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A quasistatic elastic-viscoplastic contact problem with wear and frictionless

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Abstract. We consider here a frictionless contact problem for elastic-viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. The damage the elastic deformations of the material is modeled by an internal variable of the body called the damage field. The problem formula is given as a system that includes a variational equation with respect to the displacement field, and a variational inequality of the parabolic type with respect to the damage field. We prove a weak solution existence and uniqueness theorem relating to the problem. The methods utilised are grounded in the concept of monotonic operators, followed by fixed-point arguments.
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Contents

1. Introduction

Contact-related problems, whether involving friction or not, between deformable bodies or between a rigid body and a deformable one, are frequently encountered in both industrial settings and everyday experiences. Considering the importance and the multitude of these phenomena, vast studies have been undertaken, also the literature concerning contact mechanics is vast and addresses as many different subjects as are modeling, mathematical analysis or approximation numerical contact problems, see the works [1, 2, 10, 11].

This paper explores an investigation concerning boundary conditions that mirror real-world phenomena like contact, material wear and damage. In our study, we adopt an elastic-viscoplastic constitutive law to describe the behavior of the material.

To illustrate the procedure of deformation of an elastic-viscoplastic body with wear when it contacts with a rigid body foundation, been touched on many quasi-static elastic-viscoplastic frictional Contact problems involving wear have been introduced and investigated under various conditions. For further details, we direct the reader to [5, 6] and the cited references therein.

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Chen et al.[4] were among the first to provide error estimates for fully discrete schemes designed to solve quasi-static viscoplastic frictional contact problems with wear. Gasinski et al. [7] introduced a mathematical model to describe quasi-static frictional contact with wear between a thermo-viscoelastic body and a moving foundation. In a recent development, Jureczka and Ochal [9] conducted numerical analysis and simulations for the quasi-static elastic frictional contact problem that accounts for wear.

There are other real phenomena which are very important. Such as material damage and body adhesion. The consideration of damage holds fundamental significance in the field of design engineering since it has a direct impact on the useful lifespan of the designed structure or component. There exists a substantial body of engineering literature devoted to this subject. Mathematical models that incorporate the influence of internal material damage on the contact process have been thoroughly examined. In [8], novel comprehensive damage models have been derived based on the principle of virtual power. Further mathematical analyses of one-dimensional problems related to this topic can be found in [3]. the material damage is described by capacity damage. The damage function α varies between 0 and 1. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 \prec \alpha \prec 1$ the damage is partial. This work is a continuation in this line of research to the mathematical study of a frictionlessly contact problem for Viscoplastic materials, in a quasi-static process. The contact with a rigid base is modeled without friction with condition of wear and damage. Our focus is to establish the existence of a unique weak solution for the abstract problem with regularized boundary conditions. The structure of the remainder of this paper is as follows: In Section 2, we provide an inventory of notations and outline the assumptions concerning the problem data. Additionally, we state our primary result regarding the existence and uniqueness of solutions. In Section 3, we delve into the proof of the theorem, where we consider the existence and uniqueness of the solution, utilizing arguments derived from the theory of monotonic operators and the Banach fixed-point theorem. In Section 4, we present an illustrative example that demonstrates the practical application of the abstract result.

Problem $\mathcal{P}$

Find the displacement field $\mathbf{u} : [0, T] \to V$, the stress field $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$, the damage field $\alpha : [0, T] \to \mathbb{R}$.

$$\begin{aligned}(A\dot{\mathbf{u}}(t), \mathbf{v})_V + (B\mathbf{u}(t), \mathbf{v})_V + \left(\int_0^t F(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(s), \alpha(s))\, ds, \mathbf{v}\right)_{\mathcal{H}} \\ = (\boldsymbol{f}(t), \mathbf{v})_V \quad a.e.\ t \in (0, T),\end{aligned} \tag{1.1}$$

$$\begin{aligned}&(\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\quad \geq (S(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(t), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}, \xi \in K, \text{a.e } t \in (0, T),\end{aligned} \tag{1.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0. \tag{1.3}$$

We have three spaces denoted as V, $\mathcal{H}$, and K. These spaces correspond to admissible displacements, stress, and damage, and they are all Hilbert spaces. Notably, K is a nonempty, closed, and convex set within the space V. It is defined as follows:

$$K = \{\zeta \in V \mid \quad 0 \leq \zeta(x) \leq 1 \text{ a.e. } x \in \Omega\}.$$

The operators A, B, and F are associated with the constitutive law governing an elastic-viscoplastic material with damage. The functional S is determined by the source function of the damage and the friction occurring on part Γ_3. The data $\boldsymbol{f}$ relates to both traction forces and body forces. The functions $\mathbf{u}_0$ and α_0 represent the initial data for displacement and damage, respectively. We denote the displacement field as $\mathbf{u}$ and the stress tensor field as $\boldsymbol{\sigma}$. The constitutive law applied here pertains to an elastic-viscoplastic material with damage. The interval $[0, T]$ signifies the time span of observation. A dot above $\mathbf{u}$ and α indicates the derivative of displacement $\mathbf{u}$ and the derivative of damage α with respect to the variable t.

2. Preliminaries and notion

In this section, we introduce important tools for our main results. Specifically, we denote:

$\mathbb{S}^d$ as the space comprising second-order symmetric tensors defined on $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3$), and with a smooth boundary $\partial\Omega = \Gamma$. We designate Γ_3 as the boundary contact .

We define $\boldsymbol{\nu} = (\nu_i)$ as the unit outward normal vector, and $x \in \overline{\Omega} = \Omega \cup \partial\Omega$ represents the position vector. It's worth noting that unless specified otherwise, the indices i, j range from 1 to d, and we apply the summation convention to repeated indices. For the sake of simplicity, we do not explicitly indicate the variables' dependence on x.

The inner products and norms for $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted as follows:

$$\begin{aligned} &\mathbf{u} \cdot \mathbf{w} = u_i w_i \quad \|\mathbf{w}\|_{\mathbb{R}^d} = (\mathbf{w}, \mathbf{w})^{1/2} \text{ for all } \mathbf{u} = (u_i)\,, \mathbf{w} = (w_i) \in \mathbb{R}^d, \\ &\boldsymbol{\sigma}.\boldsymbol{\vartheta} = \sigma_{ij}\vartheta_{ij} \quad \|\boldsymbol{\vartheta}\|_{\mathbb{S}^d} = (\boldsymbol{\vartheta}, \boldsymbol{\vartheta})^{1/2} \text{ for all } \boldsymbol{\sigma} = (\sigma_{ij})\,, \boldsymbol{\vartheta} = (\vartheta_{ij}) \in \mathbb{S}^d, \end{aligned}$$

We denote the following quantities:

$\mathbf{u} = (u_i)$ represents the displacement vector.

$\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor.

$\varepsilon(\mathbf{u}) = (\varepsilon(\mathbf{u}_{ij}))$ represents the linear strain tensor.

Furthermore, we use the following notation for components of displacement $\mathbf{u}$ on Γ:

Normal component: $u_\nu = u.\nu$

Tangential component: $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$

Similar notation is applied to $\dot{u_\nu}$ and $\dot{\mathbf{u}}_\tau$, which represent the normal and tangential velocities on the boundary, respectively.

Regarding the stress field $\boldsymbol{\sigma}$ on the boundary, we define its components as:

Normal component: $\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu}$

Tangential component: $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$

We use the following notations

$$\begin{aligned} &H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\right\}, \quad H_1 = \{\mathbf{u} = (u_i) \mid \varepsilon(\mathbf{u}) \in \mathcal{H}\}, \\ &\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}. \end{aligned}$$

The deformation operator ε and the divergence operator Div are defined as follows:

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))\,, \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})\,.$$

The spaces H, H_1, $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces equipped with the canonical inner products defined as follows:

$$\begin{aligned} &(\mathbf{u}, \mathbf{w})_H = \int u_i w_i dx, \quad \forall \mathbf{u}, \mathbf{w} \in H, \\ &(\mathbf{u}, \mathbf{w})_{H_1} = (\mathbf{u}, \mathbf{w})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \forall \mathbf{u}, \mathbf{w} \in H_1, \\ &(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} = \int \sigma_{ij}\vartheta_{ij} dx, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H}, \\ &(\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\vartheta})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\vartheta})_H, \ \forall \boldsymbol{\sigma}, \boldsymbol{\vartheta} \in \mathcal{H}_1. \end{aligned}$$

The associated norm in the space H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$, is denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$, respectively.

When $\boldsymbol{\sigma}$ is a regular function. The following Green-type formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{w}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \mathbf{w})_H = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{w} da \quad \forall \mathbf{w} \in H_1. \tag{2.1}$$

For the displacement field, we necessitate the closed subspace of H_1 defined as

$$V = \{\mathbf{w} \in H_1 \mid \mathbf{w} = \mathbf{0},\ on\ \Gamma_1\}.$$

Given that $meas(\Gamma_1) > 0$, Korn's inequality is satisfied, and there exists a positive constant C_k, which solely depends on Ω and Γ_1, such that

$$\|\varepsilon(\mathbf{w})\|_{\mathcal{H}} \geq C_k \|\mathbf{w}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{w} \in V.$$

We define inner product on V by

$$(\mathbf{u}, \mathbf{w})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \quad \|\mathbf{w}\|_V = \|\varepsilon(\mathbf{w})\|_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{w} \in V, \tag{2.2}$$

and let $\|.\|_V$ be the associated norm. Consequently, the norms $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent on V, and as a result, $(V, (,)_V)$ forms a real Hilbert space. Furthermore, in accordance with the Sobolev trace theorem, there exists a constant $\tilde{C}_0$, which relies solely on Ω, Γ_1, and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V. \tag{2.3}$$

We recall some spaces $W^{k,p}(0,T;V), H^k(0,T;V)$, $C(0;T;V)$ and $C^1(0;T;V)$ for a Banach space V equipped with the norm $\|.\|_V$ for $1 < p < +\infty$ and $k \geq 1$. Let $W^{k,p}(0,T;V)$ be the space of all functions from $[0,T]$ to V with the norm

$$\|\omega\|_{W^{k,p}(0,T;V)} = \begin{cases} \left(\displaystyle\int_0^T \sum_{1 \leq l \leq k} \left\| \partial_t^l \omega \right\|_V^p dt \right)^{1/p}, & \text{if } 1 \leq p < +\infty \\ \max_{0 \leq l \leq k_0 \leq t \leq T} \sup_t \left\| \partial_t^l \omega \right\|_V, & \text{if } p = +\infty. \end{cases}$$

When $p = 2$ or $k = 0$, $W^{k,2}([0,T];V)$ is written as $H^k([0,T];V)$ or $L^p([0,T];V)$, respectively. We denote by $C([0,T];V)$ the space of continuous functions from $[0,T]$ to V, and by $C^1(0,T;V)$ the space of continuously differentiable functions from $(0,T)$ to V. These spaces are equipped with the following norms:

$$\|\omega\|_{C([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V.$$

$$\|\omega\|_{C^1([0,T];V)} = \max_{t \in [0,T]} \|\omega(t)\|_V + \max_{t \in [0,T]} \|\dot{\omega}(t)\|_V.$$

Clearly, $C([0,T];V)$, $W^{k,p}([0,T];V)$ and $H^k([0,T];V)$ are all Banach spaces when V is a Banach space.

In order to solve Problem $\mathcal{P}$, we impose the following assumptions.

We consider operators $A, B : V \to V$, $F : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to V$, the damage source function $S : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to \mathbb{R}$, and two initial values $u_0 \in V$ and $\alpha_0 \in K$. These operators and values satisfy the following properties

There exists a constant $M_A \succ 0$ such that

$$(A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \geq M_A \|\mathbf{v}_1 - \mathbf{v}_2\|^2, \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.4}$$

There exists a constant $L_A \succ 0$ such that

$$\|A\mathbf{v}_1 - A\mathbf{v}_2\|_{V'} \leq L_A \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.5}$$

There exists a constant $L_B \succ 0$ such that

$$\|B\mathbf{v}_1 - B\mathbf{v}_2\|_V \leq L_B \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{2.6}$$

The $\boldsymbol{f}$ function satisfies:

$$\boldsymbol{f} \in L^2\left(0, T; V\right). \tag{2.7}$$

There exists a constant $L_F \succ 0$ such that

$$\|F(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - F(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq L_F (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.8}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \zeta_i \in H^1(\Omega),\ i = 1, 2.$

There exists $M_S \succ 0$ such that

$$\|S(\boldsymbol{\sigma}_1, \mathbf{u}_1, \zeta_1) - S(\boldsymbol{\sigma}_2, \mathbf{u}_2, \zeta_2)\| \leq M_S (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\zeta_1 - \zeta_2\|), \tag{2.9}$$

for all $\boldsymbol{\sigma}_i \in \mathcal{H},\ \mathbf{u}_i \in V,\ \forall \zeta_i \in H^1(\Omega),\ i = 1, 2.$

Now let problem $\mathcal{P}_1$ as it follows

Problem $\mathcal{P}_1$

Find $\mathbf{u} \in C^1(0, T; V)$ such that

$$\begin{cases} A\mathbf{u}(t) = \mathbf{f}, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.10}$$

Theorem 2.1. *If conditions (2.4),(2.5) and (2.7) are satisfied Then there exists $\mathbf{u} \in C^1(0, T; V)$ solution to the problem $\mathcal{P}_1$ satisfying*

$$\mathbf{u} \in H^1(0, T; V) \cap C^1(0, T; H). \tag{2.11}$$

The previous result is a special case of the Minty-Browder Theorem.

Problem $\mathcal{P}_2$

Find $\alpha(t) \in K$ such that

$$(\dot{\alpha}(t), \rho - \alpha(t))_{V' \times V} + a(\dot{\alpha}(t), \rho - \alpha(t)) \geq (S(t), \rho - \alpha(t))_{L^2(\Omega)},\ \forall \rho \in K, \tag{2.12}$$

$$\alpha(0) = \alpha_0. \tag{2.13}$$

We consider two real Hilbert spaces, denoted as V and H. It is important to note that V is densely embedded in H, and this injection map is continuous. Furthermore, we identify the space H with both its own dual and as a subspace of the dual space V' of V. In other words, we express this relationship as $V \subset H \subset V'$, and this set of inclusions is what defines a Gelfand triple.

The following is a well-established result for parabolic variational inequalities, and you can find it in standard references such as [12].

Theorem 2.2. *Consider a Gelfand triple $V \subset H \subset V'$, where K is a nonempty, closed, and convex set in V. Assume the existence of a continuous and symmetric bilinear form $a(.,.) : V \times V \to \mathbb{R}$ satisfying the following inequality for constants λ and γ:*

$$a(\alpha, \alpha) + \gamma\|\alpha\|_H^2 \geq \lambda\|\alpha\|_V^2, \quad \forall \alpha \in V.$$

Under these conditions, for any initial value $\alpha_0 \in K$ and source function $S \in L^2(0, T; H)$, there exists a unique function $\alpha \in H^1(0, T; H) \cap L^2(0, T; V)$ such that $\alpha(0) = \alpha_0$ and $\alpha(t) \in K$ for all $t \in [0, T]$. This α is the unique solution to Problem $\mathcal{P}_2$.

The next section is dedicated to investigating the existence of a unique solution to Problem $\mathcal{P}$.

3. Proof of the main result

Theorem 3.1. *Under the assumptions (2.4)-(2.9), there exists a unique solution of the problem $\mathcal{P}$, Moreover the solution satisfies:*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \tag{3.1}$$
$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;H), \tag{3.2}$$
$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.3}$$

The proof of Theorem 3.1 is conducted through several sequential steps and relies on the subsequent abstract result concerning evolutionary variational inequalities.

Suppose we have $\eta \in L^2(0,T;V)$, and let's consider the following problem

Problem $\mathcal{P}_\eta$

Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$, such that

$$\begin{cases} (A\dot{\mathbf{u}}_\eta(t), \mathbf{v})_V + (\boldsymbol{\eta}(t), \mathbf{v})_V = (\boldsymbol{f}, \mathbf{v})_V, \\ \qquad \text{a.e. } t \in (0,T), \quad \forall \mathbf{v} \in V, \\ \qquad \mathbf{u}_\eta(0) = \mathbf{u}_0. \end{cases} \tag{3.4}$$

Here is the given result concerning $\mathcal{P}_\eta$.

Lemma 3.2. *A unique solution $\mathbf{u}_\eta \in C^1(0,T;V)$ to the problem $\mathcal{P}_\eta$ exists, and it satisfies the condition (3.1).*

Proof. We apply Theorem 2.1, The Riesz representation theorem allows us to define $\boldsymbol{f}_\eta : [0,T] \to V$, by $\left(\boldsymbol{f}_\eta(t), \mathbf{v}\right)_V = (f(t) - \boldsymbol{\eta}(t), \mathbf{v})_V$. Using hypotheses (2.4)-(2.7), and $\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_\eta(s)ds, \quad \forall t \in (0,T)$, we directly find the result. ■

Subsequently, introduce $\theta \in L^2(0,T;L^2(\Omega))$, and let's examine the following problem

Problem $\mathcal{P}_\theta$

Find the damage field $\alpha_\theta : [0,T] \to \mathbb{R}$,

$$\begin{aligned} \alpha_\theta(t) \in K, (\dot{\alpha}_\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)} + a\left(\alpha_\theta(t), \rho - \alpha_\theta(t)\right) \\ \geq (\theta(t), \rho - \alpha_\theta(t))_{L^2(\Omega)}, \forall \rho \in K, \text{ a.e.t } \in (0,T), \end{aligned} \tag{3.5}$$

$$\alpha_\theta(0) = \alpha_0. \tag{3.6}$$

Lemma 3.3. *problem $\mathcal{P}_\theta$ has a unique solution α_θ such that*

$$\alpha_\theta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{3.7}$$

For the proof, we apply Theorem 2.2.

Finally, in the concluding step, formulate the subsequent Cauchy problem for the stress field

Problem $\mathcal{P}_{\eta,\theta}$

Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} : (0,T) \to \mathcal{H}$, solution of the problem

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \text{ a.e.t } \in (0,T). \tag{3.8}$$

Lemma 3.4. *The problem $\mathcal{P}_{\eta,\theta}$ has a unique solution. Additionally, if $\mathbf{u}_{\eta_i}$, α_{θ_i}, and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$ represent the solutions to problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, and $\mathcal{P}_{\eta,\theta}$ for $i = 1,2$, then there exists a positive constant C such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|_{\mathcal{H}}^2 \leq & C\left(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 + \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V^2 \, ds\right. \\ & \left. + \int_0^t \|\alpha_{\theta_1}(s) - \alpha_{\theta_2}(s)\|_{L^2(\Omega)}^2 \, ds\right).\end{aligned} \tag{3.9}$$

Proof. Consider the mapping $\sum\limits_{\eta,\theta} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined as

$$\sum_{\eta,\theta} \boldsymbol{\sigma}_{\eta,\theta}(t) = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}_\eta(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds. \tag{3.10}$$

let $\boldsymbol{\sigma}_i \in L^2(0,T;\mathcal{H})$, $\quad i = 1,2$ and $t_1 \in (0,T)$, we use the assumption (2.8) and the HÖlder inequality we find

$$\left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 \leq L_F^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds. \tag{3.11}$$

We have more

$$\begin{aligned}&\left\|\sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1)\right) - \sum_{\eta,\theta}\left(\sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right)\right\|_{\mathcal{H}}^2 \\ &\qquad\qquad \leq L_F^2 T \int_0^{t_1} \left\|\sum_{\eta,\theta} \boldsymbol{\sigma}_1(t_1) - \sum_{\eta,\theta} \boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 dt_2 \\ &\qquad\qquad \leq L_F^4 T^2 \int_0^{t_1} \int_0^{t_2} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds dt_2.\end{aligned}$$

By extending the inequality through recurrence, we deduce that for all $t_1, t_2, ..., t_n \in (0,T)$,

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1(t_n) - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2(t_n)\right\|_{\mathcal{H}}^2 \leq L_F^{2n} T^n \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_n} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds dt_n \ldots dt_2.$$

Thus, we can deduce by integrating with respect to $(0,T)$ the following inequality

$$\left\|\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_1 - \sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_2\right\|_{\mathcal{H}}^2 \leq \frac{L_F^{2n} T^{2n}}{n!} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{H}}^2. \tag{3.12}$$

Then from (3.12), for n sufficiently large, the operator $\sum_{\eta,\theta}^{(n)}$, is a contraction on space $L^2(0,T;\mathcal{H})$ and according to the Banach fixed point theorem, there is a single element $\boldsymbol{\sigma}_{\eta,\theta} \in L^2(0,T;\mathcal{H})$ such that

$\sum_{\eta,\theta}^{(n)} \boldsymbol{\sigma}_{\eta,\theta} = \boldsymbol{\sigma}_{\eta,\theta}$, which represents the unique solution of problem $\mathcal{P}_{\eta,\theta}$. Moreover, if $\mathbf{u}_{\eta_i}$, α_{θ_i} and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$, represents the solutions of problem $\mathcal{P}_{\eta_i}$,$\mathcal{P}_{\theta_i}$ and $\mathcal{P}_{\eta_i,\theta_i}$ respectively. For $i = 1, 2.$ designate $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i, \alpha_{\theta_i} = \alpha_i$.

We have

$$\boldsymbol{\sigma}_i(t) = B\mathbf{u}_i(t) + \int_0^t F\left(\boldsymbol{\sigma}_i(s) - A\dot{\mathbf{u}}_i(s),\ \mathbf{u}_i(s), \alpha_i(s)\right) ds,\ \text{a.e. } t \in (0, T),$$

we use the assumption (2.6),(2.8)), we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}^2 \leq C \bigg(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\, ds \\ &+ \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2\, ds\bigg).\end{aligned}$$

We employ the Gronwall argument within the resulting inequality to derive (3.9). ■

Now, let's contemplate the mapping

$$\Lambda : L^2(0, T; \mathcal{H} \times L^2(\Omega)) \to L^2(0, T; \mathcal{H} \times L^2(\Omega)),$$

$$\Lambda\left(\boldsymbol{\eta}, \theta\right)(t) = \left(\Lambda^1\left(\boldsymbol{\eta}, \theta\right)(t), \Lambda^2\left(\boldsymbol{\eta}, \theta\right)(t)\right), \tag{3.13}$$

defined by equalities

$$\Lambda^1(\boldsymbol{\eta}, \theta)(t), = B\mathbf{u}_\eta(t) + \int_0^t F\left(\boldsymbol{\sigma}_{\eta,\theta}(s) - A\dot{\mathbf{u}}(s), \mathbf{u}_\eta(s), \alpha_\theta(s)\right) ds, \tag{3.14}$$

$$\Lambda^2(\boldsymbol{\eta}, \theta)(t) = S\left(\left(\boldsymbol{\sigma}_{\eta,\theta}(t), \mathbf{u}_\eta(t)\right), \alpha_\theta(t)\right). \tag{3.15}$$

We have the following result.

Lemma 3.5. *For $(\boldsymbol{\eta}, \theta) \in L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$, the operator $\Lambda(\boldsymbol{\eta}, \theta) : [0, T] \to \mathcal{H} \times L^2(\Omega)$ have a unique fixed point denoted as $(\boldsymbol{\eta}^*, \theta^*) \in L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$, satisfying*

$$\Lambda\left(\boldsymbol{\eta}^*, \theta^*\right) = (\boldsymbol{\eta}^*, \theta^*).$$

Proof. Let $t \in (0, T)$ and $(\boldsymbol{\eta}_1, \theta_1), (\boldsymbol{\eta}_2, \theta_2) \in L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i,\ \dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i,\ \alpha_{\eta_i} = \alpha_i,\ \boldsymbol{\sigma}_{\boldsymbol{\eta}_i,\theta_i} = \boldsymbol{\sigma}_i$,For $i = 1, 2$ and using the assumptions (2.5),(2.6) and (2.8)

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}}^2 \\ &\qquad = \|B\mathbf{u}_1(t) + \int_0^t F\left(\boldsymbol{\sigma}_1(s) - A\dot{\mathbf{u}}_1(s), \mathbf{u}_1(s), \alpha_1(s)\right) ds \\ &\qquad - B\mathbf{u}_2(t) - \int_0^t F\left(\boldsymbol{\sigma}_2(s) - A\dot{\mathbf{u}}_2(s), \mathbf{u}_2(s), \alpha_2(s)\right) ds\|_{\mathcal{H}}^2 \\ &\qquad \leq L_B\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + L_F \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 + \\ &\qquad L_A\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2) ds.\end{aligned}$$

We utilise the estimate (3.9) to derive

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}}^2 \\ &\qquad \leq C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ &\qquad + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)) ds.\end{aligned}$$

On the other hand, we know that $\mathbf{u}_i(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_i(s)ds$, for all $t \in (0,T)$

$$\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \leq \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 ds. \tag{3.16}$$

By Apply the inequality (3.16) becomes

$$\begin{aligned}\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \leq C\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned} \tag{3.17}$$

By a similar argument, from (3.9),(3.15) and (2.9) it follows that

$$\begin{aligned}\|\Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t)\|_{L^2(\Omega)}^2 \leq C(\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds \\ + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2 \leq C(\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds \\ + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.19}$$

Combine the inequality (3.16) with (3.19) to obtain

$$\begin{aligned}\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2 \leq C\int_0^t (\|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \\ + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2)ds.\end{aligned} \tag{3.20}$$

Using the inequality (3.4), by adding the results obtained we have

$$(A\dot{\mathbf{u}}_1(t) - A\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))_V = (\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))_V,\ t \in (0,T), \tag{3.21}$$

using inequality (2.4), we find

$$M_A\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V^2 \leq \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_V\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V.$$

Therefore

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq C\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V, \quad \forall t \in [0,T].$$

Let's use (3.16)

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq C\int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V ds, \quad \forall t \in [0,T]. \tag{3.22}$$

Using (3.5) we find

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \leq (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)},$$

$$a \cdot e \cdot t \in (0,T),$$

By integrating the inequality with respect to time and incorporating the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, along with the inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$, we combine this inequality with Gronwall's lemma, resulting in the following result

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds, \forall t \in [0, T]. \tag{3.23}$$

From the previous inequality and estimates (3.20), (3.22) and (3.23) it follows that now

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)} \\
&\qquad \leq C\left(\int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)} ds\right).
\end{aligned}$$

Let is introduce the following notations

$$\begin{cases}
I_1 = \displaystyle\int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|_{\mathcal{H}\times L^2(\Omega)} ds, \\
\vdots \\
I_k = \displaystyle\int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} \|(\boldsymbol{\eta}_1, \theta_1)(r) - (\boldsymbol{\eta}_2, \theta_2)(r)\|_{\mathcal{H}\times L^2(\Omega)},
\end{cases}$$

Through an inductive process, denoting the m^{th} power of the operator Λ as Λ^m, we arrive at the following conclusion

$$\begin{aligned}
&\|\Lambda^m(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)} \\
&\qquad \leq C^m\left(\sum_{k=1}^m C_m^k I^{m-k} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|_{\mathcal{H}\times L^2(\Omega)}\right),
\end{aligned} \tag{3.24}$$

for all $t \in [0, T]$ and $m \in \mathbb{N}$,

$$\begin{aligned}
I^{m-k}((\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)) &= \int_{(m-k)\text{ fois}} . \int \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\| \\
&\leq \int_0^s \int \cdots \int_{(m-k)\text{ fois}} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\
&\leq \frac{t^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\
&\leq \frac{T^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|_{L^2(0,T;\mathcal{H}\times L^2(\Omega))},
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\Lambda^m(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))} \\
&\qquad \leq C^m\left(\sum_{k=1}^m C_m^k \frac{T^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))}\right) \\
&\qquad \leq \frac{(CT)^m}{m!} \|(\boldsymbol{\eta}_1, \theta_1)(t) - (\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{L^2(0,T;\mathcal{H}\times L^2(\Omega))},
\end{aligned}$$

this implies that for m large enough, the operator Λ^m of Λ is a contraction on Banach space $L^2(0, T; \mathcal{H} \times L^2(\Omega))$. So Λ^m has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0, T; \mathcal{H} \times L^2(\Omega))$, and therefore $(\boldsymbol{\eta}^*, \theta^*)$ is the only fixed point of Λ. ∎

Existence

Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2\left(0, T; \mathcal{H} \times L^2(\Omega)\right)$, be the fixed point of Λ defined by (3.14)-(3.15) and let $\mathbf{u}_\eta,\ \alpha_\theta$, be the solutions of problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$, for $\eta = \eta^*,\ \theta = \theta^*,\ \mathbf{u} = \mathbf{u}_{\eta^*},\ \alpha = \alpha_{\theta^*}$, we find $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ is a solution of problem $\mathcal{P}$. properties (3.1)-(3.3) follow from lemma 3.2, 3.3, 3.4.

Uniqueness

The uniqueness of the solution is a result of the uniqueness of the fixed point of operator Λ.

4. Application

In this section, we will utilise the main result from Section 3 to analyse a problem of contact without friction with condition of wear and damage. between an elastic-viscoplastic body and a rigid base in a quasistatic process. We provide the physical context for the contact problem and introduce certain notations that will be employed in the subsequent discussion. We consider a elastic-viscoplastic body which occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, such that the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The boundary $\partial\Omega$ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $meas(\Gamma_1) > 0$. We are interested in an evolution of the body in a finite time interval $(0, T)$.

We consider the following classical formulation of the problem

Problem P

Find a displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, the damage field $\alpha : \Omega \times [0, T] \to \mathbb{R}$.

$$0 = \mathrm{Div}\boldsymbol{\sigma} + f_0, \qquad \text{in } \Omega \times (0, T), \tag{4.1}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(\mathbf{u}(s)\right), \alpha(s)\right) ds \qquad \text{in } \Omega \times (0, T), \tag{4.2}$$

$$\dot{\alpha} - k_0\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \alpha), \qquad \text{in } \Omega \times (0, T), \tag{4.3}$$

$$\mathbf{u} = 0, \qquad \text{on } \Gamma_1 \times (0, T), \tag{4.4}$$

$$\sigma\nu = f_2, \qquad \text{on } \Gamma_2 \times (0, T), \tag{4.5}$$

$$\begin{cases} -\sigma_\nu = k\|\dot{u}_\nu\| \\ \boldsymbol{\sigma}_\tau = 0 \end{cases} \qquad \text{on } \Gamma_3 \times (0, T), \tag{4.6}$$

$$\frac{\partial\alpha}{\partial\nu} = 0, \qquad \text{on } \Gamma \times (0, T), \tag{4.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{4.8}$$

Equation (4.1) describes the equation of motion, where f_0 stands for the density of the voluminal forces exerted upon the deformable body Ω. Equation (4.2) describes the constitutive law applicable to an elastic-viscoplastic material with damage, (4.3) represents a differential inclusion describing the evolution of the damage field where S is a damage source function. φ_K is the sub-differential of the indicator function of the set of admissible damage functions K. The conditions (4.4) and (4.5) are displacement-traction conditions, (4.6) represent the boundary contact conditions with wear and without friction. (4.7) represents the boundary condition of Neumann, Finally,(4.8) represents the initial conditions.

Next, we outline the assumptions concerning the data of the problem, starting with the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \|\mathcal{A}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{v}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{v}_2)).(\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq m_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{v}) \text{ is lebesgue measurable on } \Omega, \forall \boldsymbol{v} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{4.9}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \|\mathcal{B}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{v}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{B}} > 0 \quad \text{such that} \\ (\mathcal{B}(\mathbf{x}, \boldsymbol{v}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{v}_2)).(\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq m_{\mathcal{B}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2, \ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{v}) \text{ is lebesgue measurable on } \Omega, \\ \qquad\qquad \forall \boldsymbol{v} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \ \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{4.10}$$

The relaxation function $\mathcal{F} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$, satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{F}} > 0 \quad \text{such that} \\ \|\mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{v}_1, \alpha_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{v}_2, \alpha_2)\| \leq \\ \qquad\qquad L_{\mathcal{F}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + |\boldsymbol{v}_1 - \boldsymbol{v}_2\| + \|\alpha_1 - \alpha_2\|) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall t \in [0, T], \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{v}, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{v} \in \mathbb{S}^d, \ \forall t \in [0, T], \ \forall \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } x \mapsto \mathcal{F}(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in \mathcal{H}, \forall t \in [0, T]. \end{cases} \tag{4.11}$$

The function describing the source of damages, denoted as $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$, is satisfied

$$\begin{cases} (a) \text{ There exists } M_{\phi} > 0 \quad \text{such that} \\ \|\phi(\mathbf{x}, \boldsymbol{v}_1, \alpha_1) - \phi(\mathbf{x}, \boldsymbol{v}_2, \alpha_2)\| \leq M_{\phi}(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \ \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{v}, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \forall \boldsymbol{v} \in \mathbb{S}^d, \forall \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \ \mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, 0) \in L^2(\Omega). \end{cases} \tag{4.12}$$

The body force $\mathbf{f}_0$, surface traction $\mathbf{f}_2$, coefficient of friction k, initial conditions u_0, have the following properties

$$\begin{cases} \mathbf{f}_0 \in L^2\left(0, T; H\right), \\ \mathbf{f}_2 \in L^2\left(0, T; L^2\left(\Gamma_2\right)^d\right), \\ k \in L^{\infty}\left(\Gamma_3\right), \quad k(x) \geq 0 \text{ for a.e. } x \in \Gamma_3, \\ \mathbf{u}_0 \in V. \end{cases} \tag{4.13}$$

We establish the bilinear form $a : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ as follows

$$a(\xi,\zeta)=k_0\int_\Omega \nabla\xi\nabla\zeta dx \tag{4.14}$$

and the micro crack diffusion coefficient verifies $k_0>0$.

The initial damage α_0 field satisfies

$$\alpha_0\in K. \tag{4.15}$$

To consider the field of displacements, we require the closed subspace V within the space H_1, defined by:

$$V=\{\mathbf{u}\in H_1 \mid \mathbf{u}=\mathbf{0}\ \ on\,\Gamma_1\ \}. \tag{4.16}$$

Using Riesz's representation theorem, we find

$$(\mathbf{f}(t),\mathbf{v})_V=\int_\Gamma \mathbf{f}_0\cdot\mathbf{v}dx+\int_{\Gamma_2}\mathbf{f}_2\cdot\mathbf{v}dx,\quad \forall\mathbf{v}\in V,t\in[0,T]. \tag{4.17}$$

It's important to observe that condition (4.13) results in the implication that

$$\mathbf{f}\in L^2\left(0,T;V\right). \tag{4.18}$$

Now, consider the application $j : V\times V\to\mathbb{R}$, defined as follows

$$j\left(\mathbf{u},\mathbf{v}\right)=\int_{\Gamma_3} k\left\|u_\nu\right\|v_\nu da. \tag{4.19}$$

The variational formulation for problem P is presented as follows

$$\begin{aligned}\boldsymbol{\sigma}(t)&=\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))+\mathcal{B}\varepsilon(\mathbf{u}(t))\\&\quad+\int_0^t\mathcal{F}\left(\boldsymbol{\sigma}(s)-\mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)),\varepsilon\left(\mathbf{u}(s)\right),\alpha(s)\right)ds\quad \text{a.e }.t\in(0,T),\end{aligned} \tag{4.20}$$

$$(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}}+j(\dot{\mathbf{u}}(t),\mathbf{v})=(\mathbf{f},\mathbf{v})_V,\quad\forall\mathbf{v}\in V, \tag{4.21}$$

$$\begin{aligned}\alpha(t)\in K,\quad &(\dot{\alpha}(t),\zeta-\alpha(t))_{L^2(\Omega)}+a(\alpha(t),\zeta-\alpha(t))\\&\geq(\phi(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{u}(t)),\alpha(t)),\zeta-\alpha(t))_{L^2(\Omega)},\quad\forall\zeta\in K,t\in[0,T],\end{aligned} \tag{4.22}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad \alpha(0)=\alpha_0. \tag{4.23}$$

Utilising Riesz's representation theorem, we define the operator $A : V\to V$ as follows:

$$(A\mathbf{u},\mathbf{v})_V=(\mathcal{A}(\varepsilon(\mathbf{u})),\varepsilon(\mathbf{v}))_{\mathcal{H}}+j(\mathbf{u},\mathbf{v}),\quad\forall\mathbf{u},\mathbf{v}\in V. \tag{4.24}$$

We will verify the hypotheses (2.4),(2.5). Let $\mathbf{u}_1,\mathbf{u}_2\in V$. Using (4.9),(4.24) and the definition of j given by (4.19), we let's find

$$\begin{aligned}\|A\mathbf{u}_1-A\mathbf{u}_2\|_V&=\|\mathcal{A}\varepsilon(\mathbf{u}_1)-\mathcal{A}\varepsilon(\mathbf{u}_2)\|_{\mathcal{H}}+C_0^2\|k\|_{L^\infty(\Gamma_3)}\|\mathbf{u}_1-\mathbf{u}_2\|_V\\&\leq L_{\mathcal{A}}\|\varepsilon(\mathbf{u}_1)-\varepsilon(\mathbf{u}_2)\|_{\mathcal{H}}+C_0^2\|k\|_{L^\infty(\Gamma_3)}\|\mathbf{u}_1-\mathbf{u}_2\|_V\\&=\left(L_{\mathcal{A}}+C_0^2\|k\|_{L^\infty(\Gamma_3)}\right)\|\mathbf{u}_1-\mathbf{u}_2\|_V,\quad\forall\mathbf{u}_1,\mathbf{u}_2\in V.\end{aligned} \tag{4.25}$$

Similarly for all $\mathbf{u}_1,\mathbf{u}_2\in V$ we have

$$(A\mathbf{u}_1-A\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)_V\geq\left(m_{\mathcal{A}}-C_0^2\|k\|_{L^\infty(\Gamma_3)}\right)\|\mathbf{u}_1-\mathbf{u}_2\|_V^2,\quad\forall\mathbf{u}_1,\mathbf{u}_2\in V. \tag{4.26}$$

Let $\gamma_0=\dfrac{m_{\mathcal{A}}}{C_0^2}$, it is clear that γ_0 is positive which depends on Ω_1,Γ_3, and $\mathcal{A}$. Then A is strongly monotonic on V if

$$\|k\|_{L^\infty(\Gamma_3)}<\gamma_0.$$

After confirming that all the assumptions of Theorem 3.1 are met, we can conclude that a unique weak solution to problem P exists, satisfying (4.20)-(4.23), along with the regularity conditions (3.1)-(3.3).

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