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Local isometry of the generalized helicoidal surfaces family in 4-space

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Abstract. In this paper, we consider the helicoidal surfaces family in four dimensional Euclidean space $\mathbb{E}^4$. We calculate normal pair and the curvatures of the surface family. Moreover, we find the local isometry from helicoidal surface family to the rotational surface family by using Bour's theorem in $\mathbb{E}^4$.

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Contents

1. Introduction

The deformation of parametric surfaces family is determined by

$$\mathrm{x}_\beta(u,v)=\begin{pmatrix}\cos\beta\sin u\sinh v+\sin\beta\cos u\cosh v\\ -\cos\beta\cos u\sinh v+\sin\beta\sin u\cosh v\\ u\cos\beta+v\sin\beta\end{pmatrix}.$$

Here, $u,\beta\in(-\pi,\pi]$, $v\in(-\infty,\infty)$, β is the parameter of deformation. x_β is minimal, i.e., has zero mean curvature. x_0 is the helicoid, $\mathrm{x}_{\pi/2}$ is the catenoid. Therefore, the surfaces are locally isometric, have the same Gauss map.

In addition, helices of x_0 match to parallel circles of $\mathrm{x}_{\pi/2}$. Finally, we meet the classical theorem of the French mathematician Edmond Bour.

Bour's Theorem [1]. *A helicoidal surface is locally isometric to a rotational surface so that helices of the helicoidal surface match to parallel circles of the rotational surface.*

Some other Euclidean and also Lorentz-Minkowski versions of it were studied by [2]-[14].

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Next, we present some fundamental geometric and differential facts of four dimensional Euclidean space. Let

$$\overrightarrow{x} \cdot \overrightarrow{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

be a Euclidean inner product, and let $\mathtt{X}$: $D \subset \mathbb{E}^2 \to \mathbb{E}^4$ be a parametric representation of surface M in Euclidean 4-space $\mathbb{E}^4$. The tangent space of M at a point $\mathbf{p} = \mathtt{X}(u, v)$ is spanned by $\mathtt{X}_u$ and $\mathtt{X}_v$, where $\mathtt{X}_u = \frac{\partial \mathtt{X}}{\partial u}$, $\mathtt{X}_v = \frac{\partial \mathtt{X}}{\partial v}$.

The first fundamental form matrix of M is obtained by

$$\mathbf{I} = \begin{pmatrix} \mathtt{E} & \mathtt{F} \\ \mathtt{F} & \mathtt{G} \end{pmatrix}$$

where

$$\mathtt{E} = \mathtt{X}_u \cdot \mathtt{X}_u, \ \mathtt{F} = \mathtt{X}_u \cdot \mathtt{X}_v, \ \mathtt{G} = \mathtt{X}_v \cdot \ \mathtt{X}_v.$$

We assume the surface M is regular. That is, $W^2 = \det \mathbf{I} = \mathtt{EG} - \mathtt{F}^2 > 0$. Let $\{\eta_1, \eta_2, \zeta_1, \zeta_2\}$ be a orthonormal frame of M where η_1, η_2 are tangent to M, ζ_1, ζ_2 are normal to M. The second fundamental form matrix of M w.r.t. the unit normal vector ζ_i, $i = 1, 2$, is described by

$$\mathbf{II}^i = \begin{pmatrix} \mathtt{L}^i & \mathtt{M}^i \\ \mathtt{M}^i & \mathtt{N}^i \end{pmatrix}$$

where

$$\mathtt{L}^i = \mathtt{X}_{uu} \cdot \zeta_i, \ \mathtt{M}^i = \mathtt{X}_{uv} \cdot \zeta_i, \ \mathtt{N}^i = \mathtt{X}_{vv} \cdot \zeta_i,$$

and $\mathtt{X}_{uu} = \frac{\partial^2 \mathtt{X}}{\partial u^2}$, $\mathtt{X}_{uv} = \frac{\partial^2 \mathtt{X}}{\partial u \partial v}$, $\mathtt{X}_{vv} = \frac{\partial^2 \mathtt{X}}{\partial v^2}$.

We determine by

(a) $H_i = \frac{(\mathtt{E})(\mathtt{N}^i) + (\mathtt{G})(\mathtt{L}^i) - 2(\mathtt{F})(\mathtt{M}^i)}{2W^2}$, the mean curvature of M w.r.t. n_i, $i = 1, 2$,

(b) $\overrightarrow{H} = H_1 n_1 + H_2 n_2$, the mean curvature vector of M,

(c) $\overrightarrow{H} = 0$, the surface M is minimal,

(d) $K = \frac{(\mathtt{L}^1)(\mathtt{N}^1) - (\mathtt{M}^1)^2 + (\mathtt{L}^2)(\mathtt{N}^2) - (\mathtt{M}^2)^2}{W^2} = \frac{\det(\mathbf{II}^1) + \det(\mathbf{II}^2)}{W^2}$, the Gaussian curvature of M, respectively.

An orthonormal tangent frame field $\{\eta_1, \eta_2\}$ of M is choosen by

$$\eta_1 = \frac{1}{\sqrt{\mathtt{E}}} \mathtt{X}_u, \quad \eta_2 = \frac{1}{W\sqrt{\mathtt{E}}} \left(\mathtt{E}\mathtt{X}_v - \mathtt{F}\mathtt{X}_u \right),$$

with its Gauss map

$$\mathcal{G} = \frac{1}{W} \left(\mathtt{X}_u \wedge \mathtt{X}_v \right).$$

In this paper, we generalized the work of The Hieu and Ngoc Thang [14].

2. Generalized helicoidal surfaces family in $\mathbb{E}^4$

A vector (a,b,c,d) of $\mathbb{E}^4$ will be identified with its transpose in the rest of this work.

Let $\gamma : I \subset \mathbb{R} \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, ℓ be a line in Π. A generalized rotational surface family in $\mathbb{E}^4$ is described by rotating a profile curve γ about a line (i.e., axis) ℓ.

When γ rotates about ℓ, it simultaneously matches parallel lines perpendicular to the ℓ, so the displacement speed is proportional to the rotation speed. Therefore, the final surface is named the *generalized helicoidal surface family* with axis ℓ and pitch $a \in \mathbb{R}\backslash\{0\}$.

Parametrization of the profile curve is given by

$$\gamma(u) = (f(u), 0, g(u), h(u)),$$

where $f, g, h : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are the differentiable functions for all $u \in I$. So, in $\mathbb{E}^4$, a generalized helicoidal surface family with pitch $a \in \mathbb{R}\backslash\{0\}$ is defined by

$$\mathcal{H}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) + av \\ h(u) \end{pmatrix}, \tag{2.1}$$

where f, g, h are the differentiable functions, $u, a \in \mathbb{R} \setminus \{0\}$, and $0 \leq v < 2\pi$. When $a = 0$, it is just a rotational surface in $\mathbb{E}^4$.

By taking the first derivatives w.r.t. u and v, respectively, of the generalized helicoidal surfaces family defined by Eq. (2.1), we find the following first quantities of the family

$$\mathtt{E} = f'^2 + g'^2 + h'^2, \quad \mathtt{F} = ag', \quad \mathtt{G} = f^2 + a^2,$$

where $f'^2 = \left(\frac{df}{du}\right)^2$, $g'^2 = \left(\frac{dg}{du}\right)^2$, $h'^2 = \left(\frac{dh}{du}\right)^2$.

We compute two normals of the generalized helicoidal surface family described by Eq. (2.1) as follows

$$\zeta_1 = \frac{1}{T}\begin{pmatrix} h'\cos v \\ h'\sin v \\ 0 \\ -f' \end{pmatrix}, \tag{2.2}$$

$$\zeta_2 = \frac{1}{WT}\begin{pmatrix} -ff'g'\cos v + a\left(f'^2 + h'^2\right)\sin v \\ -a\left(f'^2 + h'^2\right)\cos v - ff'g'\sin v \\ f\left(f'^2 + h'^2\right) \\ -fg'h' \end{pmatrix}, \tag{2.3}$$

respectively. Here, $T = \sqrt{f'^2 + h'^2}$, $W = \sqrt{a^2\left(f'^2 + h'^2\right) + f^2\left(f'^2 + g'^2 + h'^2\right)}$.

Using the second derivatives of the helicoidal surface defined by Eq. (2.1) w.r.t. u and v, respectively,

$$\begin{aligned} \mathcal{H}_{uu} &= (f''\cos v, f''\sin v, g'', h''), \\ \mathcal{H}_{uv} &= (-f'\sin v, f'\cos v, 0, 0), \\ \mathcal{H}_{vv} &= (-f\cos v, -f\sin v, 0, 0), \end{aligned}$$

where $f'' = \frac{\partial^2 f}{\partial u^2}$, $g'' = \frac{\partial^2 g}{\partial u^2}$, $h'' = \frac{\partial^2 h}{\partial u^2}$, and the normals determined by Eq. (2.2) and Eq. (2.3), we have the following second quantities of the generalized helicoidal surfaces family described by Eq. (2.1):

$$\mathtt{L}^1 = \frac{f''h' - f'h''}{T}, \ \mathtt{M}^1 = 0, \ \mathtt{N}^1 = -\frac{fh'}{T},$$

$$\mathtt{L}^2 = \frac{f\left(-\left(f'^2 + h'^2\right)g'' + g'\left(f'f'' + h'h''\right)\right)}{WT}, \ \mathtt{M}^2 = \frac{af'\left(f'^2 + h'^2\right)}{WT}, \ \mathtt{N}^2 = -\frac{f^2f'g'}{WT}.$$

Hence, the mean curvatures H_i $(i = 1, 2)$ and the Gaussian curvature K of the generalized helicoidal surfaces family defined by Eq. (2.1) are given by as follows

$$H_1 = \frac{\left(f^2 + a^2\right) h' f'' - f \left(f'^2 + g'^2\right) h' - f h'^3 - \left(a^2 + f^2\right) f' h''}{2W^2 \sqrt{f'^2 + h'^2}},$$

$$H_2 = \frac{\left\{ \begin{array}{l} f \left(a^2 + f^2\right) \left\{ f' g' f'' - \left(f'^2 + h'^2\right) g'' + g' h' h'' \right\} \\ - \left\{ 2a^2 \left(f'^2 + h'^2\right) + f^2 \left(f'^2 + g'^2 + h'^2\right) \right\} f' g' \end{array} \right\}}{2W^3 \sqrt{f'^2 + h'^2}},$$

$$K = \frac{\left\{ \begin{array}{l} - \left(f^3 f'^2 g'^2 W^2 + f h'^2\right) f'' + \left(f^3 f' g' \left(f'^2 + h'^2\right) W^2\right) g'' \\ - \left(f^3 f' g'^2 h' W^2 - f f' h'\right) h'' - a^2 f'^2 \left(f'^2 + h'^2\right)^2 W^2 \end{array} \right\}}{W^4 \left(f'^2 + h'^2\right)}.$$

3. Bour's theorem on generalized helicoidal-rotational surfaces family in $\mathbb{E}^4$

Next, we generalize the Bour's theorem for the generalized helicoidal-rotational surfaces family in four dimensional Euclidean space.

Theorem 1. *Let $\mathcal{H}$ be the generalized helicoidal surfaces family described by Eq. (2.1), and let $p(u)$, $q(u)$, $u > 0$ are the differentiable functions supplying the equation*

$$p^2 + q^2 = \frac{a^2 + f^2 f'^2 + \left(f^2 + a^2\right) h'^2}{f^2}. \tag{3.1}$$

Therefore, the generalized helicoidal surface family $\mathcal{H}$ defined by Eq. (2.1) is locally isometric to the following generalized rotational surfaces family

$$\mathcal{R}(u,v) = \begin{pmatrix} \sqrt{f^2 + a^2} \cos \left(v + \int \frac{a g'}{f^2 + a^2} du\right) \\ \sqrt{f^2 + a^2} \sin \left(v + \int \frac{a g'}{f^2 + a^2} du\right) \\ \int \frac{f\, p(u)}{\sqrt{f^2 + a^2}} du \\ \int \frac{f\, q(u)}{\sqrt{f^2 + a^2}} du \end{pmatrix} \tag{3.2}$$

so that helices on the generalized helicoidal surface correspond to parallel circles on the generalized rotational surfaces.

Proof. The arc lenght element of the generalized helicoidal surface given by Eq. (2.1) is described by as follows

$$ds^2 = \left(f'^2 + g'^2 + h'^2\right) du^2 + 2 a g' du dv + \left(a^2 + f^2\right) dv^2.$$

Setting $\overline{u} = u$, $\overline{v} = v + \int \frac{a g'}{f^2 + a^2} du$, the generalized helicoidal surface determined by Eq. (2.1) transforms to $\mathcal{H}(\overline{u}, \overline{v})$. Considering the new parameters of the surface, its arc lenght element reduces to

$$ds^2 = \left(\left(f'^2 + h'^2\right) + \frac{f^2 g'^2}{a^2 + f^2}\right) d\overline{u}^2 + \left(a^2 + f^2\right) d\overline{v}^2.$$

On the other side, in $\mathbb{E}^4$, the following generalized rotational surfaces family

$$\mathcal{R}(\mathfrak{s}, \mathfrak{t}) = \begin{pmatrix} \mathfrak{f}(\mathfrak{s}) \cos \mathfrak{t} \\ \mathfrak{f}(\mathfrak{s}) \sin \mathfrak{t} \\ \mathfrak{g}(\mathfrak{s}) \\ \mathfrak{h}(\mathfrak{s}) \end{pmatrix}$$

has the following arc lenght element

$$ds^2 = \left(\mathfrak{f}'^2 + \mathfrak{g}'^2 + \mathfrak{h}'^2\right) d\mathfrak{s}^2 + \mathfrak{f}^2 d\mathfrak{t}^2. \tag{3.3}$$

Again setting $\mathfrak{f} = \sqrt{a^2 + f^2}$, $p(u) = \mathfrak{g}'$, $q(u) = \mathfrak{h}'$, we get the following functions

$$\mathfrak{g} = \int \frac{f\, p(u)}{\sqrt{a^2 + f^2}} du, \quad \mathfrak{h} = \int \frac{f\, q(u)}{\sqrt{a^2 + f^2}} du.$$

Hence, differential Eq. (3.1) determines that the generalized helicoidal surfaces family given by Eq. (2.1) is locally isometric to the generalized rotational surfaces family determined by Eq. (3.2).

The helices of $\mathcal{H}$ are defined by $u = u_0$; where u_0 is a constant, those are match to the curves of $\mathcal{R}$ determined by $\mathfrak{f} = \sqrt{u_0^2 + a^2}$; that is, those are the circles of the plane $\{x_3 = \mathfrak{g}(\mathfrak{s}),\ x_4 = \mathfrak{h}(\mathfrak{s})\}$.

We now taking the isometric surfaces in Theorem 1, consider the following.

Theorem 2. *Let $\mathcal{H}$ and $\mathcal{R}$ be the generalized surfaces family related by Theorem 1. When the family have the same Gauss map, those are hyperplanar, minimal.*

Proof. Let $\{k_1, k_2, k_3, k_4\}$ be the canonical basis in $\mathbb{E}^4$ and denote $k_{ij} = k_i \wedge k_j$, $i, j = 1, 2, 3, 4$, $i < j$. So, the Gauss map of the generalized helicoidal surface (2.1) is

$$\mathcal{G}_{\mathcal{H}} = \frac{1}{W} \left\{ \begin{array}{c} f f' k_{12} \\ +\left(a f' \cos v + f g' \sin v\right) k_{13} \\ +f h' \sin v k_{14} \\ +\left(a f' \sin v - f g' \cos v\right) k_{23} \\ -f h' \cos v k_{24} \\ -a h' k_{34} \end{array} \right\}, \tag{3.4}$$

and also the Gauss map of the generalized rotational surface (3.2) is as follows

$$\mathcal{G}_{\mathcal{R}} = \frac{1}{W} \left\{ \begin{array}{c} f f' k_{12} \\ +f p \sin\left(v + \int \frac{a g'}{f^2 + a^2} du\right) k_{13} \\ +f q \sin\left(v + \int \frac{a g'}{f^2 + a^2} du\right) k_{14} \\ -f p \cos\left(v + \int \frac{a g'}{f^2 + a^2} du\right) k_{23} \\ -f q \cos\left(v + \int \frac{a g'}{f^2 + a^2} du\right) k_{24} \end{array} \right\}, \tag{3.5}$$

where

$$W = \sqrt{a^2\left(f'^2 + h'^2\right) + f^2\left(f'^2 + g'^2 + h'^2\right)}.$$

When $\mathcal{G}_{\mathcal{H}}$ is equal to $\mathcal{G}_{\mathcal{R}}$, identically, Eq. (3.4) and Eq. (3.5) give rise to the following

$$a f' \cos v + f g' \sin v = f p \sin\left(v_{\mathcal{R}}\right), \tag{3.6}$$

$$a f' \sin v - f g' \cos v = -f p \cos\left(v_{\mathcal{R}}\right), \tag{3.7}$$

$$f h' \sin v = f q \sin\left(v_{\mathcal{R}}\right), \tag{3.8}$$

$$-f h' \cos v = -f q \cos\left(v_{\mathcal{R}}\right), \tag{3.9}$$

$$-a h' = 0, \tag{3.10}$$

where $v_{\mathcal{R}} = v + \int \frac{a g'}{f^2 + a^2} du$. Using Eqs. $(3.8) - (3.10)$, we have

$$h' = 0 \text{ and } q = 0.$$

That is, generalized surfaces determined by Eq. (2.1) and Eq. (3.2) are hyperplanar.

Now, we prove surfaces given by Eq. (2.1) and Eq. (3.2) are minimal. For this, since $q=0$, then $p\neq 0$ by using $((3.6)\,.\cos v+(3.7)\,.\sin v)$, it gives

$$af'=fp\sin\left(\int\frac{ag'}{f^2+a^2}du\right).$$

Also, $((3.6)\,.\sin v-(3.7)\,.\cos v)$ reduces to

$$fg'=fp\cos\left(\int\frac{ag'}{f^2+a^2}du\right).$$

Hence, we have

$$arc\cot\left(\frac{fg'}{af'}\right)=\int\frac{ag'}{f^2+a^2}du.$$

Derivativing the last equation w.r.t. u, we obtain the following

$$\left(f^2+a^2\right)\left(f'^2g'+ff'g''-ff''g'\right)+\left(a^2f'^2+f^2g'^2\right)g'=0. \tag{3.11}$$

The mean curvatures of the generalized helicoidal surfaces family given by Eq. (2.1) w.r.t. the following normals

$$\zeta_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ \zeta_2=\frac{1}{\sqrt{\left(a^2+f^2\right)f'^2+f^2g'^2}}\begin{pmatrix}-fg'\cos v+af'\sin v\\-fg'\sin v-af'\cos v\\ff'\\0\end{pmatrix}$$

are described by, respectively,

$$\begin{aligned}H_1&=0,\\H_2&=\frac{\left(f^2+a^2\right)\left(fg'f''-ff'g''-f'^2g'\right)-g'\left(a^2f'^2+f^2g'^2\right)}{2\left(\left(a^2+f^2\right)f'^2+f^2g'^2\right)^{3/2}},\end{aligned}$$

And also, the mean curvatures of the generalized rotational surfaces family defined by Eq. (3.2) w.r.t. the following normals

$$\zeta_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ \zeta_2=\frac{1}{\sqrt{\left(a^2+f^2\right)f'^2+f^2g'^2}}\begin{pmatrix}-\sqrt{f^2g'^2+a^2f'^2}\cos\left(v+\int\frac{ag'}{f^2+a^2}du\right)\\-\sqrt{f^2g'^2+a^2f'^2}\sin\left(v+\int\frac{ag'}{f^2+a^2}du\right)\\ff'\\0\end{pmatrix}$$

are determined by, respectively,

$$\begin{aligned}H_1&=0,\\H_2&=\frac{f^2g'\left[\left(f^2+a^2\right)\left(fg'f''-ff'g''-f'^2g'\right)-g'\left(a^2f'^2+f^2g'^2\right)\right]}{2\sqrt{f^2+a^2}\sqrt{f^2g'^2+a^2f'^2}\left(\left(a^2+f^2\right)f'^2+f^2g'^2\right)^{3/2}}.\end{aligned}$$

From Eq. (3.11), the helicoidal-rotational surfaces family have the mean curvatures $H_2=0$. Finally, the generalized helicoidal surface family determined by Eq. (2.1) and the generalized rotational surfaces family described by Eq. (3.2) are minimal. That is, $\overrightarrow{H}=0$.

Theorem 3. *Let the generalized helicoidal surfaces family defined by Eq.* (2.1), *and the generalized rotational surfaces family given by Eq.* (3.2) *having the same Gauss map be the locally isometric surfaces family related by Theorem 1. Therefore, the parametrizations of the family are described by*

$$\mathcal{H}(u,v)=\begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u)+av \\ c \end{pmatrix},$$

$$\mathcal{R}(u,v)=\begin{pmatrix} \sqrt{f^2+a^2}\cos\left(v+\int \frac{ag'}{f^2+a^2}du\right) \\ \sqrt{f^2+a^2}\sin\left(v+\int \frac{ag'}{f^2+a^2}du\right) \\ b\ \arg\cosh\left(\frac{\sqrt{f^2+a^2}}{b}\right) \\ d \end{pmatrix},$$

respectively. Here,

$$g(u)=\sqrt{b^2-a^2}\ln\sqrt{\frac{\sqrt{f^2+a^2}+\sqrt{f^2+a^2-b^2}}{\sqrt{f^2+a^2}-\sqrt{f^2+a^2-b^2}}}-a\arctan\left(\sqrt{\frac{(b^2-a^2)(f^2+a^2)}{a^2(f^2+a^2-b^2)}}\right),$$

and $a,b,c,d\in\mathbb{R}$, $b\geq a$, $f>\sqrt{b^2-a^2}$.

Proof. Generalized surfaces $\mathcal{H}$ and $\mathcal{R}$ are the hyperplanar from Theorem 2. Assume $\mathcal{H}$ covered by the hyperplane $\mathfrak{f}(\mathfrak{s})=c$, and also $\mathcal{R}$ covered by the hyperplane $\mathfrak{f}(\mathfrak{s})=d$. Since $\mathcal{R}$ is minimal, it is just a catenoid. Thus, $\mathfrak{g}(\mathfrak{s})=b\arg\cosh\left(\frac{\mathfrak{s}}{b}\right)$, where $b\neq 0$. Therefore,

$$b\ \arg\cosh\left(\frac{\sqrt{f^2+a^2}}{b}\right)=\int\sqrt{\frac{f^2g'^2+a^2f'^2}{f^2+a^2}}du.$$

Then, we get

$$g'=\frac{\sqrt{b^2-a^2}\sqrt{f^2+a^2}}{f\sqrt{f^2+a^2-b^2}}. \tag{3.12}$$

Finally, after some computations, we obtain

$$g'=\sqrt{b^2-a^2}\ln\sqrt{\frac{w+1}{w-1}}-a\arctan\left(\frac{\sqrt{b^2-a^2}}{a}w\right),$$

where $w=\sqrt{\frac{f^2+a^2}{f^2+a^2-b^2}}>0$.

4. Conclusion

Considering the findings in the previous section, we obtain the following results.

Corollary 1. *When* $g'=0$, *generalized helicoidal surfaces family* $\mathcal{H}$ *describe the helicoid. The mean curvature of the generalized rotational surfaces family* $\mathcal{R}$ *is zero. That is, the generalized rotational surfaces family is transform to the catenoid. By using Eq.* (3.12), *the pitch* a *is equals to* b.

Example 1. *Taking $f(u) = u$, $c = d = 0$, $b = 2$, $a = 1$ in Theorem 3, the other function is described by*

$$g(u) = \sqrt{3}\ln\left(\sqrt{\frac{\sqrt{u^2+1}+\sqrt{u^2-3}}{\sqrt{u^2+1}-\sqrt{u^2-3}}}\right) - \arctan\left(\sqrt{\frac{3\left(u^2+1\right)}{u^2-3}}\right).$$

Then, we have the projection of the isometric helicoidal-rotational surfaces from dimension four to three. See Figure 1 for the graphics of the helicoidal surface, and also see Figure 2 for the rotational surface.

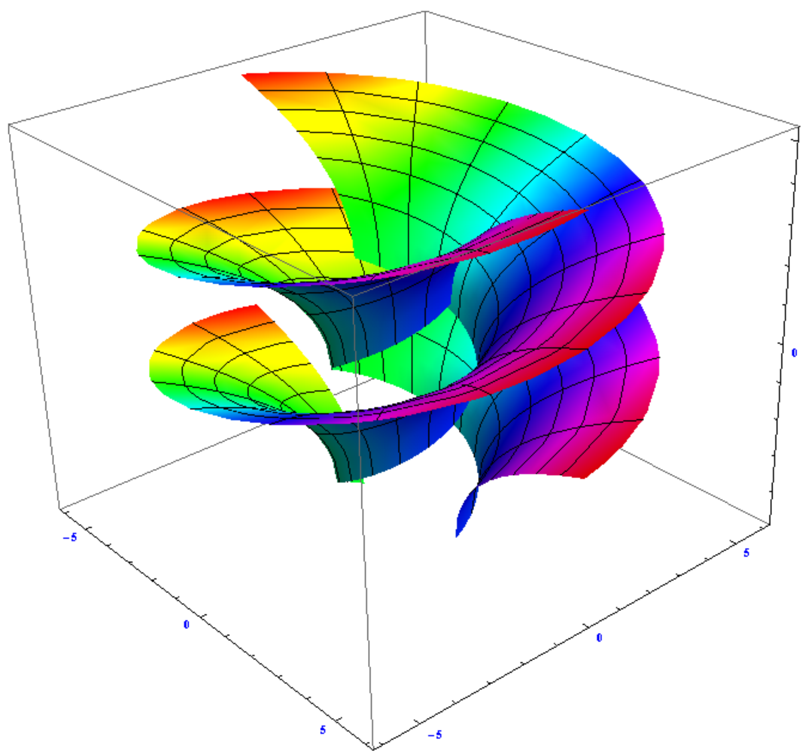

Figure 1: Helicoidal surface

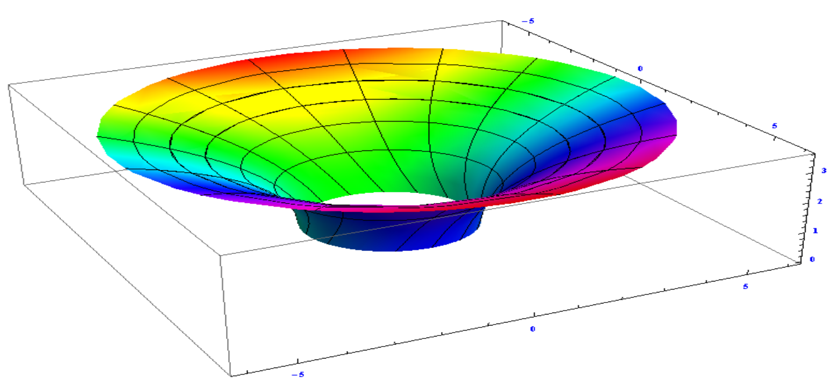

Figure 2: Rotational surface

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