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The sequence of the hyperbolic k-Padovan quaternions

RENATA VIEIRA*[1], FRANCISCO REGIS ALVES[2] AND PAULA CATARINO[3]

[1] *Post-Graduate Program in Education of the Nordeste Education Network – Polo RENOEN-UFC, Federal University of Ceara, Brazil.*
[2] *Federal Institute of Science and Technology Education of the State of Ceará, Brazil.*
[3] *University of Trás-os-Montes and Alto Douro, Portugal.*

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Abstract. This work introduces the hyperbolic k-Padovan quaternion sequence, performing the process of complexification of linear and recurrent sequences, more specifically of the generalized Padovan sequence. In this sense, there is the study of some properties around this sequence, deepening the investigative mathematical study of these numbers.

AMS Subject Classifications: 11B37, 11B39.

Keywords: hyperbolic numbers, quaternions, k-Padovan sequence.

Contents

1. Introduction and Background

Studies of recursive linear sequences have been noticed in the mathematical literature. Based on this, there is the concern to carry out an investigative study on the process of complexification of certain sequences. So soon, in this work, the hyperbolic quaternion k-Padovan sequence is introduced, presenting algebraic properties around these numbers.

The Padovan sequence is a linear and recurrent third-order sequence, named after the Italian architect Richard Padovan. Thus, its recurrence is given by: $P_n = P_{n-2} + P_{n-3}$, $n \geq 3$ and being $P_0 = P_1 = P_2 = 1$ your initial conditions [13–16].

The quaternions were developed by Willian Rowan Hamilton (1805-1865), arose from the attempt to generalize complex numbers in the form $z = a + bi$ in three dimensions [10]. Thus are presented as formal sums of scalars with usual vectors of three-dimensional space, existing four dimensions. Second Halici (2012) [8], a quaternion is a hyper-complex number and is described by:

$$q = a + bi + cj + dk$$

*Corresponding author. Email address: **re.passosm@gmail.com** (Renata Vieira)

where a, b, c are real numbers or scalar and i, j, k the orthogonal part at the base $\mathbb{R}^3$. The quaternionic product being $i^2 = j^2 = k^2 = ijk = -1, ij = k = -ji, jk = i = -kj$ and $ki = j = -ik$.

Being $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ two distinct quaternions. The addition, equality and multiplication scalar operations between them are:

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k.$$

$q_1 = q_2$ only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$. And for $\alpha \in \mathbb{R}$, we have $\alpha q_1 = \alpha a_1 + \alpha b_1 i + \alpha c_1 j + \alpha d_1 k$. The conjugate of the quaternion is denoted by $\overline{q} = a - bi - cj - dk$.

There are also other works, such as [3, 6, 7, 9] that address the quaternions in the scope of numerical sequences, which are also used as a basis for this research.

As for hyperbolic numbers, the set of these numbers $\mathbb{H}$ can be described as:

$$\mathbb{H} = \left\{ z = x + hy | h \notin \mathbb{R}, h^2 = 1, x, y \in \mathbb{R} \right\}.$$

The addition and multiplication of two of these hyperbolic numbers n_1 e n_2,are given by [12]:

$$\begin{aligned} n_1 \pm n_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2) \\ n_1 n_2 &= (x_1 + hy_1)(x_2 + hy_2) = (x_1 x_2) + h(y_1 y_2) + h(x_1 y_2 + x_2 y_1) \end{aligned}$$

In this sense, there are works on hyperbolic numbers and the quaternion sequence, used as a basis for this investigative process [1, 2, 4, 5, 11].

2. The hyperbolic k-Padovan quaternions

The sequence of k-Padovan is defined by $P_{k,n} = P_{k,n-2} + kP_{k,n-3}, n \geq 3, k \geqslant 1$ with initial values $P_{k,0} = P_{k,1} = P_{k,2} = 1$. In turn, we have the characteristic polynomial of this sequence as being $x^3 - x - k = 0$.

Definition 2.1. *The hyperbolic k-Padovan quaternions are given by:*

$$\mathbb{H}P_{k,n} = P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3},$$

where $i^2 = j^2 = k^2 = -1$,$ij = k = -ji$,$jk = i = -kj$,$ki = j = -ik$.

According to the definitions presented, a study is carried out on the operations of addition, subtraction, and multiplication of hyperbolic k-Padovan quaternions.

$$\begin{aligned} \mathbb{H}P_{k,n} \pm \mathbb{H}P_{k,m} = {} & (P_{k,n} \pm P_{k,m}) + i(P_{k,n+1} \pm P_{k,m+1}) + j(P_{k,n+2} \pm P_{k,m+2}) \\ & + k(P_{k,n+3} \pm P_{k,m+3}), \end{aligned}$$

$$\begin{aligned} \mathbb{H}P_{k,n}\mathbb{H}P_{k,m} = {} & (P_{k,n}P_{k,m} + P_{k,n+1}P_{k,m+1} + P_{k,n+2}P_{k,m+2} + P_{k,n+3}P_{k,m+3}) \\ & + i(P_{k,n}P_{k,m+1} + P_{k,n+1}P_{k,m} + P_{k,n+2}P_{k,m+3} - P_{k,n+3}P_{k,m+2}) \\ & + j(P_{k,n}P_{k,m+2} + P_{k,n+2}P_{k,m} - P_{k,n+1}P_{k,m+3} + P_{k,n+3}P_{k,m+1}) \\ & + k(P_{k,n}P_{k,m+3} + P_{k,n+3}P_{k,m} + P_{k,n+1}P_{k,m+2} - P_{k,n+2}P_{k,m+1}) \\ & \neq \mathbb{H}P_{k,m}\mathbb{H}P_{k,n} \end{aligned}$$

The conjugate of the hyperbolic k-Padovan quaternary numbers is represented by:

$$\overline{\mathbb{H}P}_{k,n} = P_{k,n} - iP_{k,n+1} - jP_{k,n+2} - kP_{k,n+3}.$$

Theorem 2.2. *Let $P_{k,n}$ be the nth term of the k-Padovan sequence and $\mathbb{H}P_{k,n}$ the nth term of the quaternionic k-Padovan sequence hyperbolic, for $n \geqslant 1$ the following relations are given:*

$$(i)\mathbb{H}P_{k,n+3} = \mathbb{H}P_{k,n+1} + k\mathbb{H}P_{k,n};$$
$$(ii)\mathbb{H}P_{k,n} - i\mathbb{H}P_{k,n+1} + j\mathbb{H}P_{k,n+2} - k\mathbb{H}P_{k,n+3} = P_{k,n} + P_{k,n+2} + P_{k,n+4} + P_{k,n+6}.$$

Proof. (i) Based on Definition 2.1, we have:

$$\begin{aligned}\mathbb{H}P_{k,n+1} + k\mathbb{H}P_{k,n} &= P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + kP_{k,n+4} \\ &+ k(P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3}) \\ &= (P_{k,n+1} + kP_{k,n}) + i(P_{k,n+2} + kP_{k,n+1}) + j(P_{k,n+3} + kP_{k,n+2}) \\ &+ k(P_{k,n+4} + kP_{k,n+3}) \\ &= P_{k,n+3} + iP_{k,n+4} + jP_{k,n+5} + kP_{k,n+6} \\ &= \mathbb{H}P_{k,n+3}\end{aligned}$$

For (ii), we have:

$$\begin{aligned}\mathbb{H}P_{k,n} - i\mathbb{H}P_{k,n+1} + j\mathbb{H}P_{k,n+2} - k\mathbb{H}P_{k,n+3} &= P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3} \\ &- i(P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + kP_{k,n+4}) \\ &- j(P_{k,n+2} + iP_{k,n+3} + jP_{k,n+4} + kP_{k,n+5}) \\ &- k(P_{k,n+3} + iP_{k,n+4} + jP_{k,n+5} + kP_{k,n+6}) \\ &= P_{k,n} + P_{k,n+2} - kP_{k,n+3} + jP_{k,n+4} + kP_{k,n+3} \\ &+ P_{k,n+4} - iP_{k,n+5} - jP_{k,n+4} + iP_{k,n+5} + P_{k,n+6} \\ &= P_{k,n} + P_{k,n+2} + P_{k,n+4} + P_{k,n+6}\end{aligned}$$

■

Theorem 2.3. *Let $\overline{\mathbb{H}P}_{k,n}$ be the quaternionic conjugate of hyperbolic k-Padovan, then:*

$$\mathbb{H}P_{k,n} + \overline{\mathbb{H}P}_{k,n} = 2P_{k,n}$$

Proof. According to Definition 2.1, we have:

$$\begin{aligned}\mathbb{H}P_{k,n} + \overline{\mathbb{H}P}_{k,n} &= P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + kP_{k,n+3} \\ &+ P_{k,n} - iP_{k,n+1} - jP_{k,n+2} - kP_{k,n+3} \\ &= 2P_{k,n}\end{aligned}$$

■

3. Some properties

Hereinafter, some properties of the hyperbolic quaternion k-Padovan sequence are studied, based on the definitions discussed in the previous section.

Theorem 3.1. *The generating function of the hyperbolic k-Padovan quaternions is given by:*

$$g(\mathbb{H}P_{k,n}, x) = \frac{\mathbb{H}P_{k,0} + \mathbb{H}P_{k,1}x + (\mathbb{H}P_{k,2} - \mathbb{H}P_{k,0})x^2}{1 - x^2 - kx^3}.$$

Proof. Performing the multiplication of the function by x^2, kx^3 in the equations below, we have:

$$g(\mathbb{H}P_{k,n},x)=\sum_{n=0}^{\infty}\mathbb{H}P_{k,n}x^n=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+\mathbb{H}P_{k,2}x^2+\ldots+\mathbb{H}P_{k,n}x^n+\ldots \tag{3.1}$$

$$x^2g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}x^2+\mathbb{H}P_{k,1}x^3+\mathbb{H}P_{k,2}x^4+\ldots+\mathbb{H}P_{k,n-2}x^n+\ldots \tag{3.2}$$

$$kx^3g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}kx^3+\mathbb{H}P_{k,1}kx^4+\mathbb{H}P_{k,2}kx^5+\ldots+\mathbb{H}P_{k,n-3}kx^n+\ldots \tag{3.3}$$

Based on the Equation (3.1-3.2+3.3), we have:

$$(1-x^2-kx^3)g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2+(\mathbb{H}P_{k,3}-\mathbb{H}P_{k,1}-\mathbb{H}P_{k,0})x^3$$
$$+\ldots+(\mathbb{H}P_{k,n}-\mathbb{H}P_{k,n-2}-\mathbb{H}P_{k,n-3})x^n+\ldots$$

Thus:

$$(1-x^2-kx^3)g(\mathbb{H}P_{k,n},x)=\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2$$
$$g(\mathbb{H}P_{k,n},x)=\frac{\mathbb{H}P_{k,0}+\mathbb{H}P_{k,1}x+(\mathbb{H}P_{k,2}-\mathbb{H}P_{k,0})x^2}{1-x^2-kx^3}.$$

■

Theorem 3.2. *For $n\in\mathbb{N}$, the Binet formula of the hyperbolic k-Padovan quaternions is expressed by:*

$$Q_{k,n}^{(n)}=C_1r_1^n+C_2r_2^n+C_3r_3^n,$$

where C_1,C_2,C_3 are the coefficients of the Binet formula of the sequence and r_1,r_2,r_3 the roots of the characteristic polynomial ($x^3-x-k=0$).

Proof. Based on the k-Padovan sequence recurrence formula, its respective defined initial values and its characteristic polynomial whose roots are r_1,r_2,r_3, it is possible to obtain, by solving the linear system of equations, the values of coefficients C_1,C_2,C_3.

The discriminant $\Delta=\frac{(-k)^2}{4}-\frac{1}{27}$, referring to the 3rd degree polynomial, determines how the roots of the polynomial will be. Thus, when $\Delta\neq 0$ all roots will be distinct, concluding that $k^2\neq\frac{64}{27}$. Note also that $r_1r_2r_3=k$, $r_1+r_2+r_3=0$ and that when $k\neq 0$, there will be at least one root equal to zero, there being no Binet formula for this case. ■

Theorem 3.3. *For $n\ geqslant 2$ and $n\ in\ mathbbN$, the matrix form of the hyperbolic k-Padovan quaternions is given by:*

$$\begin{bmatrix}0&1&k\\1&0&0\\0&1&0\end{bmatrix}^n\begin{bmatrix}Q_{k,2}&Q_{k,1}&Q_{k,0}\\Q_{k,1}&Q_{k,0}&Q_{k,-1}\\Q_{k,0}&Q_{k,-1}&Q_{k,-2}\end{bmatrix}=\begin{bmatrix}\mathbb{H}_{k,n+2}&\mathbb{H}_{k,n+1}&\mathbb{H}_{k,n}\\\mathbb{H}_{k,n+1}&\mathbb{H}_{k,n}&\mathbb{H}_{k,n-1}\\\mathbb{H}_{k,n}&\mathbb{H}_{k,n-1}&\mathbb{H}_{k,n-2}\end{bmatrix}.$$

Proof. Through the finite induction principle, for $n=2$, we have:

$$\begin{bmatrix}0&1&k\\1&0&0\\0&1&0\end{bmatrix}^2\begin{bmatrix}\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\\\mathbb{H}_{k,1}&\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}\\\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}&\mathbb{H}_{k,-2}\end{bmatrix}=\begin{bmatrix}1&k&0\\0&1&k\\1&0&0\end{bmatrix}\begin{bmatrix}\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\\\mathbb{H}_{k,1}&\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}\\\mathbb{H}_{k,0}&\mathbb{H}_{k,-1}&\mathbb{H}_{k,-2}\end{bmatrix}$$
$$=\begin{bmatrix}\mathbb{H}_{k,2}+k\mathbb{H}_{k,1}&\mathbb{H}_{k,1}+k\mathbb{H}_{k,0}&\mathbb{H}_{k,0}+k\mathbb{H}_{k,-1}\\\mathbb{H}_{k,1}+k\mathbb{H}_{k,0}&\mathbb{H}_{k,0}+k\mathbb{H}_{k,-1}&\mathbb{H}_{k,-1}+k\mathbb{H}_{k,-2}\\\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\end{bmatrix}$$
$$=\begin{bmatrix}\mathbb{H}_{k,4}&\mathbb{H}_{k,3}&\mathbb{H}_{k,2}\\\mathbb{H}_{k,3}&\mathbb{H}_{k,2}&\mathbb{H}_{k,1}\\\mathbb{H}_{k,2}&\mathbb{H}_{k,1}&\mathbb{H}_{k,0}\end{bmatrix}.$$

Checking the validity for any $n = z, z \in \mathbb{N}$, one has:

$$\begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^z \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} = \begin{bmatrix} \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \\ \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-2} \end{bmatrix}.$$

Therefore, it turns out to be valid for $n = z + 1 = 1 + z$:

$$\begin{aligned} \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{1+z} \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^z \begin{bmatrix} \mathbb{H}_{k,2} & \mathbb{H}_{k,1} & \mathbb{H}_{k,0} \\ \mathbb{H}_{k,1} & \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} \\ \mathbb{H}_{k,0} & \mathbb{H}_{k,-1} & \mathbb{H}_{k,-2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \\ \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{k,z+1} + k\mathbb{H}_{k,z} & \mathbb{H}_{k,z} + k\mathbb{H}_{k,z-1} & \mathbb{H}_{k,z-1} + k\mathbb{H}_{k,z-2} \\ \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{H}_{k,z+3} & \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z+1} \\ \mathbb{H}_{k,z+2} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z} \\ \mathbb{H}_{k,z+1} & \mathbb{H}_{k,z} & \mathbb{H}_{k,z-1} \end{bmatrix}. \end{aligned}$$

■

4. Conclusion

The study allowed for a mathematical analysis of the k-Padovan sequence and its complex form. Thus, the hyperbolic k-Padovan quaternion sequence was introduced, addressing some mathematical properties and theorems. It is noteworthy that for the particular case of $k = 1$, it is possible to notice that we have the hyperbolic quaternionic Padovan sequence.

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