \in \mathcal{X}_n^{'}$, we have $ghg^{-1} = h$ if and only if $2 \mid i$ and hence, $Stab_{U_{6n}}(\mathcal{X}_n^{'}) \subseteq \langle a^2 \rangle$. ■

In the upcoming theorem we obtain a 2-distinguishing labelling for the conjugation action of U_{6n} on itself.

Theorem 4.3. *The map $\phi : U_{6n} \longrightarrow \{1, 2\}$ defined by $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{X}_n \\ 2\,; & \text{otherwise} \end{cases}$, is a 2-distinguishing labelling of U_{6n}.*

Proof. Define a map, $\phi : U_{6n} \longrightarrow \{1, 2\}$, by $\phi(x) = 1$, for all $x \in \mathcal{X}_n$ and $\phi(x) = 2$, otherwise. In order to prove that ϕ is a 2-distinguishing labelling, it is sufficient to show that $\{g \in U_{6n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in U_{6n}\} = \langle a^2 \rangle$. Now, in light of Proposition 4.2, it is equivalent to prove that for every $g \in U_{6n} \setminus \langle a^2 \rangle$, there exists an element $h \in \mathcal{X}_n^{'}$ such that $ghg^{-1} \in \mathcal{X}_n$. Let $g \in U_{6n} \setminus \langle a^2 \rangle$. Then, g will be of the form $g = a^i;\ i \notin \{0, 2, ..., 2n-2\}$. Fix $h = ab^2 \in \mathcal{X}_n^{'}$ and we have $ghg^{-1} = ab$ and certainly, $ab \in \mathcal{X}_n$. Otherwise, the element g will be of the form $g = a^i b$ or $g = a^i b$; $0 \leq i \leq 2n-1$. In either case, if $2 \nmid i$, set $h = b^2 \in \mathcal{X}_n^{'}$, then by using Lemma 4.1 (2), we get $ghg^{-1} = a^i b^2 a^{-i} = b$, as $2 \nmid i$. Clearly, $ghg^{-1} = b \notin \mathcal{X}_n$. Further, if $2 \mid i$, then for $g = a^i b$ choose $h = ab^2$, we have $ghg^{-1} = (a^i b) ab^2 (b^{-1} a^{-i}) = a^i baba^{-i} = a^{i+1} b^{-1} ba^{-i} = a$. Furthermore, for $g = a^i b^2$, take $h = ab^2 \in \mathcal{X}_n^{'}$ and we get $ghg^{-1} = (a^i b^2) ab^2 (b^{-1} a^{-i}) = a^i b^2 a^{-i+1} = ab \notin \mathcal{X}_n^{'}$. Thus $\{g \in U_{6n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in U_{6n}\} \subseteq \langle a^2 \rangle$ and the result follows by using Lemma 4.1 (3). ■

The following theorem is a direct consequence of Theorem 1.3 and Theorem 4.3.

Theorem 4.4. *If U_{6n} acts on itself by conjugation, then $\mathcal{D}_{U_{6n}}(U_{6n}) = 2$.*

5. Distinguishing Labelling for semi-dihedral group SD_{8n} action

The semi-dihedral group, SD_{8n} [3] is a non-Abelian group of order $8n$. For $n \geq 2$, this group can be presented as follows:

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle.$$

Clearly, the elements of the semi-dihedral group are of the form a^r or ba^r; $r = 0, 1, ..., 4n-1$. As observed in the case of V_{8n}, the group SD_{8n} has $2n+3$ or $2n+6$ conjugacy classes, when n is even or odd respectively. We begin with the following definition

Definition 5.1. *[3] Define $C^{even} := C_1 \cup C_2^{even} \cup C_3^{even}$ and $C^{odd} := C_1 \cup C_2^{odd} \cup C_3^{odd}$, where $C_1 := \{0, 2, 4, ..., 2n\}, C_2^{even} := \{1, 3, ..., n-1\}, C_3^{even} := \{2n+1, 2n+3, 2n+5, ..., 3n-1\}$ and $C_2^{odd} := \{1, 3, 5, ..., n\}, C_3^{odd} := \{2n+1, 2n+3, 2n+5, ..., 3n\}$. Also, define $C_{even}^{\dagger} := C_1 \setminus \{0, 2n\}, C_{odd}^{\dagger} := C_2^{even} \cup C_3^{even}, C_{2,3}^{odd} := C_2^{odd} \cup C_3^{odd}$ and $C_*^{even} := C^{even} \setminus \{0, 2n\}, C_*^{odd} := C^{odd} \setminus \{0, n, 2n, 3n\}$.*

The next proposition provides us with the conjugacy classes of SD_{8n}.

Proposition 5.2. *[3] The conjugacy classes of SD_{8n}, $n \geq 2$, are as follows:*

- *If n is even, then there are $2n+3$ conjugacy classes. Precisely,*
  - *2 classes of size one being $[1] = \{1\}$ and $[a^{2n}] = a^{2n}$,*

- $(2n-1)$ *classes of size two being* $[a^r] = \{a^r, a^{(2n-1)r}\}$*, where* $r \in C_*^{even}$ *and*
- 2 *classes of size* $2n$ *being* $[b] = \{ba^{2t} : t = 0, 1, ..., 2n-1\}$ *and* $[ba] = \{ba^{2t+1} : t = 0, 1, ..., 2n-1\}$.

- *If* n *is odd, then there are* $2n+6$ *conjugacy classes. Precisely,*
  - 4 *classes of size one being* $[1] = \{1\}, [a^n] = \{a^n\}, [a^{2n}] = \{a^{2n}\}$ *and* $[a^{3n}] = \{a^{3n}\}$,
  - $(2n-2)$ *classes of size two being* $[a^r] = \{a^r, a^{(2n-1)r}\}$*, where* $r \in C_*^{odd}$ *and*
  - 4 *classes of size* n *being* $[b] = \{ba^{4t} : t = 0, 1, ..., n-1\}, [ba] = \{ba^{4t+1} : t = 0, 1, ..., n-1\}, [ba^2] = \{ba^{4t+2} : t = 0, 1, ..., n-1\}$*, and* $[ba^3] = \{ba^{4t+3} : t = 0, 1, ..., n-1\}$.

Lemma 5.3. *[3] Let* $\mathcal{G} = SD_{8n} = \langle a, b : a^{4n} = 1 = b^2, bab = a^{2n-1}\rangle$*. Then*

(1) $a^k b = ba^{(2n-1)k}$.

(2) $Z(V_{8n}) = \begin{cases} \langle a^n \rangle; & \text{if } 2 \nmid n \\ \langle a^{2n} \rangle; & \text{if } 2 \mid n \end{cases}$.

For n **even**, we can partition the set of elements of SD_{8n} as a disjoint union of two subsets $\mathcal{X}_n^{'}$ and $\mathcal{X}_n$ of SD_{8n}, where $\mathcal{X}_n^{'} = \{a^{r(2n-1)} : r = 2, 4, 6, ..., 2n-2, 1, 3, 5, ..., n-1, 2n+1, 2n+3, ..., 3n-1\} \cup \{ba^2, ba^3\}$ and $\mathcal{X}_n = SD_{8n} \setminus \mathcal{X}_n^{'}$. This partition will be used to define a 2-distinguishing labelling for the conjugation action of the group SD_{8n}, for an even n.

Proposition 5.4. *The stabilizer* $Stab_{SD_{8n}}(\mathcal{X}_n^{'})$ *of the subset* $\mathcal{X}_n^{'}$ *is* $\{1, a^{2n}\}$.

Proof. Let $g \in Stab_{SD_{8n}}(\mathcal{X}_n^{'}) \cap SD_{8n}$. Then by definition $ghg^{-1} = h$, for all $h \in \mathcal{X}_n^{'}$. Also, an element g in the group SD_{8n} will be of the form $a^i b^j$, where $0 \leq i \leq 4n-1$ and $j = 0, 1$. We claim that if $g \in Stab_{SD_{8n}}(\mathcal{X}_n^{'})$, then $j \neq 1$. If not, then $g = a^i b$ and we have $h = a^{2n-1} \in \mathcal{X}_n^{'}$ such that $ghg^{-1} = a^i ba^{2n-1} b^{-1} a^{-i} = a^i (ba^{2n-1} b) a^{-i} = a^i a a^{-i} = a$. Now, $ghg^{-1} = h = a^{2n-1}$ if and only if $n = 1$, which is not possible, as $n \geq 2$. Therefore, an element $g \in Stab_{SD_{8n}}(\mathcal{X}_n^{'})$ should be of the form a^i, for some $0 \leq i \leq 4n-1$. Next, we show that either $i = 0$ or $i = 2n$. For this, set $h = ba^2 \in \mathcal{X}_n^{'}$, then $ghg^{-1} = a^i ba^2 a^{-i} = a^i ba^{2-i} = ba^{(2n-1)i+2-i} = ba^{2i(n-1)+2}$. Again, $ghg^{-1} = h = ba^2$ if and only if $2i(n-1) \equiv 0 (mod\, 4n)$ if and only if $2i \equiv 0 (mod\, 4n)$. Therefore, we have $i = 0$ or $2n$. Thus, $Stab_{SD_{8n}}(\mathcal{X}_n^{'}) \subseteq \{1, a^{2n}\}$. Obviously, $\{1, a^{2n}\} \subseteq Stab_{SD_{8n}}(\mathcal{X}_n^{'})$, as the center $Z(SD_{8n}) = \{1, a^{2n}\}$. Hence $Stab_{SD_{8n}}(\mathcal{X}_n^{'}) = \{1, a^{2n}\}$. ■

Theorem 5.5. *Let* SD_{8n}*, n even, acts on itself by conjugation. Then the map* $\phi : SD_{8n} \longrightarrow \{1, 2\}$ *defined by* $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{X}_n \\ 2\,; & \text{otherwise} \end{cases}$*, is a 2-distinguishing labelling of* SD_{8n}.

Proof. Define a labelling, $\phi : V_{8n} \longrightarrow \{1, 2\}$ of this action by $\phi(x) = 1$, for all $x \in \mathcal{X}_n$ and $\phi(x) = 2$, otherwise. From the Definition 1.1, for ϕ to be a 2-distinguishing labelling, it suffices to show that $\{g \in SD_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in SD_{8n}\} = \{1, a^{2n}\}$. Now, by Proposition 5.4, it is equivalent to prove that if $g \in SD_{8n} \setminus \{1, a^{2n}\}$, then there exists an element $h \in \mathcal{X}_n^{'}$ such that $ghg^{-1} \notin \mathcal{X}_n^{'}$. For this, let $g \in SD_{8n} \setminus \{1, a^{2n}\}$ and $g = a^i; i \in \{0, 1, 2, ..., 4n-1\} \setminus \{0, 2n\}$. Then choose $h = ab^2 \in \mathcal{X}_n^{'}$, so that $ghg^{-1} = a^i ba^2 a^{-i} = ba^{2i(n-1)+2}$. Now, $ba^{2i(n-1)+2} \in \mathcal{X}_n^{'}$, if and only if $a^{2i(n-1)+2} = a^2$ or a^3 if and only if $2i(n-1)+2 \equiv 2$ or 3 $(mod\, 4n)$. Clearly, $2i(n-1)+2 \equiv 3 (mod\, 4n)$, is not possible, as n is even. Also, $2i(n-1)+2 \equiv 2 (mod\, 4n)$ holds if and only if $2i(n-1) \equiv 0 (mod\, 4n)$ if and only if $i = 0, 2n$, which is not possible. On the other hand, if $g = ba^i$; $0 \leq i \leq 4n-1$, then in this case, fix $h = a^{2n-1} \in \mathcal{X}_n^{'}$, which leads to $ghg^{-1} = a^i ba^{2n-1} b^{-1} a^{-i} = a^i a a^{-i} = a$ and certainly $ghg^{-1} = a \notin \mathcal{X}_n^{'}$. Thus $\{g \in SD_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in SD_{8n}\} \subseteq \{1, a^{2n}\}$ and the reverse inclusion follows by using Lemma 5.3 (2). ■

Next, we shall examine the distinguishing labelling for the conjugation action of the semi-dihedral group, when n is an odd positive integer.

For n odd, we can partition the group SD_{8n} as follows:

$$SD_{8n} = \mathcal{Y}_n^{'} \cup \mathcal{Y}_n,$$

where $\mathcal{Y}_n^{'} = \{a^{r(2n-1)} : r = 2, 4, 6, ..., 2n-2, 1, 3, 5, ..., n-2, 2n+1, 2n+3, ..., 3n-2\} \cup \{ba^4, ba^5, ba^6, ba^7\}$ and $\mathcal{Y}_n = SD_{8n} \setminus \mathcal{Y}_n^{'}$. Also, we use this partition to define a 2-distinguishing labelling for the conjugation action of the group SD_{8n} acting on itself.

Proposition 5.6. *The stabilizer $Stab_{SD_{8n}}(\mathcal{Y}_n^{'})$ of the subset $\mathcal{Y}_n^{'}$ is $\{1, a^n, a^{2n}, a^{3n}\}$.*

Proof. Let $g \in Stab_{SD_{8n}}(\mathcal{Y}_n^{'}) \cap SD_{8n}$. Then by definition $ghg^{-1} = h$, for all $h \in \mathcal{Y}_n^{'}$. Also, an element g in SD_{8n} will be of the form $a^i b^j$, where $0 \leq i \leq 4n-1$ and $j = 0, 1$. We claim that if $g \in Stab_{SD_{8n}}(\mathcal{Y}_n^{'})$, then $j \neq 1$. If not, then $g = a^i b$ and we have $h = a^{2n-1} \in \mathcal{Y}_n^{'}$ such that $ghg^{-1} = a^i ba^{2n-1}b^{-1}a^{-i} = a^i(ba^{2n-1}b)a^{-i} = a^i aa^{-i} = a$. Now, $ghg^{-1} = h$ if and only if $a = a^{2n-1}$ if and only if $n = 1$, which is not possible, as $n \geq 2$. This infers that an element $g \in Stab_{SD_{8n}}(\mathcal{X}_n^{'})$ will be of the form a^i, for some $0 \leq i \leq 4n-1$. Next, we show that $i \in \{0, n, 2n, 3n\}$. For this, choose $h = ba^4 \in \mathcal{Y}_n^{'}$ and we have $ghg^{-1} = a^i ba^4 a^{-i} = a^i ba^{4-i} = ba^{(2n-1)i+4-i} = ba^{2i(n-1)+4}$. Again, $ghg^{-1} = h = ba^4$ if and only if $2i(n-1) \equiv 0 (mod\, 4n)$ if and only if $i \equiv 0 (mod\, n)$, as n is odd. This implies that $i = 0$ or n or $2n$ or $3n$. Thus $Stab_{SD_{8n}}(\mathcal{Y}_n^{'}) \subseteq \{1, a^n, a^{2n}, a^{3n}\}$. Furthermore, $\{1, a^n, a^{2n}, a^{3n}\} \subseteq Stab_{SD_{8n}}(\mathcal{Y}_n^{'})$, since $Z(SD_{8n}) = \{1, a^n, a^{2n}, a^{3n}\}$. Hence $Stab_{SD_{8n}}(\mathcal{Y}_n^{'}) = \{1, a^n, a^{2n}, a^{3n}\}$. ■

Theorem 5.7. *If SD_{8n}, n odd, acts on itself by conjugation, then the map $\phi : SD_{8n} \longrightarrow \{1, 2\}$ defined by $\phi(x) = \begin{cases} 1\,; & \text{if } x \in \mathcal{Y}_n \\ 2\,; & \text{otherwise} \end{cases}$, is a 2-distinguishing labelling of SD_{8n}.*

Proof. Suppose that the group SD_{8n}, n odd acts on itself by conjugation. Define a labelling, $\phi : V_{8n} \longrightarrow \{1, 2\}$ of this action by $\phi(x) = 1$, for all $x \in \mathcal{Y}_n$ and $\phi(x) = 2$, otherwise. Again, for ϕ to be a 2-distinguishing labelling, it is required to show that $\{g \in SD_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in SD_{8n}\} = \{1, a^n, a^{2n}, a^{3n}\}$. Now, by the definition of ϕ and Proposition 5.6, it is equivalent to prove that for $g \in SD_{8n} \setminus \{1, a^n, a^{2n}, a^{3n}\}$, there exists an element $h \in \mathcal{Y}_n^{'}$ such that $ghg^{-1} \notin \mathcal{Y}_n^{'}$. For this, assume that $g \in SD_{8n} \setminus \{1, a^n, a^{2n}, a^{3n}\}$ and $g = a^i; i \in \{0, 1, 2, ..., 4n-1\} \setminus \{0, n, 2n, 3n\}$. One can choose $h = ab^4 \in \mathcal{Y}_n^{'}$ and we get $ghg^{-1} = a^i ba^4 a^{-i} = ba^{2i(n-1)+4}$. Now, $ba^{2i(n-1)+4} \in \mathcal{Y}_n^{'}$, if and only if $2i(n-1)+4 \equiv 4$ or 5 or 6 or 7 $(mod\, 4n)$. Clearly, $2i(n-1)+4 \equiv 5, 6, 7 (mod\, 4n)$, is not possible, as n is an odd positive integer. Also, $2i(n-1)+4 \equiv 4 (mod\, 4n)$ holds if and only if $2i(n-1) \equiv 0 (mod\, 4n)$ if and only if $i = 0, n, 2n, 3n$, which is not possible. On the other hand, if $g = ba^i$; $0 \leq i \leq 4n-1$, then in this case, we fix $h = a^{2n-1} \in \mathcal{Y}_n^{'}$, then $ghg^{-1} = a^i ba^{2n-1}b^{-1}a^{-i} = a^i aa^{-i} = a$ and certainly $ghg^{-1} = a \notin \mathcal{Y}_n^{'}$. Thus $\{g \in SD_{8n} : \phi(ghg^{-1}) = \phi(h), \text{ for all } h \in SD_{8n}\} \subseteq \{1, a^n, a^{2n}, a^{3n}\}$ and by using Lemma 5.3 (2) the reverse inclusion follows immediately. ■

Finally in light of Theorem 1.3, Theorem 5.5 and Theorem 5.7, we have the following theorem which provides the distinguishing number for the conjugation action of SD_{8n} on itself, for arbitrary n.

Theorem 5.8. *If SD_{8n} acts on itself by conjugation action, then $\mathcal{D}_{SD_{8n}}(SD_{8n}) = 2$.*

6. Some Characterizations

Theorem 2.5 in [6], along with the conclusions from each section discussed so far, arise a natural question that

Whether there exists a non-Abelian group acting via conjugation action on itself with the distinguishing number other than 2?

The main purpose of this section is to find the answer of the aforementioned question and surprisingly, we answer it in negative at the end of this section. Before reaching a conclusion, we need some characterizations for the existence of a 2-distinguishing labelling, in general, for the action of a group $\mathcal{G}$ on a $\mathcal{G}$-set $\mathcal{X}$, not necessarily the conjugation action of $\mathcal{G}$ on itself. First, we give some basic definitions:

Definition 6.1. *Let $\mathcal{G}$ be a group acting on a set $\mathcal{X}$. A subset $\mathcal{A}$ of the $\mathcal{G}$-set $\mathcal{X}$ is called $\mathcal{G}$-invariant, if the subset $\{ga : g \in \mathcal{G}, a \in \mathcal{A}\} \subseteq \mathcal{A}$, under the action of $\mathcal{G}$ restricted to $\mathcal{A}$.*

It is easy to see that the orbits of a $\mathcal{G}$-set $\mathcal{X}$ are $\mathcal{G}$-invariant subsets of $\mathcal{X}$. Moreover, if $\mathcal{A}$ is a $\mathcal{G}$-invariant subset of a $\mathcal{G}$-set $\mathcal{X}$, then it is always a union of orbits of a $\mathcal{G}$-set $\mathcal{X}$.

Definition 6.2. *Let $\mathcal{G}$ be a group acting on a set $\mathcal{X}$ and $g \in \mathcal{G}$. A subset $\mathcal{A}$ of the $\mathcal{G}$-set $\mathcal{X}$ is called g-invariant, if the subset $\{ga : g \in \mathcal{G}, a \in \mathcal{A}\} \subseteq \mathcal{A}$, under the action of $\mathcal{G}$ restricted to $\mathcal{A}$.*

Theorem 6.3. *Let $\mathcal{G}$ be a group acting on a set $\mathcal{X}$. If the $\mathcal{G}$-set $\mathcal{X}$ can be partitioned into a disjoint union of two subsets (say) $\mathcal{X}_1$ and $\mathcal{X}_2$, which are not g-invariant, for every $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, then there exists a 2-distinguishing labelling for the action of $\mathcal{G}$ on the set $\mathcal{X}$.*

Proof. Assume that $\mathcal{X}$ can be partitioned as a disjoint union of two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that for each $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, the subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ are not invariant under the action of g. Define a labelling $\phi : \mathcal{X} \longrightarrow \{1,2\}$ by $\phi(x) = 1$, if $x \in \mathcal{X}_1$ and $\phi(x) = 2$, otherwise. Claim that the labelling $\phi : \mathcal{X} \longrightarrow \{1,2\}$, is a 2-distinguishing labelling. For this, it suffices to prove that $\{g \in \mathcal{G} : \phi \circ g\ (x) = \phi(x)$ for all $x \in \mathcal{X}\} \subseteq Stab_{\mathcal{G}}(\mathcal{X})$. Equivalently, we will prove that if $g \in \mathcal{G} \setminus Stab_{\mathcal{G}}(\mathcal{X})$, then the element g can not be a member of the set $\{g \in \mathcal{G} : \phi \circ g\ (x) = \phi(x),$ for all $x \in \mathcal{X}\}$. If not, then clearly by the definition of ϕ and the fact that $\mathcal{X}$ is a disjoint union of $\mathcal{X}_1$ and $\mathcal{X}_2$, we have $g.x \in \mathcal{X}_1$ or $\mathcal{X}_2$ for $x \in \mathcal{X}_1$ or $\mathcal{X}_2$ respectively, for some $g \in \mathcal{G} \setminus Stab_{\mathcal{G}}(\mathcal{X})$. Therefore, for some $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, the subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ are g-invariant, which is a contradiction to the given hypothesis. Thus, $\{g \in \mathcal{G} : \phi \circ g\ (x) = \phi(x)$ for all $x \in \mathcal{X}\} \subseteq Stab_{\mathcal{G}}(\mathcal{X})$ and hence ϕ is a 2-distinguishing labelling. ■

The upcoming theorem provides a characterization for the existence of a 2-distinguishing labelling for a group acting on a set. Also, it is observed that the conclusion of the above theorem still remains true if at least one of the partitioning components is not invariant under the action of each element of the group.

Theorem 6.4. *Let $\mathcal{X}$ be a $\mathcal{G}$-set. Then the following statements are equivalent:*

(1) There is a 2-distinguishing labelling for the action of the group $\mathcal{G}$ on the set $\mathcal{X}$.

(2) The $\mathcal{G}$-set $\mathcal{X}$ can be partitioned as a disjoint union of two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that for every $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, at least one of them is not g-invariant.

(3) The $\mathcal{G}$-set $\mathcal{X}$ can be partitioned as a disjoint union of two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ that are not g-invariant, for every $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$.

Proof. $(1) \Rightarrow (2)$: Assume that there exists a 2-distinguishing labelling ϕ, for the action of the group $\mathcal{G}$ on the set $\mathcal{X}$. Partition the $\mathcal{G}$-set $\mathcal{X}$ as the disjoint union of two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_i = \{x \in \mathcal{X} : \phi(x) = i\}$, for $i = 1,2$. Next, we shall show that for every $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, the set $\mathcal{X}_1$ is not g-invariant. On the contrary, let us assume that $\mathcal{X}_1$ is $Stab_{\mathcal{G}}(\mathcal{X})^c$-invariant. Therefore, by definition, for $g \notin Stab_{\mathcal{G}}(\mathcal{X})$ and $x \in \mathcal{X}_1$, we have $gx \in \mathcal{X}_1$. Then clearly, if $x \in \mathcal{X}_2$ we have $gx \in \mathcal{X}_2$. Otherwise, for some $x \in \mathcal{X}_2$, we have $y = gx \in \mathcal{X}_1$. Note that since $Stab_{\mathcal{G}}(\mathcal{X})$ is a group, so if $g \notin Stab_{\mathcal{G}}(\mathcal{X})$, then $g^{-1} \notin Stab_{\mathcal{G}}(\mathcal{X})$. Consequently, we have $y \in \mathcal{X}_1$, while $g^{-1}(y) = g^{-1}gx = x \in \mathcal{X}_2$, which is not possible. Thus, there is an element g outside the set $Stab_{\mathcal{G}}(\mathcal{X})$ that preserves the labelling, which is a contradiction, as ϕ is a 2-distinguishing labelling.

$(2) \Rightarrow (3)$: Suppose that the $\mathcal{G}$-set $\mathcal{X}$ can be partitioned into a disjoint union of two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{X}$ such that for every $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$, at least one of the $\mathcal{X}_i$ is not g-invariant. Without loss of generality, we can assume that $\mathcal{X}_1$ is not g-invariant. Therefore, there exists an element $x \in \mathcal{X}_1$ such that for some $g \notin Stab_{\mathcal{G}}(\mathcal{X})$, we have $x \neq y = gx \notin \mathcal{X}_1$. Thus $y = gx \in \mathcal{X}_2$. Note that $g \notin Stab_{\mathcal{G}}(\mathcal{X})$ if and only if $g^{-1} \notin Stab_{\mathcal{G}}(\mathcal{X})$. Clearly, there is $y \in \mathcal{X}_2$ and $g^{-1} \notin Stab_{\mathcal{G}}(\mathcal{X})$ satisfying $g^{-1}y = x \in \mathcal{X}_1$. Thus $\mathcal{X}_2$ is not g-invariant, for all $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$. Similarly, for all $g \in Stab_{\mathcal{G}}(\mathcal{X})^c$ if $\mathcal{X}_2$ is not g-invariant, then so is $\mathcal{X}_1$. Hence, the $\mathcal{G}$-set $\mathcal{X}$ can be partitioned as a union of two disjoint subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ which are not g-invariant.

$(3) \Rightarrow (1)$: Follows from Theorem 6.3. ∎

An immediate consequence of the above theorem is

Corollary 6.5. *Let $\mathcal{G}$ be a group acting on a set $\mathcal{X}$ and $\phi : \mathcal{X} \longrightarrow \{1, 2\}$ be a 2-distinguishing labelling for the action of $\mathcal{G}$ on the set $\mathcal{X}$. Then the subset $\mathcal{X}_i = \{x \in \mathcal{X} : \phi(x) = i\}$, $i = 1, 2$, cannot be a union of orbits.*

Also, we have

Corollary 6.6. *Let $\mathcal{X} = P_1 \sqcup P_2$ be a partition of a $\mathcal{G}$-set $\mathcal{X}$. If the partitioning subsets P_1 and P_2 are a union of orbits of $\mathcal{G}$, then the action of $\mathcal{G}$ on $\mathcal{X}$ can not be 2-distinguishable.*

In the forthcoming results, we completely characterize the distinguishing number for conjugation action. In fact, we find a connection of the distinguishing number for the above specified group action depending on the fact whether the group is an Abelian group or not.

Theorem 6.7. *If a group $\mathcal{G}$ acts on itself by conjugation, then $\mathcal{D}_{\mathcal{G}}(\mathcal{G}) = 1$ if and only if $\mathcal{G}$ is an Abelian group.*

Proof. Note that $\mathcal{G}$ is Abelian if and only if $\mathcal{G} = Z(\mathcal{G}) = \{g : ghg^{-1} = h, \text{ for all } h \in \mathcal{G}\}$ if and only if under the conjugation action, the group $\mathcal{G}$ acts on itself by fixing each of its elements. Thus, the result follows by using Proposition 2.1 [6]. ∎

Finally, when a group $\mathcal{G}$ acts on itself by conjugation, we have

Theorem 6.8. *A group $\mathcal{G}$ is non-Abelian if and only if $\mathcal{D}_{\mathcal{G}}(\mathcal{G}) = 2$.*

Proof. Clearly, Theorem 6.7 guarantees that if the distinguishing number for the conjugation action of a group $\mathcal{G}$ on itself is 2, that is, $\mathcal{D}_{\mathcal{G}}(\mathcal{G}) = 2$, then the group $\mathcal{G}$ is non-Abelian. This completes the sufficient part of the present theorem.

For the necessary part, assume that a non-Abelian group $\mathcal{G}$ acts on itself by conjugation. Then, in view of Theorem 1.3, it is sufficient to prove that there exists a 2-distinguishing labelling for the action of the group $\mathcal{G}$. However, using Theorem 6.4, it is enough to prove that the non-Abelian group $\mathcal{G}$ can be partitioned as the disjoint union of two subsets $\mathcal{X}$ and $\mathcal{Y}$ such that for every $g \notin Z(\mathcal{G})$, at least one of the partitioning subsets is not g-invariant. Next, we construct such a partition of $\mathcal{G}$ as follows:

Let $\mathcal{X}$ be the set constructed by taking exactly one element from each conjugacy class and $\mathcal{Y} = \mathcal{G} \setminus \mathcal{X}$. Clearly, $\mathcal{G} = \mathcal{X} \sqcup \mathcal{Y}$. Note that an element of a group belongs to its center if and only if its conjugacy class contains exactly one element. Moreover, a group is non-Abelian if and only if it has a conjugacy class containing at least two elements. Thus, we conclude that the partitioning components $\mathcal{X}$ and $\mathcal{Y}$ are non empty and satisfy the property that $Z(\mathcal{G}) \subsetneq \mathcal{X}$ and $\mathcal{Y} \neq \mathcal{G} \setminus Z(\mathcal{G})$, as $\mathcal{G}$ is non-Abelian.

We claim that the partition $\mathcal{X} \sqcup \mathcal{Y}$ of $\mathcal{G}$ satisfies the condition: for every $g \notin Z(\mathcal{G})$, at least one of the partitioning subset $\mathcal{X}$ or $\mathcal{Y}$ is not g-invariant. If not, then for every $g \notin Z(\mathcal{G})$, both the partitioning subsets $\mathcal{X}$ and $\mathcal{Y}$ are g-invariant. In particular, for every $g \notin Z(\mathcal{G})$ and $x \in \mathcal{X}$, we have $gxg^{-1} \in \mathcal{X}$. Now, we shall show that $\mathcal{X} = \mathcal{G}$. For this, let $g \in \mathcal{G}$. Clearly, if $g \in Z(\mathcal{G})$, then $g \in \mathcal{X}$. However, if $g \in \mathcal{G} \setminus (Z(\mathcal{G}) \cup \mathcal{X})$, then by the construction of $\mathcal{X}$, we can find an element $x \in \mathcal{X}$ in the conjugacy class of g, as the set $\mathcal{X}$ has a non-trivial intersection with each conjugacy class of $\mathcal{G}$. Therefore, there exists an element $k \in \mathcal{G} \setminus Z(\mathcal{G})$ such that $g = k^{-1}xk$ with $k \neq x$. Since, $Z(\mathcal{G})$ is a subgroup of $\mathcal{G}$, so we conclude that $k^{-1} \notin Z(\mathcal{G})$. Finally, one can choose $h = k^{-1} \notin Z(\mathcal{G})$ and $x = kgk^{-1} \in \mathcal{X}$, so that $hxh^{-1} = k^{-1}(kgk^{-1})k = g \in \mathcal{X}$, as the partitioning subset $\mathcal{X}$ is g-invariant. Thus, we conclude that $\mathcal{X} = \mathcal{G}$, which is a contradiction, as $\mathcal{Y} \neq \phi$. On the other hand, if the partitioning subset $\mathcal{Y}$ is g-invariant for every $g \notin Z(\mathcal{G})$, then in a similar way, this assumption leads to a contradiction that $\mathcal{Y} = \mathcal{G} \setminus Z(\mathcal{G})$. Hence, for every $g \notin Z(\mathcal{G})$, at least one of the partitioning subsets $\mathcal{X}$ or $\mathcal{Y}$ is not g-invariant. This completes the proof. ∎

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