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Permuting Tri-derivations in MV-algebras

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Abstract. An MV-algebra is an algebraic structure with a binary operation $\oplus$, a unary operation $'$ and the constant 0 satisfying certain axioms. MV-algebras are the algebraic semantics of Lukasiewicz logic. This work includes a type of derivation research on MV-algebras. Our aim is to introduce the concept of permuting tri-derivation on MV-algebras and to discuss some results.

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Contents

1. Introduction

The concept of derivation has an important place in the research of the structure and properties of algebraic systems. In prime rings, the notion of derivation was introduced by Posner [16]. In [17], Szasz applied the derivation concept to lattices. Xin et al. developed derivation for a lattice and they offered some equivalent conditions under which a derivation is isotone for lattices with a greatest element, modular lattices and distributive lattices, in [18] and [19]. Later, different derivations and properties in lattices were examined, for example [5], [6]. In [15], Öztürk achieved some results by introducing the idea of permuting tri-derivations in rings. After, Öztürk et al. studied the permuting tri-derivations in lattices [14]. Further, permuting skew 3-derivations, permuting skew n-derivations in rings have studied and commutativity of a ring satisfying certain identities involving the trace of permuting n-derivations (see [3], [9], [10]).

When dealing with information and uncertainty, non-classical logic is useful in terms of uncertain and fuzzy information in computer science. MV-algebras as the algebraic counterpart of many-valued prepositional calculus were proposed by Chang [7]. Classical two-valued logic makes it meaningful to study Boolean algebras, and while every Boolean algebra is an MV-algebra, the reverse is not true. MV-algebras have many applications as they are generalization of Boolean algebras. Also, MV-algebras are categorically equivalent to some mathematical structures. For example, perfect MV-algebras categorically equivalent to abelian

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lattice-groups with strong unit and to bounded commutative BCK-algebras (see [12], [13]). In [1], Alshehri presented the concept of derivation in MV-algebras and examined some properties of the derivation in MV-algebras with the help of isotone derivations. Recently, several authors studied different derivations in MV-algebras, for example [2], [11], [20].

In this paper, we introduce the notion of permuting tri-derivations in MV-algebras. This article is organized as follows: In the next section, some results and basic concepts about MV-algebras are reminded. In section 3, permuting tri-derivation structure in MV-algebras is characterized and some results are obtained. Also, fixed point set structure of isotone permutig tri-derivations is established.

2. Preliminaries

Definition 2.1. *[7] Let us define $\oplus$ binary operation, $'$ a unary operation on the set Δ and 0 be a constant in Δ. If the following axioms are satisfied, then we say $(\Delta, \oplus, ', 0)$ is MV-algebra:*

(i) $(\Delta, \oplus, 0)$ is a commutative monoid,
(ii) $(\delta^{'})^{'} = \delta$,
(iii) $0^{'} \oplus \delta = 0^{'}$,
(iv) $(\delta^{'} \oplus \eta)^{'} \oplus \eta = (\eta^{'} \oplus \delta)^{'} \oplus \delta$ for all $\delta, \eta \in \Delta$.

In the remainder of the article, we denote an MV-algebra $(\Delta, \oplus, ', 0)$ by Δ.

Define the operations $\odot$ and $\ominus$ and the constant 1 as follows: $1 = 0^{'}$, $\delta \odot \eta = (\delta^{'} \oplus \eta^{'})^{'}$, $\delta \ominus \eta = \delta \odot \eta^{'}$. If we define $\delta \leq \eta$ if and only if $\delta^{'} \oplus \eta = 1$, then "$\leq$" is a partial order which called the natural order of Δ. This order determines a bounded distributive lattice structure. For the elements δ and η, the join $\delta \vee \eta$ and the meet $\delta \wedge \eta$ defined by: $\delta \vee \eta = (\delta \odot \eta^{'}) \oplus \eta = (\delta \ominus \eta) \oplus \eta$ and $\delta \wedge \eta = \delta \odot (\delta^{'} \oplus \eta) = \delta \ominus (\delta \ominus \eta) = (\delta^{'} \vee \eta^{'})^{'}$. Also, Δ is called linearly ordered, if the order relation "$\leq$" is total.

Example 2.2. *[8] Let $\Delta = [0, 1]$ be the real unit interval. For all $\delta, \eta \in \Delta$, if we define $\delta \oplus \eta = \min\{1, \delta + \eta\}$, $\delta \odot \eta = \max\{0, \delta + \eta - 1\}$ and $\delta^{'} = 1 - \delta$, then $(\Delta, \oplus, ', 0)$ is an MV-algebra. For each integer $n \geq 2$, the n-element set $\Delta_n = \left\{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\right\}$ is a linearly ordered MV-algebra which called MV-chain.*

Proposition 2.3. *[4, 8] Suppose that Δ is an MV-algebra and $\delta, \eta, \sigma \in \Delta$. Thus the followings hold:*

(1) $\delta \oplus \delta^{'} = 1$, $\delta \odot \delta^{'} = 0$, $\delta \oplus 1 = 1$,
(2) Provided that $\delta \oplus \eta = 0$, then $\delta = \eta = 0$,
(3) Provided that $\delta \odot \eta = 1$, then $\delta = \eta = 1$,
(4) If $\delta \leq \eta$, then $\delta \oplus \sigma \leq \eta \oplus \sigma$ and $\delta \odot \sigma \leq \eta \odot \sigma$,
(5) $\delta \odot \eta \leq \delta \wedge \eta \leq \delta, \eta \leq \delta \vee \eta \leq \delta \oplus \eta$,
(6) $\delta \leq \eta$ iff $\eta^{'} \leq \delta^{'}$,
(7) $\delta \oplus \eta = \eta$ iff $\delta \odot \eta = \delta$,
(8) $\delta \odot (\eta \vee \sigma) = (\delta \odot \eta) \vee (\delta \odot \sigma)$,
(9) $\delta \oplus (\eta \wedge \sigma) = (\delta \oplus \eta) \wedge (\delta \oplus \sigma)$,
(10) $\delta \oplus \eta = \eta$ iff $\delta \wedge \eta^{'} = 0$,
(11) If $\delta \odot \eta = \delta \odot \sigma$ and $\delta \oplus \eta = \delta \oplus \sigma$, then $\eta = \sigma$.

MV-algebras that do not satisfy idempotent conditions are generalizations of Boolean algebras. For any MV-algebra Δ, if we define $B(\Delta) = \{\delta \in \Delta | \, \delta \odot \delta = \delta\} = \{\delta \in \Delta | \, \delta \oplus \delta = \delta\}$, then $(B(\Delta), \oplus, ', 0)$ is a largest subalgebra of Δ, which is called Boolean center of Δ.

Theorem 2.4. *[8] Let Δ be an MV-algebra. Then for each element δ in Δ, the following conditions are equivalent:*

(1) $\delta \in B(\Delta)$,
(2) $\delta \vee \delta^{'} = 1$,
(3) $\delta \wedge \delta^{'} = 0$,
(4) $\delta \oplus \delta = \delta$,
(5) $\delta \odot \delta = \delta$,
(6) $\delta \oplus \eta = \delta \vee \eta$ for all $\eta \in \Delta$,
(7) $\delta \odot \eta = \delta \wedge \eta$ for all $\eta \in \Delta$.

Theorem 2.5. *[7] Assume that Δ is an MV-algebra. Therefore, the following expressions are equivalent:*

(i) $\delta \leq \eta$,
(ii) $\eta \oplus \delta^{'} = 1$,
(iii) $\delta \odot \eta^{'} = 0$.

Definition 2.6. *[7] Suppose Δ be an MV-algebra and $\emptyset \neq I \subseteq \Delta$. If the following situations are satisfied,*

(1) $0 \in I$,
(2) Provided that $\delta, \eta \in I$, then $\delta \oplus \eta \in I$,
(3) Provided that $\eta \in I$, $\delta \in \Delta$ and $\delta \leq \eta$, then $\delta \in I$
then I is called an ideal of Δ.

Proposition 2.7. *[7] Assume that Δ is a linearly ordered MV-algebra. Then $\delta \oplus \eta = \delta \oplus \sigma$ and $\delta \oplus \sigma \neq 1$ imply that $\eta = \sigma$.*

Definition 2.8. *[1] Assume that Δ is an MV-algebra. A mapping $D : \Delta \to \Delta$ is called a derivation on Δ if it provides*

$$D(\delta_1 \odot \delta_2) = (D(\delta_1) \odot \delta_2) \oplus (\delta_1 \odot D(\delta_2))$$

for all $\delta_1, \delta_2 \in \Delta$.

3. Permuting tri-derivations on MV-algebras

We begin with the following definition.

Definition 3.1. *Suppose that Δ is an MV-algebra. A map $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is called permuting if $\Gamma(\delta, \eta, \sigma) = \Gamma(\delta, \sigma, \eta) = \Gamma(\eta, \delta, \sigma) = \Gamma(\eta, \sigma, \delta) = \Gamma(\sigma, \delta, \eta) = \Gamma(\sigma, \eta, \delta)$ holds for all $\delta, \eta, \sigma \in \Delta$.*

A mapping $\gamma : \Delta \to \Delta$ defined by $\gamma(\delta) = \Gamma(\delta, \delta, \delta)$ is called the trace of Γ, where $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is a permuting mapping. In that follows, we often abbreviate $\gamma(\delta)$ to $\gamma\delta$.

Definition 3.2. *Suppose that Δ is an MV-algebra and $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is a permuting mapping. If Γ satisfies the following*

$$\Gamma(\delta \odot \rho, \eta, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\delta \odot \Gamma(\rho, \eta, \sigma))$$

for all $\delta, \eta, \sigma, \rho \in \Delta$, then Γ is called a permuting tri-derivation. Clearly, if Γ is a permuting tri-derivation on Δ, then the relations hold: for all $\delta, \eta, \sigma, \rho \in \Delta$,

$$\Gamma(\delta, \eta \odot \rho, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\eta \odot \Gamma(\delta, \rho, \sigma))$$

and

$$\Gamma(\delta, \eta, \sigma \odot \rho) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\sigma \odot \Gamma(\delta, \eta, \rho)).$$

Example 3.3. *Let $\Delta = \{0, \delta, \eta, 1\}$. Consider the tables given below:*

$\oplus$	0	δ	η	1
0	0	δ	η	1
δ	δ	δ	1	1
η	η	1	η	1
1	1	1	1	1

$'$	0	δ	η	1
	1	η	δ	0

Then $(\Delta, \oplus, ', 0)$ is an MV- algebra. Define a mapping $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \delta, \ x_1, x_2, x_3 \in \{1, \delta\} \\ 0, \ otherwise \end{cases}$.

It appears that Γ is a permuting tri-derivation on Δ.

Proposition 3.4. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ. For all $\delta \in \Delta$, we have*

(1) $\gamma 0 = 0$,
(2) $\gamma\delta \odot \delta^{'} = \delta \odot \gamma\delta^{'} = 0$,
(3) $\gamma\delta = \gamma\delta \oplus (\delta \odot \Gamma(\delta, \delta, 1))$,
(4) $\gamma\delta \leq \delta$,
(5) If I is an ideal of Δ, then $\gamma(I) \subseteq I$.

Proof. (1) We can write

$$\begin{aligned} \gamma 0 &= \Gamma(0,0,0) = \Gamma(0 \odot 0, 0, 0) \\ &= (\Gamma(0,0,0) \odot 0) \oplus (0 \odot \Gamma(0,0,0)) \\ &= 0 \oplus 0 = 0. \end{aligned}$$

(2) For all $\delta \in \Delta$,

$$\begin{aligned} \Gamma(\delta, \delta, 0) &= \Gamma(\delta, \delta, 0 \odot 0) \\ &= (\Gamma(\delta, \delta, 0) \odot 0) \oplus (0 \odot \Gamma(\delta, \delta, 0)) \\ &= 0 \oplus 0 = 0. \end{aligned}$$

Then, we get

$$\begin{aligned} 0 = \Gamma(\delta, \delta, 0) &= \Gamma(\delta, \delta, \delta \odot \delta^{'}) \\ &= (\Gamma(\delta, \delta, \delta) \odot \delta^{'}) \oplus (\delta \odot \Gamma(\delta, \delta, \delta^{'})). \end{aligned}$$

By the property (2) of Proposition 2.3, $\gamma\delta \odot \delta^{'} = 0$ and $\delta \odot \Gamma(\delta, \delta, \delta^{'}) = 0$. We can see that $\delta \odot \gamma\delta^{'} = 0$ for all $\delta \in \Delta$, similarly.

(3) For all $\delta \in \Delta$,

$$\begin{aligned} \gamma\delta &= \Gamma(\delta, \delta, \delta) = \Gamma(\delta, \delta, \delta \odot 1) \\ &= (\Gamma(\delta, \delta, \delta) \odot 1) \oplus (\delta \odot \Gamma(\delta, \delta, 1)) \\ &= \gamma\delta \oplus (\delta \odot \Gamma(\delta, \delta, 1)). \end{aligned}$$

(4) For all $\delta \in \Delta$,

$$1 = 0^{'} = (\gamma\delta \odot \delta^{'})^{'} = [((\gamma\delta)^{'} \oplus (\delta^{'})^{'})^{'}]^{'} = (\gamma\delta)^{'} \oplus \delta.$$

Then, by Theorem 2.5, we have $\gamma\delta \leq \delta$ for all $\delta \in \Delta$.

(5) If $\eta \in \gamma(I)$, then $\eta = \gamma(\delta)$ for some $\delta \in I$. From (4), we have $\gamma(\delta) \leq \delta$. Since I is an ideal of Δ, we get $\eta \in I$ and so $\gamma(I) \subseteq I$. ■

Remark 3.5. *We have* $\delta \odot \Gamma(\delta, \delta, \delta^{'}) = 0$ *for all* $\delta \in \Delta$*. Thus,* $\Gamma(\delta, \delta, \delta^{'}) \leq \delta^{'}$ *and* $\delta \leq (\Gamma(\delta, \delta, \delta^{'}))^{'}$*. For all* $\delta, \eta, \sigma \in \Delta$*,*

$$0 = \Gamma(\delta \odot \delta^{'}, \eta, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \delta^{'}) \oplus (\delta \odot \Gamma(\delta^{'}, \eta, \sigma))$$

and hence $\Gamma(\delta, \eta, \sigma) \leq \delta$ *and* $\Gamma(\delta^{'}, \eta, \sigma) \leq \delta^{'}$*.*

Proposition 3.6. *Let* Δ *be an MV-algebra,* Γ *be a permuting tri-derivation on* Δ *and* γ *be the trace of* Γ*. For* $\delta, \eta \in \Delta$*, if* $\delta \leq \eta$ *then*

(1) $\gamma(\delta \odot \eta^{'}) = 0$,
(2) $\gamma\eta^{'} \leq \delta^{'}$,
(3) $\gamma\delta \odot \gamma\eta^{'} = 0$.

Proof. (1) We assume $\delta \leq \eta$ for $\delta, \eta \in \Delta$. By the property (4) of Proposition 2.3, we have $\delta \odot \eta^{'} \leq \eta \odot \eta^{'} = 0$. Then, $\delta \odot \eta^{'} = 0$ and so $\gamma(\delta \odot \eta^{'}) = 0$, since $\gamma 0 = 0$.

(2) We have $\delta \odot \gamma\eta^{'} \leq \eta \odot \gamma\eta^{'} \leq \eta \odot \eta^{'} = 0$ since $\delta \leq \eta$. From here we obtain that $\delta \odot \gamma\eta^{'} = 0$ and $\gamma\eta^{'} \leq \delta^{'}$.

(3) We have $\gamma\delta \leq \eta$ since $\delta \leq \eta$. Hence $\gamma\delta \odot \gamma\eta^{'} \leq \eta \odot \gamma\eta^{'} \leq \eta \odot \eta^{'} = 0$ and so $\gamma\delta \odot \gamma\eta^{'} = 0$. ■

Proposition 3.7. *Suppoese that* Δ *is an MV-algebra,* Γ *is a permuting tri-derivation on* Δ *and* γ *is the trace of* Γ*. Then,*

(1) $\gamma\delta \odot \gamma\delta^{'} = 0$,
(2) $\gamma\delta^{'} = (\gamma\delta)^{'}$ iff γ is the identity on Δ.

Proof. (1) From Proposition 3.6(3), $\gamma\delta \odot \gamma\eta^{'} = 0$. Taking η by δ, we have $\gamma\delta \odot \gamma\delta^{'} = 0$.

(2) Since $\delta \odot \gamma\delta^{'} = 0$, we get $\delta \odot (\gamma\delta)^{'} = 0$. Then, $\gamma\delta \leq \delta$ and $\delta \leq \gamma\delta$ i.e., $\gamma\delta = \delta$. Thus, γ is identity on Δ. Conversely, if γ is identity on Δ, then $\gamma\delta^{'} = (\gamma\delta)^{'}$, $\forall \delta \in \Delta$. ■

Definition 3.8. *Suppose that* Δ *is an MV-algebra and* Γ *is a permuting tri-derivation on* Δ*. If* $\delta \leq \rho$ *implies* $\Gamma(\delta, \eta, \sigma) \leq \Gamma(\rho, \eta, \sigma)$ *for all* $\delta, \eta, \sigma, \rho \in \Delta$*, then* Γ *is called an isotone. If* γ *is the trace of* Γ *and* Γ *is an isotone, then* $\delta \leq \eta$ *implies* $\gamma\delta \leq \gamma\eta$ *for all* $\delta, \eta \in \Delta$*.*

Example 3.9. *Let* $\Delta = \{0, \delta_1, \delta_2, \delta_3, \delta_4, 1\}$*. Consider the following tables:*

$\oplus$	0	δ_1	δ_2	δ_3	δ_4	1
0	0	δ_1	δ_2	δ_3	δ_4	1
δ_1	δ_1	δ_3	δ_4	δ_3	1	1
δ_2	δ_2	δ_4	δ_2	1	δ_4	1
δ_3	δ_3	δ_3	1	δ_3	1	1
δ_4	δ_4	1	δ_4	1	1	1
1	1	1	1	1	1	1

$'$	0	δ_1	δ_2	δ_3	δ_4	1
	1	δ_4	δ_3	δ_2	δ_1	0

Then $(\Delta, \oplus, ', 0)$ is an MV-algebra. Let us define a map $\Gamma : \Delta \times \Delta \times \Delta \rightarrow \Delta$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \delta_2, & x_1, x_2, x_3 \in \{\delta_2, \delta_4, 1\} \\ 0, & otherwise \end{cases}$. We can see that Γ is an isotone permuting tri-derivation on Δ.

Example 3.10. *Consider* $\Delta_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ *as in Example 2.2 and define* $\Gamma : \Delta_4 \times \Delta_4 \times \Delta_4 \rightarrow \Delta_4$ *by* $\Gamma(x_1, x_2, x_3) = \begin{cases} \frac{1}{3}, & (x_1, x_2, x_3) \in \{(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\} \\ 0, & otherwise \end{cases}$.

Then Γ is a permuting tri-derivation on Δ_4, but Γ is not isotone, because $\Gamma(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \nleq \Gamma(1, \frac{2}{3}, \frac{2}{3})$.

Proposition 3.11. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ. If $\gamma\delta^{'} = \gamma\delta$ for all $\delta \in \Delta$, then the followings hold:*

(1) $\gamma 1 = 0$,
(2) $\gamma\delta \odot \gamma\delta = 0$,
(3) Provided that Γ is an isotone on Δ, then $\gamma = 0$.

Proof. (1) Replacing δ by 0 in hypothesis, we have $\gamma 1 = 0$.
(2) For all $\delta \in \Delta$, $\gamma\delta \odot \gamma\delta = \gamma\delta \odot \gamma\delta^{'} = 0$, by Proposition 3.7.
(3) Suppose that Γ is an isotone on Δ. For $\delta \in \Delta$, since $\gamma\delta \leq \gamma 1 = 0$, we have $\gamma\delta = 0$ and so $\gamma = 0$. ■

Proposition 3.12. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ and $\delta \in B(\Delta)$. Then the followings hold:*

(1) If $\delta \leq \Gamma(1, \eta, \sigma)$ for all $\eta, \sigma \in \Delta$, then $\Gamma(\delta, \eta, \sigma) = \delta$,
(2) $\delta \wedge \Gamma(\delta, \delta, 1) \wedge (\gamma\delta)^{'} = 0$,
(3) If $\delta \leq \Gamma(\delta, \delta, 1)$, then $\gamma\delta = \delta$.

Proof. (1) We have

$$\begin{aligned}\Gamma(\delta, \eta, \sigma) &= \Gamma(\delta \odot 1, \eta, \sigma)\\ &= (\Gamma(\delta, \eta, \sigma) \odot 1) \oplus (\delta \odot \Gamma(1, \eta, \sigma))\\ &= \Gamma(\delta, \eta, \sigma) \oplus \delta = \delta.\end{aligned}$$

(2) Since $\gamma\delta = \gamma\delta \oplus (\delta \odot \Gamma(\delta, \delta, 1))$, it follows that $(\delta \odot \Gamma(\delta, \delta, 1)) \wedge (\gamma\delta)^{'} = 0$. Then, by Theorem 2.4, we obtain $\delta \wedge \Gamma(\delta, \delta, 1) \wedge (\gamma\delta)^{'} = 0$.
(3) Let $\delta \leq \Gamma(\delta, \delta, 1)$. Then, we get $\delta \odot (\gamma\delta)^{'} = 0$ by (2). Thus, $\delta \leq \gamma\delta \leq \delta$ and so $\gamma\delta = \delta$. ■

Theorem 3.13. *Suppose that Δ is an MV-algebra. We define a map by $\Gamma(\delta, \eta, \sigma) = \delta \odot \eta \odot \sigma$ for all $\delta, \eta, \sigma \in \Delta$. Then Γ is a permuting tri-derivation on $B(\Delta)$.*

Proof. We have

$$\Gamma(\delta \odot \rho, \eta, \sigma) = (\delta \odot \rho) \odot \eta \odot \sigma$$

for all $\delta, \eta, \sigma, \rho \in B(\Delta)$. Moreover,

$$\begin{aligned}(\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\delta \odot \Gamma(\rho, \eta, \sigma)) &= ((\delta \odot \eta \odot \sigma) \odot \rho) \oplus (\delta \odot (\rho \odot \eta \odot \sigma))\\ &= (\delta \odot \rho) \odot \eta \odot \sigma.\end{aligned}$$

Thus, Γ is a permuting tri-derivation on $B(\Delta)$. ■

Definition 3.14. *Suppose that Δ is an MV-algebra, Γ is a permuting mapping on Δ. If $\Gamma(\delta \oplus \rho, \eta, \sigma) = \Gamma(\delta, \eta, \sigma) \oplus \Gamma(\rho, \eta, \sigma)$ for all $\delta, \eta, \sigma, \rho \in \Delta$, then Γ is said to be tri-additive mapping.*

Theorem 3.15. *Suppose that Δ is an MV-algebra, Γ is a tri-additive mapping on Δ and γ is the trace of Γ. Thus, $\gamma(B(\Delta)) \subseteq B(\Delta)$.*

Proof. Let $\delta \in \gamma(B(\Delta))$. Then, $\delta = \gamma(\eta)$ for some $\eta \in B(\Delta)$. Hence, $\delta \oplus \delta = \gamma\eta \oplus \gamma\eta = \Gamma(\eta \oplus \eta, \eta, \eta) = \gamma\eta = \delta$. Therefore, $\delta \in B(\Delta)$ i.e., $\gamma(B(\Delta)) \subseteq B(\Delta)$. ■

Theorem 3.16. *Suppose that Δ is a linearly ordered MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then $\gamma = 0$ or $\gamma 1 = 1$.*

Proof. For all $\delta \in \Delta$, we have $\delta \oplus \delta^{'} = 1$ and $\delta \oplus 1 = 1$. Thus,

$$\gamma 1 = \Gamma(1,1,1) = \Gamma(\delta \oplus \delta^{'},1,1) = \Gamma(\delta,1,1) \oplus \Gamma(\delta^{'},1,1)$$

and

$$\gamma 1 = \Gamma(1,1,1) = \Gamma(\delta \oplus 1,1,1) = \Gamma(\delta,1,1) \oplus \gamma 1.$$

If $\gamma 1 \neq 1$, then we have $\gamma 1 = \Gamma(\delta^{'},1,1)$ by Proposition 2.7. Replacing δ by 1, we have $\gamma 1 = 0$. For all $\delta \in \Delta$,

$$0 = \gamma 1 = \Gamma(\delta,1,1) \oplus \gamma 1 = \Gamma(\delta,1,1)$$

and

$$\begin{aligned}\Gamma(\delta,1,1) &= \Gamma(\delta,1,\delta \oplus 1) = \Gamma(\delta,1,\delta) = \Gamma(\delta,\delta \oplus 1,\delta)\\ &= \gamma\delta \oplus \Gamma(\delta,1,\delta) = \gamma\delta.\end{aligned}$$

Therefore, $\gamma\delta = 0$ for all $\delta \in \Delta$. In this case, we have $\gamma = 0$. ■

Proposition 3.17. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ. Then,*

(1) Γ is an isotone,
(2) If γ is trace of Γ, then $\gamma\delta = \delta \odot \Gamma(\delta,\delta,1)$ for all $\delta \in B(\Delta)$.

Proof. (1) Let $\delta \leq \rho$. Then,

$$\begin{aligned}\Gamma(\rho,\eta,\sigma) &= \Gamma(\rho \vee \delta,\eta,\sigma) = \Gamma((\rho \odot \delta^{'}) \oplus \delta,\eta,\sigma)\\ &= \Gamma(\rho \odot \delta^{'},\eta,\sigma) \oplus \Gamma(\delta,\eta,\sigma) \geq \Gamma(\delta,\eta,\sigma)\end{aligned}$$

for all $\delta,\eta,\sigma,\rho \in \Delta$.

(2) Since Γ is an isotone, we have $\gamma\delta \leq \Gamma(\delta,\delta,1)$. Thus

$$\delta \odot \gamma\delta \leq \delta \odot \Gamma(\delta,\delta,1) \leq \gamma\delta \oplus (\delta \odot \Gamma(\delta,\delta,1)) = \gamma\delta.$$

Also, $\delta \in B(\Delta)$ implies that $\delta \odot \gamma\delta = \delta \wedge \gamma\delta = \gamma\delta$. Hence $\gamma\delta = \delta \odot \Gamma(\delta,\delta,1)$. ■

Remark 3.18. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. If $\gamma\delta = 0$ for all $\delta \in \Delta$, then $\Gamma(\delta,\delta,\eta) = 0$ for all $\eta \in \Delta$. Indeed, we have*

$$0 = \gamma\delta = \Gamma(\delta,\delta,\delta) = \gamma\delta \oplus \Gamma(1,\delta,\delta) = \Gamma(1,\delta,\delta)$$

and so

$$0 = \Gamma(1,\delta,\delta) = \Gamma(\eta \oplus 1,\delta,\delta) = \Gamma(\eta,\delta,\delta).$$

Theorem 3.19. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then,*

$$\ker\gamma = \gamma^{-1}(0) = \{\delta \in \Delta |\, \gamma\delta = 0\}$$

is an ideal of Δ.

Proof. We have $\gamma 0 = 0$, by Proposition 3.4(1). This yields that $0 \in \gamma^{-1}(0)$. Assume that $\delta,\eta \in \gamma^{-1}(0)$. Then,

$$\begin{aligned}\gamma(\delta \oplus \eta) &= \Gamma(\delta \oplus \eta,\delta \oplus \eta,\delta \oplus \eta)\\ &= \gamma\delta \oplus \Gamma(\delta,\delta,\eta) \oplus \Gamma(\delta,\eta,\delta) \oplus \Gamma(\delta,\eta,\eta)\\ &\oplus \Gamma(\eta,\delta,\delta) \oplus \Gamma(\eta,\delta,\eta) \oplus \Gamma(\eta,\eta,\delta) \oplus \gamma\eta.\end{aligned}$$

Using Remark 3.8, $\gamma(\delta \oplus \eta) = 0$ which ensures that $\delta \oplus \eta \in \gamma^{-1}(0)$. Suppose $\delta \in \gamma^{-1}(0)$ and $\eta \leq \delta$. Since Γ is an isotone, we get $\gamma\eta \leq \gamma\delta = 0$. Thus $\gamma\eta = 0$ and so $\eta \in \gamma^{-1}(0)$. ■

Now, we discuss the structures and some properties of fixed points set of isotone permuting tri-derivations. Let Γ be an isotone permuting tri-derivation on Δ. We denote by $Fix_\gamma(\Delta)$ the set of all fixed points of Δ for γ. That is,

$$Fix_\gamma(\Delta) = \{\delta \in \Delta | \, \gamma\delta = \delta\}.$$

Theorem 3.20. *Suppose that Δ is an MV-algebra and Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then,*

(1) $\gamma\delta = \gamma 1 \odot \delta$ for any $\delta \in Fix_\gamma(\Delta)$,
(2) $\gamma^2(\delta) = \gamma(\delta)$ for any $\delta \in Fix_\gamma(\Delta)$; where $\gamma^2(\delta) = \gamma(\gamma(\delta))$,
(3) $Fix_\gamma(\Delta) = \gamma(Fix_\gamma(\Delta))$.

Proof. (1) We have $\gamma\delta = \delta$. Thus, $\gamma 1 \odot \gamma\delta = \gamma 1 \wedge \gamma\delta = \gamma\delta$ implies that $\gamma\delta = \gamma 1 \odot \delta$ for all $\delta \in Fix_\gamma(\Delta)$.

(2) If δ is a fixed point of γ, then $\gamma(\delta) = \delta$, so $\gamma(\gamma(\delta)) = \gamma(\delta) = \delta$.

(3) If $\delta \in Fix_\gamma(\Delta)$, then $\delta \in \gamma(Fix_\gamma(\Delta))$. If $\delta \in \gamma(Fix_\gamma(\Delta))$, then for some $\eta \in Fix_\gamma(\Delta)$, $\delta = \gamma\eta = \eta$. Thus, we have $\delta \in Fix_\gamma(\Delta)$. Therefore, $Fix_\gamma(\Delta) = \gamma(Fix_\gamma(\Delta))$. ■

Example 3.21. *In Example 2.2, considering $\Delta_3 = \left\{0, \frac{1}{2}, 1\right\}$ and defining $\Gamma : \Delta_3 \times \Delta_3 \times \Delta_3 \to \Delta_3$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \frac{1}{2}, \, x_1 = x_2 = x_3 = \frac{1}{2} \\ 0, \, otherwise \end{cases}$. One can check that Γ is a permuting tri-derivation on Δ_3 and $Fix_\gamma(\Delta_3) = \left\{0, \frac{1}{2}\right\}$. Since $\frac{1}{2} \oplus \frac{1}{2} = 1 \notin Fix_\gamma(\Delta_3)$. Then, we have $Fix_\gamma(\Delta_3)$ is not an ideal of Δ_3.*

As can be seen from the example above, we encounter the following open problem:

For any ideal I of a MV-algebra Δ, whether there is a permuting tri-derivation Γ such that $Fix_\gamma(\Delta_3) = I$.

4. Conclusion

The concept of derivation was presented by Posner in 1957. In the following years, many mathematicians used derivations to examine the properties of algebraic structures. In their studies, on different derivations, the conditions for the ring to be commutative are examined. Some characterizations of algebraic structures are determined by the trace of permuting tri-additive mappings. With the help of the trace of permuting tri-derivation, the commutativity conditions of rings and how the elements are ordered in some structures such as lattices are investigated. The derivation type used in this article was put forward by Öztürk in rings. In this study, we obtained some results by presenting permuting tri-derivations on MV-algebras. The first aim of this study is to give the notion of permuting tri-derivation on this algebraic structure. Then some features provided by this derivation are listed. Fixed set structure has been investigated for such derivations by defining isotone permuting tri-derivations. After this study, permuting tri-f-derivations and permuting tri-(f, g)-derivations can be studied on MV-algebras. Also, since MV-algebras are BL-algebras that provide double negation property, permuting tri-derivation structure can be examined in BL-algebras.

applicable.

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