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Existence results on nonautonomous partial functional differential equations with state-dependent infinite delay

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Abstract. The aim of this work is to establish the existence of mild solutions for some nondensely nonautonomous partial functional differential equations with state-dependent infinite delay in Banach space. We assume that, the linear part is not necessarily densely defined and generates an evolution family under the hyperbolique conditions. We use the classic Shauder Fixed Point Theorem, the Nonlinear Alternative Leray-Schauder Fixed Point Theorem and the theory of evolution family, we show the existence of mild solutions. Secondly, we obtain the existence of mild solution in a maximal interval using Banach's Fixed Point Theorem which may blow up at the finite time, we show that this solution depends continuously on the initial data under the global Lipschitz condition on the second argument of F and we get the existence of global mild solution. We propose some model arising in dynamic population for the application of our results.

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Contents

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1. Introduction

Partial differential equations play a crucial role in providing mathematical answers to natural phenomena and they continue to be an indispensable tool in scientific investigations of real-world problems. The future behaviors of many phenomenas are therefore supposed to be described by the solutions of an ordinary or partial differential equations. These have long played important roles in the history of mathematical modeling and will undoubtedly continue to serve as indispensable tools in future investigations. They are encountered in a variety of problems in physics, chemistry, biology, medicine, economics, engineering, climate and disease modeling and many others.

In this work, we study the existence of mild solutions for the following partial functional differential equation with state-dependent infinite delay

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x_{\rho(t,x_t)}); \quad t \in J := [0, b], \\ \\ x_0 = \varphi \in \mathcal{B} \end{cases} \tag{1.1}$$

in a Banach space $(X, \|\cdot\|)$. Here $(A(t))_{t\geq 0}$ is a given family of closed linear operators in X with non necessarily dense domain and satisfying the hyperbolic conditions ($\mathbf{A}_1$) through ($\mathbf{A}_3$) introduce by Tanaka in [45, 46] which will be specified later. The phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X satisfying some Axioms which will be described in the sequel. $F : J \times \mathcal{B}$ is continuous and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are appropriate functions. The history x_t ($t \geq 0$), represents the mapping defined from $(-\infty, 0]$ into X by

$$x_t(\theta) = x(t + \theta) \qquad \text{for } \theta \in (\text{-}\ \infty, 0].$$

For the nonautonomous dynamical systems, the basic law of evolution is not static in the sense that the environment change with time. Parameters in real-world situations and particularly in the life sciences are rarely constant over time. The theory of nonautonomous dynamical systems is a well-developed and successful mathematical framework to describe time-varying phenomena. Its applications in the life sciences range from simple predator-prey models to complicated signal traduction pathways in biological cells, in physics from the motion of a pendulum to complex climate models, and beyond that to further fields as diverse as chemistry (reaction kinetics), economics, engineering, sociology, demography, and biosciences. Nonautonomous differential equations has received the great attention see for instance the works [22, 26, 28, 40, 42, 47, 51] and some recent works [9, 37–39]. For some applications, we refer the reader to the handbook by Peter E. Kloeden and Christian Pötzsche [44]. Note that when $A(t) := A$ is independent of t, the theory of partial functional differential equations was studied by several authors. Hernández et al. [34] studied the existence of mild solutions of Equation (1.1) by using the classical C_0-semigroup theory. Later on, Belmekki et al. [12] obtained the existence results of the following partial functional differential equations with state-dependent delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t - \tau(x(t)))) \quad \text{for} \quad t \in [a, b]; \\ \\ x_0 = \varphi \in C([-r, 0]; X) \end{cases} \tag{1.2}$$

where the operator A satisfies the usual Hille-Yosida condition except the density of D(A) in X. They obtained their results by using the variation of constants formula which is given in terms of integrated semigroups. In the autonomous case where $\rho(t, x_t) = t$, we refer the reader to Adimy et al [2], K. Ezzinbi et al [23, 24], Hale and Lunel [30], G. F. Webb. [48, 49], Wu [50], and the papers [2, 3, 13, 14, 16–18, 18, 36].

The literature related to partial nonautonomous functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive and we refer the reader to the papers in [9, 13, 25, 37, 38, 40, 47] concerning this case. Recently Kpoumié et al in [9], investigate several results on the existence of solutions of the following nonautonomous equation :

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x_t) \quad \text{for} \quad t \geq 0, \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.3}$$

where $(A(t))_{t\geq 0}$ is a given family of closed linear operators on a Banach space $(X, \|\cdot\|)$ not necessarily densely defined satisfying the hyperbolic conditions, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ to X satisfying some Axioms and F a continuous function defined on $[0, +\infty) \times \mathcal{B}$ with values in X. In this context, they have studied the local existence of the mild solutions which may blow up at the finite time, the global existence of mild solutions are given and under sufficient conditions, the existence of the strict solutions have been obtained.

Functional differential equations with state-dependent delay appear frequently in applications as models of equations and for this reason the study of this type of equation has attracted attention in recent years and more than ten years ago we refer the reader to the handbook by Cañada et al. [5], the book [19], the papers [6, 8, 11, 12, 20, 26, 27, 31, 32] and the references therein. In [39], we investigated the existence of mild solutions of the following nonautonomous equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x(t - \rho(x(t)))) \quad \text{for} \quad t \in [0, a] \\ \\ x_0 = \varphi \in C([-r, 0], X), \end{cases} \tag{1.4}$$

where $(A(t))_{t\geq 0}$ is a given family of closed linear operators on a Banach space $(X, \|\cdot\|)$ not necessarily densely defined and satisfying the hyperbolic conditions $(\mathbf{A}_1)$ through $(\mathbf{A}_3)$ introduced by Tanaka in [46] which will be specified in Section 2. F is a given function defined on $[0, +\infty) \times X$ with values in X, the initial data $\rho : [-r; 0] \to X$ is a continuous function, ρ is a positive bounded continuous function on X and r is the maximal delay defined by

$$r = \sup_{x \in X} \rho(x)$$

.

In this paper, we study the existence of at least one mild solutions where the family of closed linear operators on a Banach space is not necessarily densely defined. Note that there are many examples where evolution equations are not densely defined. One can refer to [1, 4, 21] for references and discussion on this subject. Our work is motivated by [9, 34]. The results obtained is a continuation of work done by Hernãndez et al in [34], Belmekki et al. [12] and Kpoumié et al in [39].

In the whole of this work we employ an axiomatic definition for the phase space $\mathcal{B}$ due to Hale and Kato [29]. We assume that $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ to X endowed with a normed $|\cdot|_{\mathcal{B}}$ and satisfying the following Axioms:

($\mathbf{B}_1$) There exist a positive constant H and functions $K(\cdot); M(\cdot) : [0, +\infty) \to [0; +\infty)$, with K continuous and M locally bounded, and the are independent of x, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if x is a function mapping $(-\infty, \sigma + a[$ into X, $a > 0$, such that $x_\sigma \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma + a[$, then for every t in $[\sigma, \sigma + a[$ the following conditions hold :

- (i) $x_t \in \mathcal{B}$,
- (ii) $\|x(t)\|_X \leq H\|x_t\|_{\mathcal{B}}$ which is equivalent to
- $(ii)'$ $\|\varphi(0)\|_X \leq H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$.
- (iii) $\|x_t\|_{\mathcal{B}} \leq M(t - \sigma)\|x_\sigma\|_{\mathcal{B}} + K(t - \sigma) \sup_{\sigma \leq s \leq t} \|x(s)\|_X$

($\mathbf{B}_2$) For the function $x(\cdot)$ in $(\mathbf{B}_1)$, $t \mapsto x_t$ is a $\mathcal{B}-$valued continuous function for $t \in [\sigma; \sigma + a[$.

(B) The space $\mathcal{B}$ is complete.

For examples and more details on phase space, see the book by Y. Hino, S. Murakami and T. Naito [35].

The organization of this work is as follows: in Section 2, we recall some results on nonautonomous evolution family with nondensely domain theory that will be used to develop our main results. In Section 3, we use the variant of Shauder's Fixed Point Theorem and the nonlinear alternative of Leray-Schauder's to prove the existence of at least one mild solution. In Section 4, we propose an application to some models with state dependent delay.

2. Nonautonomous evolution family with nondense domain

In this section, we recall some notations, definitions and preliminary facts concerning our work. Throughout this paper we used the results which are detailed in [43, 45, 46]. We assume that $\mathcal{B}(X)$ is the Banach space of all bounded linear operators from X to itself. In this work, we assume the following hyperbolic assumptions:

($\mathbf{A}_1$) $D(A(t)) := D$ independent of t and not necessarily densely defined.

($\mathbf{A}_2$) The family $(A(t))_{t\geq 0}$ is stable that means there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that:

$$(w, +\infty) \subset \rho(A(t)) \quad \text{and} \quad \Big\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \Big\| \leq M(\lambda - w)^{-k}$$

for $t \geq 0$, $\lambda > w$ and for very finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k < +\infty$ and $k = 1, 2, \ldots$, where $\rho(A(t))$ is the resolvent set of $A(t)$ and $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$.

($\mathbf{A}_3$) The mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.

We recall here the classical result which gives us the existence and explicit formula of the evolution family generated by $(A(t))_{t\geq 0}$ due to Oka and Tanaka [43] and Tanaka [46].

Theorem 2.1. *(Oka and Tanaka [43]; Tanaka [46]) Assume that $(A(t))_{t\geq 0}$ satisfies conditions ($\mathbf{A}_1$) -($\mathbf{A}_3$). Then the limit*

$$U(t,s)x = \lim_{\lambda \to 0^+} \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x$$

exists for $x \in \overline{D}$ and $t \geq s \geq 0$, where the convergence is uniform on $\Gamma := \{(t,s) : t \geq s \geq 0\}$. Moreover, the family $\{U(t,s) : (t,s) \in \Gamma\}$ satisfies the following properties:

i) *$U(t,s) : \overline{D} \to \overline{D}$ for $(t,s) \in \Gamma$;*

ii) *$U(t,t)x = x$ and $U(t,s)x = U(t,r)U(r,s)x$ for $x \in \overline{D}$ and $t \geq r \geq s \geq 0$;*

iii) *the mapping $(t,s) \mapsto U(t,s)x$ is continuous on Γ for any $x \in \overline{D}$;*

iv) *$\|U(t,s)x\| \leq Me^{w(t-s)}\|x\|$ for $x \in \overline{D}$ and $(t,s) \in \Gamma$;*

v) *$U(t,s)D(s) \subset D(t)$ for all $t \geq s \geq 0$ where $D(t) := \{x \in D : A(t)x \in \overline{D}\}$;*

vi) *for all $x \in D(s)$ and $t \geq s \geq 0$, the function $t \mapsto U(t,s)x$ is continuously differentiable with: $\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$ and $\frac{\partial^+}{\partial s}U(t,s)x = -U(t,s)A(s)x$.*

Let $\lambda > 0$, $t \geq s \geq 0$ and $x \in X$. We define $U_\lambda(t,s)$ by:

$$U_\lambda(t,s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x$$

Remark 2.1. *For $x \in X$, $\lambda > 0$ and $t \geq r \geq s \geq 0$ one can see that*

$$U_\lambda(t,t)x = x \quad \text{and} \quad U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x.$$

We consider the following nonautonomous linear evolution equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t) \text{ for } t \in [0,a], \\ \\ x(0) = x_0 \in X \end{cases} \tag{2.1}$$

where $f : [0,a] \to X$ is a function.

Theorem 2.2. *(Tanaka [46]) Assume that ($\mathbf{A}_1$)-($\mathbf{A}_3$) hold. Let $x_0 \in \overline{D}$ and $f \in L^1([0,a],X)$. Then the limit*

$$x(t) := U(t,0)x_0 + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)f(r)dr \tag{2.2}$$

exists uniformly for $t \in [0,a]$ and x is a continuous function on $[0,a]$.

Definition 2.1. *(Tanaka [46]) For $x_0 \in \overline{D}$, a continuous function $x : [0,a] \to X$ is called a mild solution of the initial value of Equation (2.1) if x satisfies the following equation:*

$$x(t) = U(t,0)x_0 + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)f(r)dr. \tag{2.3}$$

Lemma 2.1. *(Ezzinbi, Békollè and Kpoumiè [37]) Assume $f \in L^1([0,a],X)$. If x is the mild solution of Equation (2.1), then*

$$\|x(t)\| \leq Me^{wt}\|x_0\| + \int_0^t Me^{\omega(t-s)}\|f(s)\|ds.$$

Definition 2.2. *(Kpoumiè, Ezzinbi and Békollè [38]) For $\varphi(0) \in \overline{D}$, a continuous function $x : (-\infty,b] \to X$ is a mild solution of Equation (1.3) if x satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s)F(s,x_s)ds & \text{for } 0 \leq t \leq b, \\ \\ \varphi(t) & \text{for } -\infty \leq t \leq 0. \end{cases} \tag{2.4}$$

In the whole of this work, we assume that ($\mathbf{A}_1$) - ($\mathbf{A}_3$) are true and $w > 0$.

3. Existence of mild solutions

In this section, we use some Fixed Point Theorems and the Kuratowski's measure of noncompactness to establish the existence of mild solutions of Equation (1.1). In this work, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous.

Definition 3.1. *Let $\varphi(0) \in \overline{D}$. We say that a continuous function $x : (-\infty,b] \to X$ is a mild solution of Equation (1.1) if x satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s)F(s,x_{\rho(s,x_s)})ds & \text{for } 0 \leq t \leq b, \\ \\ \varphi(t) & \text{for } -\infty \leq t \leq 0. \end{cases} \tag{3.1}$$

We introduce the Kuratowski's measure of noncompactness $\alpha(\cdot)$ of bounded sets K on a Banach space Y which is defined by:

$$\alpha(K) = \inf\{\varepsilon > 0 : \text{K has a finite cover of ball with diameter} < \varepsilon\}.$$

Some basic properties of $\alpha(\cdot)$ are given in the following Lemma.

Lemma 3.1. *(Akhmerov et al. in [7])*

(i) $\alpha(A_1) \leq diaA_1$, *where* $dia(A_1) = \sup_{x,y\in A_1} |x-y|$,

(ii) $\alpha(A_1) = 0$ *if and only if* A_1 *is relatively compact in* X,

(iii) $\alpha(A_1 \cup A_2) = max(\alpha(A_1), \alpha(A_2))$,

(iv) if $A_1 \subset A_2$, *then* $\alpha(A_1) \leq \alpha(A_2)$,

(v) $\alpha(A_1 + A_2) \leq \alpha(A_1) + \alpha(A_2)$,

(vi) $\alpha(B(0,\varepsilon)) = 2\varepsilon$ *if* $dimX = +\infty$.

The terminology and notations employed in this work coincide with those generally used in functional analysis. In particular, for Banach spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, the notation $L(X,Y)$ stands for the Banach space of bounded linear operators from X into Y, and we abbreviate this notation to $L(X)$ when $X = Y$. Moreover $B_r(z,X)$ denotes the open ball with center at z and radius $r > 0$ in X and for a bounded function $x : J \to X$ and $0 \leq t \leq b$ we employ the notation $\|x\|_{X,t}$ for $\|x\|_{X,t} := \sup\limits_{\theta\in[0,t]} \|x(\theta)\|$. We will simply write $\|x\|_t$ when no confusion arises.

To prove our main result we will use the following variant of Schauder's Theorem see Radu Precup [41] and the Nonlinear Alternative of Leray-Schauder see A. Granas [27] or W. Arendt [10].

Theorem 3.1. *(Schauder) Let X be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $\mathcal{T} : D \to D$ be a completely continuous operator. Then $\mathcal{T}$ has at least one fixed point.*

Theorem 3.2. *(Leray-Schauder) Let $\mathcal{W}$ be a convex subset of a Banach space X and assume that $0 \in \mathcal{W}$. Let $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ be a completely continuous map. Then either*

(i) $\mathcal{F}$ has a fixed point in $\mathcal{W}$, or

(ii) the set $\{x \in \mathcal{W} : \ \ x = \alpha\mathcal{F}(x), \ \ 0 < \alpha < 1\}$ is unbounded.

Theorem 3.3 (Banach's Fixed Point Theorem)**.** *Let (E, d) be a non empty complete metric space and a mapping $T : E \to E$ such that T^p is a strict contraction ($p \in \mathbb{N}^\star$). Then T admits a unique fixed point $\bar{x}$ in E (i.e. $T(\bar{x}) = \bar{x}$) and the sequence $(x_n)_n$ define by $x_n = T(x_{n-1})$ with $x_0 \in E$, converges to $\bar{x}$.*

Lemma 3.2. *(Lemma Bellman-Gronwall) Let f, g the continuous positives fonctions from $[a,b]$ to $\mathbb{R}_+$.*
If Ψ is constant, then from

$$g(t) \leq \Psi + \int_a^t f(s)g(s)ds \ \text{ for all } t \in [a,b],$$

it follows that

$$g(t) \leq \Psi \exp\Big(\int_a^t f(s)ds\Big) \ \text{ for all } t \in [a,b].$$

Let us consider the following assumptions:

$(\mathbf{C}_1)$**.** $U(t,s)_{t>s}$ is compact on D for $t > s$.

$(\mathbf{C}_2)$. The function $F : J \times \mathcal{B} \to X$ satisfies the following properties.

(a) The function $F(\cdot, \psi) : J \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.

(b) The function $F(t, \cdot) : \mathcal{B} \to X$ is continuous for each t $\in$ J.

(c) Let $L^1(J, [0, +\infty))$ be the space of integrable functions from J to $[0, +\infty)$. There exist $p \in L^1(J, [0, +\infty))$ and a continuous non-decreasing function $V : (0, +\infty) \to (0, +\infty)$ such that

$$\|F(t, \psi)\| \le p(t)V(\|\psi\|_{\mathcal{B}}) \text{ for all } (t, \psi) \in J \times \mathcal{B}.$$

$(\mathbf{C}_3)$. Let $\varphi \in \mathcal{B}$ such that $x_0 = \varphi$ and $t \longmapsto \varphi_t$ is a $\mathcal{B}$-valued well defined continuous function on ρ^- where $\rho^- = \{\rho(s, \psi) : (s, \psi) \in J \times \mathcal{B}, \rho(s, \psi) \le 0\}$, and there exists a continuous and bounded function $\eta : \rho^- \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$.

Remark 3.1. *For $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ and $\varphi = x_0$ we can see that for all $t < 0$, $\varphi_t = x_t$. In fact if for all $t < 0, \varphi_t \neq x_t$, then for all $\theta \in (-\infty, 0]$, $\varphi_t(\theta) \neq x_t(\theta)$ hence $\varphi(t + \theta) \neq x(t + \theta)$ thus for all $t \in (-\infty, 0]$, $\varphi(t) \neq x(t)$ which is absurd because $\varphi = x_0$ that means for all $t \in (-\infty; 0]$, $\varphi(t) = x(t)$. Therefore for all $t < 0$, $\varphi_t = x_t$.*

To continue with the next step we need the following Lemma due to E. Hernández.

Lemma 3.3. *(Hernández et al. [33]) Let $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ for every $t \in \rho^-$. Assume that there exists a locally bounded function $\eta : \rho^- \to [0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$ and $\zeta = \sup\{\eta(s) : s \in \rho^-\}$. If $x : (-\infty, b] \to X$ is continuous on J and $x_0 = \varphi$, then*

$$\|x_s\|_{\mathcal{B}} \le (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b \sup_{0 \le \theta \le s} \|x(\theta)\|, \qquad s \in \rho^- \cup J$$

$$\textit{Where } K_b = \sup_{t \in J} K(t) \qquad , \qquad M_b = \sup_{t \in J} M(t)$$

.

In the sequel, we prove the existence of mild solution of equation (1.1).

Theorem 3.4. *Let Ω be a nonempty open subset of $\mathcal{B}$ and the function $F : [0, b] \times \mathcal{B} \to X$ is Carathéodory mapping. Assume that $(\boldsymbol{C}_1) - (\boldsymbol{C}_3)$ and $(\boldsymbol{A}_1) - (\boldsymbol{A}_3)$ hold. Let $\varphi \in \Omega$ be such that $\varphi(0) \in \overline{D}$. Then, Equation (1.1) has at least one mild solution $x(\cdot, \varphi)$ define on $]-\infty, a] \to X$, for some $a \in]0, b]$.*

Proof. We use the classic Schauder's Fixed Point Theorem.

Step 1. Let $\varphi \in \Omega$ be such that $\varphi(0) \in \overline{D}$. Then, there exists a constants $r > 0, r < b$ such that $\overline{B}_X(\varphi, r) = \{\psi \in \mathcal{B}$ such that $\|\psi - \varphi\|_{\mathcal{B}} \le r\} \subset \Omega$ and $\|F(s, \psi)\| \le \|p\|_{L^1}V(\|\psi\|)$ for all $s \in [0, r]$ and $\psi \in \overline{B}_X(\varphi, r)$.
Define the function $y : (-\infty, b] \longrightarrow X$ defined by:

$$y(t) = \begin{cases} U(t, 0)\varphi(0) & \text{for } t \in J, \\ \\ \varphi(t) & \text{for } -\infty \le t \le 0. \end{cases}$$

By virtue of Axioms $(\mathbf{B}_1) - (i)$ and $(\mathbf{B}_2)$, $y_t \in \mathcal{B}$ and $t \mapsto y_t$ is a continuous function. Then for $\gamma \in (0, r)$ there exists $b_1 \in (0, r]$ such that $\|y_t - \varphi\|_{\mathcal{B}} \le \gamma$ for all $t \in [0, b_1]$.
Set $K_b := \sup_{t \in [0, b]} K(t)$. Let a be a constant such that:

$$0 < a \le \min\left\{b_1, \frac{r - \gamma}{Me^{wa}K_b\|p\|_{L^1}V(l)}\right\}$$

where $l = (M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr$. For $u \in C([0,a];X)$ such that $u(0) = \varphi(0)$, we define its extension on $(\infty, a]$ by :

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in [0,a], \\ \\ \varphi(t) & \text{for } -\infty \leq t \leq 0. \end{cases}$$

Let us introduce the following space:

$$\mathbb{F}_a := \Big\{ u : [0,a] \to X \text{ continuous such that } \ u_0 = \varphi \text{ and } \sup_{0 \leq t \leq a} \|\tilde{u}_t - \varphi\|_{\mathcal{B}} \leq r \Big\}$$

endowed with the uniform norm topology. $\|.\|_{\mathbb{F}_a}$ defined by:

$$\|u\|_{\mathbb{F}_a} := \|u_0\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} \|u(s)\|$$

The restriction of y to $(\infty, a]$ is an element of $\mathbb{F}_a$. In fact $\|y_t - \varphi\|_{\mathcal{B}} \leq \gamma$ for all $t \in [0, b_1]$ whereas $\gamma < r$ then $\|\tilde{y}_t - \varphi\|_{\mathcal{B}} \leq r$ for all $t \in [0,a]$ thus $y \in \mathbb{F}_a$. Therefore $\mathbb{F}_a$ is nonempty.

For all $u \in \mathbb{F}_a$, we have

$$\begin{aligned} \|u\|_{\mathbb{F}_a} &= \|u_0\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} \|u(s)\| \\ &\leq \|u_0 - \varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}} + \sup_{0 \leq s \leq a} H\|u_s\| \text{ by } (\mathbf{B}_1) - (iii) \\ &\leq \|\varphi\|_{\mathcal{B}} + H \sup_{0 \leq s \leq a} \{\|(u_s - \varphi) + \varphi\|_{\mathcal{B}}\} \text{ since } u_0 = \varphi \\ &\leq \|\varphi\|_{\mathcal{B}} + H\{\sup_{0 \leq s \leq a} \|(u_s - \varphi)\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}}\} \\ &\leq \|\varphi\|_{\mathcal{B}} + H(r + \|\varphi\|_{\mathcal{B}}). \end{aligned} \tag{3.2}$$

Then $\mathbb{F}_a$ is bounded.

By using the triangular inequality in $\mathcal{B}$ it is clear that $\lambda p + (1-\lambda)q \in \mathbb{F}_a$ for any $p, q \in \mathbb{F}_a$, with $\lambda \in [0,1]$. Indeed

$$\begin{aligned} \|\lambda\tilde{p}_t + (1-\lambda)\tilde{q}_t - \varphi\|_{\mathcal{B}} &= \|\lambda\tilde{p}_t + (1-\lambda)\tilde{q}_t - (1-\lambda)\varphi + (1-\lambda)\varphi - \varphi\|_{\mathcal{B}} \\ &= \|\lambda\tilde{p}_t + (1-\lambda)(\tilde{q}_t - \varphi) + \varphi - \lambda\varphi - \varphi\|_{\mathcal{B}} \\ &= \|\lambda(\tilde{p}_t - \varphi) + (1-\lambda)(\tilde{q}_t - \varphi)\|_{\mathcal{B}} \\ &\leq \lambda\|(\tilde{p}_t - \varphi)\| + (1-\lambda)\|(\tilde{q}_t - \varphi)\|_{\mathcal{B}} \\ &\leq \lambda r + (1-\lambda)r \\ &= r. \end{aligned}$$

Then $\mathbb{F}_a$ is convex.

Now we prove that $\mathbb{F}_a$ is closed. To prove that, consider a convergent sequence $(\tilde{u}_t^n)_{n\in\mathbb{N}}$ of $\mathbb{F}_a$ which converges to $\tilde{u}_t$. We want to show that $\tilde{u}_t \in \mathbb{F}_a$.

$$\begin{aligned} \|\tilde{u}_t - \varphi\|_{\mathcal{B}} &= \|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} + \|\tilde{u}_t^n - \varphi\|_{\mathcal{B}} \\ &\leq \|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} + r, \text{ since } \tilde{u}_t^n \in \mathbb{F}_a. \end{aligned}$$

whereas

$$\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \leq K_b \sup_{0\leq s\leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + M_b\|\tilde{u}_0 - \tilde{u_0^n}\|_{\mathcal{B}}$$

$$\leq Max(K_b, M_b) \sup_{0\leq s\leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + Max(K_b, M_b)\|\tilde{u}_0 - \tilde{u}_0^n\|_{\mathcal{B}}$$

$$\leq Max(K_b, M_b)\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a}.$$

Thus $\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \leq Max(K_b, M_b)\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a}$ as $\|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a} \to 0$ with $n \to +\infty$ hence $\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} \to 0$ with $n \to +\infty$ then $\|\tilde{u}_t - \varphi\| \leq r$, hence $\tilde{u}_t \in \mathbb{F}_a$ thus $\mathbb{F}_a$ is closed.

To continue our proof, we need the following Lemma.

Lemma 3.4. *Let $\varphi \in \mathcal{B}$ such that $\varphi_t \in \mathcal{B}$ for every $t \in \rho^-$. Assume that there exists a locally bounded function $\eta : \rho^- \to [0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \rho^-$ and $\zeta = \sup\{\eta(s) : s \in \rho^-\}$. If $u \in \mathbb{F}_a$, then*

$$\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \leq l < +\infty$$

Where $l = M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr$.

Proof.

$$\begin{aligned}\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq\rho(s,\tilde{u}_s)} \|\tilde{u}(\theta)\|, \text{ by the Lemma 3.3}\\ &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b(\|\varphi\|_{\mathcal{B}} + \sup_{0\leq\theta\leq a} \|u(\theta)\|)\\ &\leq (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|u\|_{\mathbb{F}_a} \text{ since } \|u\|_{\mathbb{F}_a} = \|\varphi\|_{\mathcal{B}} + \sup_{0\leq\theta\leq a} \|u(\theta)\|\\ &\leq (M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr. \text{ By relation (3.2)}\end{aligned}$$

■

Consider the mapping $\mathcal{K}$ defined on $\mathbb{F}_a$ by:

$$\begin{cases}(\mathcal{K}x)(t) = U(t,0)\varphi(0) + \lim\limits_{\lambda\to 0^+} \displaystyle\int_0^t U_\lambda(t,s)F(s,\tilde{x}_{\rho(s,\tilde{x}_s)})ds \quad \text{for} \quad t \in [0,a],\\ \\ \varphi(t) \qquad \text{for} \quad -\infty < t \leq 0.\end{cases} \tag{3.3}$$

From definition (3.1), theorem (2.1) and the assumptions on φ, we infer that $(\mathcal{K}x)(\cdot)$ is well defined. We claim that $\mathcal{K}(\mathbb{F}_a) \subset \mathbb{F}_a$. In fact, Axiom ($\mathbf{C}_2$) implies that for every $x \in \mathbb{F}_a$, the mapping $s \mapsto F(s, \tilde{x}_{\rho(s,\tilde{x})})$ is continuous on $[0, a]$. Hence this mapping $v := \mathcal{K}x$ is continuous on $[0, a]$. In the other hand, One has

$$\|\tilde{v}_t - \varphi\|_{\mathcal{B}} \leq \|\tilde{v}_t - y_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}}$$

$$\leq \|\tilde{v}_t - y_t\|_{\mathcal{B}} + \gamma$$

On one hand, by Axiom ($\mathbf{B}_1$) $- (iii)$, we have for any $t \in [0, a]$,

$$\|\tilde{v}_t - y_t\|_{\mathcal{B}} \leq K_b \sup_{0\leq s\leq t} \|v(s) - y(s)\|$$

For any $t \in [0, a]$

$$\begin{aligned}
\|v(t) - y(t)\| &= \left\| U(t,0)\varphi(0) - \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s) F(s, \tilde{v}_{\rho(s,\tilde{v_s})}) ds - U(t,0)\varphi(0) \right\| \\
&= \left\| \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,s) F(s, \tilde{v}_{\rho(s,\tilde{v_s})}) ds \right\| \\
&\leq \int_0^t M e^{w(t-s)} \left\| F(s, \tilde{v}_{\rho(s,\tilde{v_s})}) \right\| ds \\
&\leq M e^{wa} \int_0^t \|p\|_{L^1} V(\|\tilde{v}_{\rho(s,\tilde{v_s})}\|_{\mathcal{B}}) ds \\
&\leq M e^{wa} a \|p\|_{L^1} V(l), \text{ by Lemma 3.4} \\
&\leq \tfrac{r-\gamma}{K_b}.
\end{aligned}$$

hence $K_b \sup\limits_{0 \leq s \leq t} \|v(s) - y(s)\| \leq r - \gamma$ then $\|\tilde{v}_t - y_t\|_{\mathcal{B}} \leq r - \gamma$ we have $\|\tilde{v}_t - \varphi\|_{\mathcal{B}} \leq r$, for any $t \in [0, a]$. Therefore $v \in \mathbb{F}_a$. We have proved that $\mathbb{F}_a$ is a nonempty, bounded, convex and closed subset of $\mathbb{F}_a$:

Now we want to prove that $\mathcal{K}$ is a completely continuous operator.

Step 2. The continuity of $\mathcal{K}$. Let $(u^n)_{n \in \mathbb{N}^*}$ be a sequence in $\mathbb{F}_a$ such that $\lim\limits_{n \to \infty} u^n = u$. For $t \in [0, a]$, we have by Axiom $(\mathbf{B}_1 - iii)$:

$$\begin{aligned}
\|\tilde{u}_t - \tilde{u}_t^n\|_{\mathcal{B}} &\leq K_b \sup_{0 \leq s \leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + M_b \|\tilde{u}_0 - \tilde{u}_0^n\|_{\mathcal{B}} \\
&\leq Max(K_b, M_b) \sup_{0 \leq s \leq t} \|\tilde{u}(s) - \tilde{u}^n(s)\| + Max(K_b, M_b) \|\tilde{u}_0 - \tilde{u}_0^n\|_{\mathcal{B}} \\
&\leq Max(K_b, M_b) \|\tilde{u} - \tilde{u}^n\|_{\mathbb{F}_a}.
\end{aligned}$$

then $\lim\limits_{n \to \infty} \tilde{u}_s^n = \tilde{u_s}$. we recall that $\rho : [0, a] \times \mathcal{B} \to (-\infty, a]$ is continuous then $\lim\limits_{n \to \infty} \rho(s, \tilde{u}^n)_s = \rho(s, \tilde{u_s})$.

Let us study therefore the convergence of the sequence $(\tilde{u}^n_{\rho(s,\tilde{u}_s^n)})_{n \in \mathbb{N}}$ for $s \in [0, a]$. At first, if $s \in [0, a]$ such that $\rho(s, \tilde{u}_s) > 0$, and there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$ $\rho(s, \tilde{u}_s^n) > 0$.

In this case one has

$$\begin{aligned}
\|\tilde{u}^n_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &\leq \|\tilde{u}^n_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s^n)}\|_{\mathcal{B}} + \|\tilde{u}_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \\
&\leq K_b \sup_{0 \leq \theta \leq \rho(s,\tilde{u}_s^n)} \|u^n(\theta) - u(\theta)\| + M_b \|\varphi - \varphi\| + \|\tilde{u}_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \\
&\quad \text{by } (\mathbf{B}_1 - iii) \\
&\leq K_b \|u^n - u\|_a + \|\tilde{u}_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}
\end{aligned}$$

whereas

$$\lim_{n \to \infty} u^n = u \text{ then } \|u^n - u\|_a \to 0 \text{ for } n \to +\infty,$$

$$\|\tilde{u}_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} \to 0 \text{ for } n \to +\infty \text{ by } (\mathbf{B}_1 - iii),$$

which proves that $\tilde{u}^n_{\rho(s,\tilde{u}_s^n)} \to \tilde{u}_{\rho(s,\tilde{u}_s)}$ in $\mathcal{B}$ as $n \to +\infty$ for every $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) > 0$. Similar, if $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) < 0$, and there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$ $\rho(s,\tilde{u}_s^n) < 0$.

In this case one has

$$\begin{aligned}\|\tilde{u}^n_{\rho(s,\tilde{u}_s^n)} - \tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}} &= \|\varphi_{\rho(s,\tilde{u}_s^n)} - \varphi_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}}, \text{ by Remark 3.1}\\ &\le \eta(t)\|\varphi - \varphi\|, \text{ by } (\mathbf{C}_3) \text{ with } t<0.\end{aligned}$$

Which proves that $\tilde{u}^n_{\rho(s,\tilde{u}_s^n)} \to \tilde{u}_{\rho(s,\tilde{u}_s)}$ in $\mathcal{B}$ as $n \to +\infty$ for every $s \in [0,a]$ such that $\rho(s,\tilde{u}_s) < 0$. Then

$$\lim_{n\to\infty} \tilde{u}^n_{\rho(s,\tilde{u}_s^n)} = \tilde{u}_{\rho(s,\tilde{u}_s)}.$$

For $t \in [0,b]$, we have :

$$\begin{aligned}\|(\mathcal{K}u^n)(t) - (\mathcal{K}u)(t)\| &= \left\| \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,s)F(s,\tilde{u}^n_{\rho(s,\tilde{u}_s^n)})ds - \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds \right\|\\ &= \left\| \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,s)(F(s,\tilde{u}^n_{\rho(s,\tilde{u}_s^n)}) - F(s,\tilde{u}_{\rho(s,\tilde{u}_s)}))ds \right\|\\ &\le Me^{\omega b} \int_0^t \left\| F(s,\tilde{u}^n_{\rho(s,\tilde{u}_s^n)}) - F(s,\tilde{u}_{\rho(s,\tilde{u}_s)}) \right\| ds.\end{aligned}$$

As $\lim_{n\to\infty} \tilde{u}^n_{\rho(s,\tilde{u}_s^n)} = \tilde{u}_{\rho(s,\tilde{u}_s)}$, $F(s,\cdot)$ is continuous from assumption $(\mathbf{C}_2) - (b)$, then $\left(F(s,\tilde{u}^n_{\rho(s,\tilde{u}_s^n)})\right)_{n\in\mathbb{N}}$ converges to $F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})$, and from assumption $(\mathbf{C}_2) - (c)$ we can conclude by the Lebesgue Dominated Convergence Theorem that $\mathcal{K}u^n \to \mathcal{K}u$.

Next, we will show now that the range of $\mathcal{K}$; $Range(\mathcal{K}) := \{\mathcal{K}u, u \in \mathbb{F}_a\}$, is relatively compact in $\mathbb{F}_a$. By the Arzela–Ascoli theorem, it suffices to prove that $Range(\mathcal{K})(t)$ is relatively compact in X for each $t \in [0,a]$, and $Range(\mathcal{K})$ is equicontinuous on $[0,a]$.

Step 3. The set of fonctions $Range(\mathcal{K})(t)$ of is relatively compact on $\mathbb{F}_a$. To prove this assertion, it is sufficient to show that the set $\left\{(\mathcal{K}u)(t) - U(t,0)\varphi(0) :\ u \in \mathbb{F}_a\right\}$ is relatively compact.

Let $0 < \epsilon < t \le a$. Then

$$\begin{aligned}\mathcal{K}u(t) - U(t,0)\varphi(0) &= \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\\ &= \lim_{\lambda\to 0^+} \int_0^{t-\epsilon} U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds + \lim_{\lambda\to 0^+} \int_{t-\epsilon}^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\\ &= U(t,t-\epsilon) \lim_{\lambda\to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds + \lim_{\lambda\to 0^+} \int_{t-\epsilon}^t U_\lambda(t,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds.\end{aligned}$$

We claim that

$$\left\{ \lim_{\lambda\to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds :\ \ u \in \mathbb{F}_a \right\}$$

is a bounded. In fact, for $u \in \mathbb{F}_a$:

$$\Big\| \lim_{\lambda\to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \Big\| \leq Me^{\omega a} \int_0^{t-\epsilon} p(s)V(\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}})ds$$

$$\leq Me^{\omega a}V(l) \int_0^{t-\epsilon} p(s)ds \text{ by Lemma 3.4.}$$

Where $l = M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr$. Since $U(t, t-\epsilon)$ is a compact operator for $0 < \epsilon < t$, the set

$$U(t, t-\epsilon)\Big\{ \lim_{\lambda\to 0^+} \int_0^{t-\epsilon} U_\lambda(t-\epsilon, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds : \ \ u \in \mathbb{F}_a \Big\}$$

is relatively compact in X for every ϵ, $0 < \epsilon < t$. We know that,

$$\Big\| \lim_{\lambda\to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \Big\| \leq Me^{\omega a} \int_{t-\epsilon}^{t} p(s)V(\|\tilde{u}_{\rho(s,\tilde{u}_s)}\|_{\mathcal{B}})ds$$

$$\leq Me^{\omega a}V(l) \int_{t-\epsilon}^{t} p(s)ds \text{ by Lemma 3.4.}$$

Where $l = M_b + \zeta + K_b + K_bH)\|\varphi\|_{\mathcal{B}} + K_bHr$. Thus

$$\lim_{\lambda\to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \in B\Big(0, Me^{\omega a}V(l) \int_{t-\epsilon}^{t} p(s)ds\Big).$$

By Lemma 2.1 it follows that

$$\alpha\Big(B\Big(0, Me^{\omega a}V(l) \int_{t-\epsilon}^{t} p(s)ds\Big)\Big) = 2Me^{\omega a}V(l) \int_{t-\epsilon}^{t} p(s)ds. \tag{3.4}$$

where $\alpha(\cdot)$ is Kuratowski's measure of noncompactness of sets in X. Letting ϵ tends to 0 , we obtain in relation (3.4) that $\alpha\Big(B\Big(0, Me^{\omega a}V(l) \int_{t-\epsilon}^{t} p(s)ds\Big)\Big) = 0$. By Lemma 2.1,

$$\Big\{ \lim_{\lambda\to 0^+} \int_{t-\epsilon}^{t} U_\lambda(t, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds : \ u \in \mathbb{F}_a \Big\}$$

is relatively compact. Then

$$\Big\{(\mathcal{K}u)(t) - U(t, 0)\varphi(0) : \ u \in \mathbb{F}_a \Big\}$$

is relatively compact. Hence, $Range(\mathcal{K})(t)$ is relatively compact in X for each $t \in J$.

Step 4. The set of fonctions $Range(\mathcal{K})$ is equicontinuous on $[0, a]$. For every $0 \leq t_0 \leq t \leq a$, one has:

$$\begin{aligned}(\mathcal{K}u)(t) - (\mathcal{K}u)(t_0) &= \Big(U(t, 0) - U(t_0, 0)\Big)\varphi(0) + \lim_{\lambda\to 0^+} \int_0^{t} U_\lambda(t, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \\ &\quad - \lim_{\lambda\to 0^+} \int_0^{t_0} U_\lambda(t_0, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \\ &= \Big(U(t, 0) - U(t_0, 0)\Big)\varphi(0) + \lim_{\lambda\to 0^+} \int_{t_0}^{t} U_\lambda(t, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds \\ &\quad + \Big(U(t, t_0) - I\Big) \lim_{\lambda\to 0^+} \int_0^{t_0} U_\lambda(t_0, s)F(s, \tilde{u}_{\rho(s,\tilde{u}_s)})ds.\end{aligned}$$

This implies that

$$\|(\mathcal{K}u)(t)-(\mathcal{K}u)(t_0)\| \le \Big\|\Big(U(t,0)-U(t_0,0)\Big)\varphi(0)\Big\| + Me^{\omega b}V(l)\int_{t_0}^{t} p(s)ds + \Big\|\Big(U(t,t_0)-I\Big)\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds\Big\|.$$

Since $Range(\mathcal{K})(t_0)$ is relatively compact and

$$\Big\{\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds : \ u\in \mathbb{F}_a\Big\} \subseteq Range(\mathcal{K})(t_0)$$

. There exists a compact set G such that:

$$\Big\{\lim_{\lambda\to 0^+}\int_0^{t_0} U_\lambda(t_0,s)F(s,\tilde{u}_{\rho(s,\tilde{u}_s)})ds : \ u\in \mathbb{F}_a\Big\} \subseteq G.$$

Then

$$\lim_{\substack{t\to t_0\\ t>t_0}} \sup_{u\in G}\Big\|\Big(U(t,t_0)-I\Big)u\Big\| = 0.$$

Thus, we get

$$\lim_{\substack{t\to t_0\\ t>t_0}} \|(\mathcal{K}u)(t)-(\mathcal{K}u)(t_0)\| = 0 \ \text{ for all } u\in\mathbb{F}_a.$$

Using similar argument for $0\le t\le t_0\le b$, we conclude that $Range(\mathcal{K})$ is equicontinuous. Then by Arzelá-Ascoli's Theorem, $Range(\mathcal{K})$ is retlatively compact. Since $\mathcal{K}$ is continuous by Step 2, we can conclude that $\mathcal{K}$ is a completely continuous operator. The existence of at least one a mild solution for Equation (1.1) is now a consequence of the variant of Schauder's Fixed Point Theorem. □

Theorem 3.5. *Let $(\boldsymbol{C}_1)-(\boldsymbol{C}_3)$ be satisfied. If $\rho(t,\psi)\le t$ for every $(t,\psi)\in J\times\mathcal{B}$ and*

$$MK_b e^{\omega b}\|p\|_{L^1} < \int_N^\infty \frac{ds}{V(s)} \tag{3.5}$$

where $N=(M_b+\zeta)\|\varphi\|_{\mathcal{B}}+K_b\|\varphi(0)\|_X$ with $K_b=\sup_{t\in J} k(t)$, $M_b=\sup_{t\in J} M(t)$, $\zeta:=\sup\{\eta(s): s\in\rho^-\}$. Then there exists a mild solution of Equation (1.1).

Proof. Let $E=C(J,X)$ and $\mathcal{K}: E\to E$ be the operator defined by (3.3). In order to use Leray Schauder Alternative Theorem. We claim that the set

$$\xi := \Big\{x\in C(J,X): \ x=\mu\mathcal{K}(x), \ 0<\mu<1\Big\} \text{ is bounded. Indeed}$$

$$\begin{aligned}
\|x\| &\le Me^{\omega t}\|\varphi(0)\| + \int_0^t Me^{\omega(t-s)}\|F(s,\tilde{x}_{\rho(s,\tilde{x_s})})\|ds \\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + M\int_0^t e^{\omega(t-s)}p(s)V(\|\tilde{x}_{\rho(s,\tilde{x_s})}\|_{\mathcal{B}})ds \quad \text{by } (\mathbf{B}_1)-(ii)' \text{ and } (\mathbf{C}_2) \\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\sup_{0\le\theta\le\rho(s,\tilde{x_s})}\|\tilde{x}(\theta)\|\Big)ds \quad \text{by Lemma 3.3} \\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_{\rho(s,\tilde{x_s})}\Big)ds \\
&\le Me^{\omega t}H\|\varphi\|_{\mathcal{B}} + Me^{\omega b}\int_0^t p(s)V\Big((M_b+\zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_s\Big)ds
\end{aligned}$$

since $\rho(t, \tilde{x_t}) \leq t$ for every $t \in J$. If

$$\vartheta(t) := (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t,$$

we obtain that

$$\vartheta(t) \leq (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\vartheta(s))ds$$

since $\|x\|_t \leqslant \|x\|$ for all $t \in J$. Setting

$$\nu(t) := (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\vartheta(s))ds$$

and using the nondecreasing character of V, we have :

$$\nu(t) \leq (M_b + \zeta + K_b M e^{\omega b} H)\|\varphi\|_{\mathcal{B}} + MK_b e^{\omega b}\int_0^t p(s)V(\nu(s))ds$$

since $\vartheta(t) \leqslant \nu(t)$ for every $t \in J$. Since ν is defferentiable, we have

$$\nu'(t) \leq MK_b e^{\omega b} p(t)V(\nu(t)) \text{ for every } t \in J.$$

Thus

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} \leq MK_b e^{\omega b}\int_0^t p(s)ds.$$

Hence

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} \leq MK_b e^{\omega b}\|p\|_{L^1}.$$

Using relation (3.5), we get

$$\int_{\nu(0)=N}^{\nu(t)} \frac{ds}{V(s)} < \int_N^{+\infty} \frac{ds}{V(s)}.$$

This implies that, the set of functions $\{\nu(\cdot) : 0 < \mu < 1\}$ is bounded in $C(J : X)$. Thus the set $\{x(\cdot) : 0 < \mu < 1\}$ is also bounded in $C(J : X)$ since

$$(M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t \leq \nu(t) \text{ for all } t \in J.$$

We obtain the completely continuous property of $\mathcal{K}$ by proceeding as in the proof of Theorem 3.4. Since E is convex and $0 \in E$, then the Nonlinear Alternative Leray-Schauder's Fixed Point Theorem guaranties the existence of at least one mild solution for Equation (1.1).

Arguing as in the proof of Theorem 3.2 we can prove that $\mathcal{K}$ is completely continuous. Then by the Nonlinear Alternative Leray-Schauder's Fixed Point Theorem the exists at least one mild solution for Equation (1.1). □

4. Global existence of mild solutions and Blowing up phenomena

Let us give the following local Lipschitz condition on the nonlinear part F of Equation (1.1):

(**C**$_4$) For each $\alpha > 0$ there exists a positive constant $r_0(\alpha)$ such that for $\varphi, \psi \in \mathcal{B}$ with $|\varphi|_{\mathcal{B}}, |\psi|_{\mathcal{B}} \leq \alpha$, we have:

$$\|F(t, \varphi) - F(t, \psi)\| \leq r_0(\alpha)|\varphi - \psi|_{\mathcal{B}} \text{ for } t \geq 0.$$

Contrarily to the previous results, if we replace conditions ((**C**$_2$) by condition (**C**$_4$), the following local existence results hold.

Theorem 4.1. *Assume that (*$\boldsymbol{C}_1$*), (*$\boldsymbol{C}_3$*) and (*$\boldsymbol{C}_4$*) hold. Then, for* $\varphi \in \mathcal{B}$ *such that* $\varphi(0) \in \overline{D}$*, Equation (1.1) has a mild solution* $x(.,\varphi)$ *in a maximal interval* $(-\infty, a_{max})$ *and either*

$$a_{max} = +\infty \quad or \quad \limsup_{t\to a_{max}^-} \|x(t,\varphi)\| = +\infty.$$

Moreover, $x(.,\varphi)$ *depends continuously on the initial data* φ *in the sense that, if* $\varphi \in \mathcal{B}$, $\varphi(0) \in \overline{D}$ *and* $t \in [0, a_{max})$*, then there exist positive constants* k *and* $\varepsilon > 0$ *such that, for* $\psi \in \mathcal{B}$ *and* $|\varphi - \psi|_{\mathcal{B}} \leq \varepsilon$*, we have*

$$\|x(s,\varphi) - x(s,\psi)\| \leq k|\varphi - \psi|_{\mathcal{B}} \ \ for\ s \in]-\infty, a].$$

Proof. Let $x(.,\varphi)$ be a mild solution of Equation (1.1) in $(-\infty, b]$. We know that, $x(t) \in \overline{D}$ for all $t \in [0,a]$. Repeating the procedure used in the local existence result, this yields existence of $a > a_1$ and a function $x(., x_a(.,\varphi)) : (-\infty, a_1] \to X$ which satisfies for $t \in [a, a_1]$:

$$x(., x_a(.,\varphi)) = U(t,0)x(a,\varphi) + \lim_{\lambda\to 0^+} \int_a^t U_\lambda(t,s)F(s, \tilde{x}_{\rho(s,\tilde{x}_s)}(., x_a(.,\varphi)))ds.$$

Proceeding inductively, we obtain the maximal interval of existence $(-\infty, a_{\max})$ of the solution $x(.,\varphi)$. Assume that $a_{\max} < +\infty$ and $\lim_{t\to a_{\max}^-} \sup \|x(t,\varphi)\| < M$. We claim that $x(.,\varphi)$ is uniformly continuous and consequently $\lim_{t\to a_{\max}^-} x(.,\varphi)$ exists in X, which contradicts the maximality of $[0, a_{max}[$. In the following, we show uniform continuity of $x(.,\varphi)$. Let $t, t+h \in [0, a_{max})$ with $h > 0$. Then,

$$\begin{aligned}
&\|x(t+h,\varphi) - x(t,\varphi)\| \\
&\leq \|U(t+h,0)\varphi(0) - U(t,0)\varphi(0)\| \\
&+ \Big\| \lim_{\lambda\to 0^+} \int_0^{t+h} U_\lambda(t+h,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau - \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \Big\| \\
&\leq \|U(t+h,0)\varphi(0) - U(t,0)\varphi(0)\| + \Big\| U_\lambda(t+h,t) \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \\
&+ \lim_{\lambda\to 0^+} \int_t^{t+h} U_\lambda(t+h,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau - \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \Big\| \\
&\leq \|(U(t+h,0) - U(t,0))\varphi(0)\| + \Big\| (U(t+h,t) - I) \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \Big\| \\
&+ \lim_{\lambda\to 0^+} \Big\| \int_t^{t+h} U_\lambda(t+h,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \Big\|.
\end{aligned}$$

Since $\mathcal{W} := \Big\{ \int_0^t U_\lambda(t,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau : \ x \in \mathbb{F}_a \Big\} \subseteq G$ with G compact. We obtain that

$$\lim_{\substack{h\to 0\\ t+h>t}} \|\left(U(t+h,t) - I\right)x\| = 0 \quad \text{for } x \in \mathcal{W}.$$

Since

$$\lim_{\lambda\to 0^+} \Big\| \int_t^{t+h} U_\lambda(t+h,\tau)F(\tau, \tilde{x}_{\rho(\tau,\tilde{x}_\tau)})d\tau \Big\| \leq Me^{wh}\|p\|_{L^1}V(l)h,$$

then

$$\lim_{\substack{h\to 0\\ t+h>t}} \|x(t+h,\varphi) - x(h,\varphi)\| = 0.$$

Similarly, we show that

$$\lim_{\substack{h\to 0\\ t+h<t}} \|x(t+h,\varphi) - x(t,\varphi)\| = 0.$$

Then $x(.,\varphi)$ is uniformly continuous on $[0,a_{\max})$ and therefore, $\lim_{t\to a_{\max}} x(.,\varphi)$ exists. If we define $x(a_{\max},\varphi) := \lim_{t\to a_{\max}} x(.,\varphi)$, we can extend $x(.,\varphi)$ beyond $a_{\max}$ which contradict the maximality of $]-\infty,a_{\max})$.

We prove now that $\mathcal{K}$ is strict contraction in $\mathbb{F}_a(\varphi)$ and for this end, we consider $x,z\in\mathbb{F}_a(\varphi)$. For $t\in[0,a]$, we have

$$\|(\mathcal{K}x)-(\mathcal{K}z)\|_{\mathbb{F}_a}=\sup_{0\le t\le b}\|(\mathcal{K}x)(t)-(\mathcal{K}z)(t)\|$$

and

$$\begin{aligned}\|(\mathcal{K}x)-(\mathcal{K}z)\|_{\mathbb{F}_a}&=\Big\|\lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{x}_{\rho(s,\tilde{x}_s)})ds-\lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)F(s,\tilde{z}_{\rho(s,\tilde{z}_s)})ds\Big\|\\&\le\int_0^t Me^{w(t-s)}\|F(s,\tilde{x}_{\rho(s,\tilde{x}_s)})-F(s,\tilde{z}_{\rho(s,\tilde{z}_s)})\|ds\\&\le Me^{wb}r_0(\alpha)\int_0^t\|\tilde{x}_{\rho(s,\tilde{x}_s)}-\tilde{z}_{\rho(s,\tilde{z}_s)}\|_{\mathcal{B}}ds\\&\le K_bMe^{wb}r_0(\alpha)\int_0^t\sup_{0\le\theta\le\rho(s,\tilde{x}_s)}\|x(\theta)-z(\theta)\|_X ds\\&\le K_bMe^{wb}r_0(\alpha)a\|x-z\|_{\mathbb{F}_a}.\end{aligned}$$

Following the same reasoning, we can see that

$$\begin{aligned}\|(\mathcal{K}^2x)(t)-(\mathcal{K}^2z)(t)\|_{\mathbb{F}_a}&\le K_bMe^{wb}r_0(\alpha)\int_0^t\sup_{0\le\theta\le\rho(s,\tilde{x}_s)}\|(\mathcal{K}x)(\theta)-(\mathcal{K}z)(\theta)\|_X ds\\&\le(K_bMe^{wb}r_0(\alpha))^2\int_0^t\sup_{0\le\theta\le s}\int_0^\theta\sup_{0\le\xi\le p}\|x(\xi)-z(\xi)\|_X dpds\\&\le(K_bMe^{wb}r_0(\alpha))^2\int_0^t\int_0^s\|x-z\|_{\mathbb{F}_a}dpds\\&\le\frac{(K_bMe^{wb}r_0(\alpha))^2a^2}{2}\|x-z\|_{\mathbb{F}_a}.\end{aligned}$$

We can repeat the previous argument, and we obtain

$$\|(\mathcal{K}^nx)(t)-(\mathcal{K}^nz)(t)\|_{\mathbb{F}_a}\le\frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}\|x-z\|_{\mathbb{F}_a}.$$

Since $\frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}\to0$ as $n\to+\infty$ then $\exists n\in\mathbb{N}$ such that $\frac{(K_bMe^{wb}r_0(\alpha))^na^n}{n!}<1$. It follows that $\mathcal{K}^n$ is strict contraction and by the Banach fixe point theorem, we deduce there $\exists!x\in\mathbb{F}_a$ such that $\mathcal{K}^nx=x$. Thus $\mathcal{K}^nx=x$ implies that $\mathcal{K}^{n+1}x=\mathcal{K}x$ on the other hand $\mathcal{K}^n(\mathcal{K}x)=\mathcal{K}(x)$ it follows that $\mathcal{K}(x)$ is a fixed point of $\mathcal{K}^n$ and since fixed point is unique then we get $\mathcal{K}(x)=x$. Equation (1.1) has a unique mild solution $x(.,\varphi)$ which is defined on the interval $(-\infty,a]$. This is true for all $a>0$, then $x(.,\varphi)$ is a global solution of Equation (1.1) on $\mathbb{R}$.

Next, we prove that the solution depends continuously on initial data. Let $\varphi\in\mathcal{B}$ and $t\in[0,a[$ be fixed. Show that $x(\cdot,\varphi)$ is continuous in the sense of φ. $\forall\varepsilon>0$, look for $k(a)>0$ such that for $\psi\in\mathcal{B}$ and $|\varphi-\psi|_{\mathcal{B}}\le\varepsilon$ implies that

$$\|x(\iota,\varphi)-x(\iota,\psi)\|\le k(a)|\varphi-\psi|_{\mathcal{B}}\ \text{ for }\iota\in]-\infty,a].$$

We have by Lemme 3.3

$$\begin{aligned}
|x_s(\cdot,\varphi)-x_s(\cdot,\psi)|_{\mathcal{B}} &\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|x(\theta,\varphi)-x_s(\theta,\psi)\|_X, \qquad s\in\rho^-\cup J\\
&\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|U(\theta,0)(\varphi-\psi)\|_X\\
&+K_b\sup_{0\leq\theta\leq s}\lim_{\lambda\to 0^+}\int_0^{\theta}\|U(\theta,\tau)F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi))-F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi))\|d\tau\\
&\leq (M_b+\zeta)\,|\varphi-\psi|_{\mathcal{B}}+K_b\sup_{0\leq\theta\leq s}\|U(\theta,0)(\varphi-\psi)\|_X\\
&+K_b\sup_{0\leq\theta\leq s}\lim_{\lambda\to 0^+}\int_0^{\theta}Me^{w(\theta-\tau)}\|F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi))-F(\tau,\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi))\|d\tau\\
&\leq (M_b+\zeta+HK_bMe^{wa})\,|\varphi-\psi|_{\mathcal{B}}\\
&+K_bMe^{wa}r_0(\alpha)\int_0^{\theta}\|\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\varphi)-\tilde{x}_{\rho(\tau,\tilde{x}_\tau)}(\cdot,\psi)\|d\tau\\
&\text{using the Bellman-Gronwall Lemma it follows that}\\
&\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}|\varphi-\psi|_{\mathcal{B}}.
\end{aligned}$$

Hence we can write

$$|x_s(\vartheta,\varphi)-x_s(\vartheta,\psi)|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}|\varphi-\psi|_{\mathcal{B}}\text{ for }\vartheta\in]-\infty,0]$$

thus

$$|x(s+\vartheta,\varphi)-x(s+\vartheta,\psi)|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}\|\varphi-\psi\|_{\mathcal{B}}\text{ for }\vartheta\in]-\infty,0]$$

therefore

$$\|x(\iota,\varphi)-x(\iota,\psi)\|_{\mathcal{B}}\leq (M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}\|\varphi-\psi\|_{\mathcal{B}}\text{ for }\iota\in]-\infty,a]$$

It is clear that $(M_b+\zeta+HK_bMe^{wa})\,e^{K_bMe^{wa}r_0(\alpha)\theta}>0$ hence, we deduce the continuous dependence on the initial data. □

Corollary 4.1. *Assume that (**C**$_4$) holds. Let q_1 and q_2 be continuous fonctions from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that*

$$\|F(t,\phi)\|\leq q_1(t)|\phi|_{\mathcal{B}}+q_2(t)\quad \text{for } t\in\mathbb{R}^+ \text{ and } \phi\in\mathcal{B}.$$

Then, for $\phi\in\mathcal{B}$ such that $\phi(0)\in\overline{D}$, Equation (1.1) has a unique mild solution which is defined on $\mathbb{R}$.

Proof. Let $x(\cdot,\phi)$ the solution of Equation (1.1) defined on a maximal interval $(-\infty,a_{max})$. Then by the Theorem 4.1

$$a_{max}=+\infty\ \text{ or }\ \limsup_{t\to a_{max}^-}\|x(t,\varphi)\|=+\infty.$$

We assume that $a_{max}<+\infty$ and $\limsup_{t\to a_{max}^-}\|x(t,\varphi)\|=+\infty$.

For all $t\in[0,a_{max}[$:

On has

$$\|x(t,\phi)\| \le \|U(t,0)\|\|\phi(0)\| + \lim_{\lambda\to 0^+}\int_0^t \|U_\lambda(t,s)\|\|F(s,\tilde{x}_{\rho(s,\tilde{x_s})})\|ds$$
$$\le Me^{\omega t}\|\phi(0)\| + \int_0^t Me^{\omega t}\Big(q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}} + q_2(t)\Big)ds$$
$$\le Me^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^t q_2(\theta)d\theta\Big) + Me^{\omega a_{max}}\int_0^t q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds$$

By Lemma 3.3

$$|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} \le (M_b+\zeta)\,\|\phi\|_{\mathcal{B}} + K_b \sup_{0\le\theta\le_{\rho(s,\tilde{x_s})}}\|x(\theta,\phi)\|$$

.

Thus

$$|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} \le (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_b \sup_{0\le\theta\le_{\rho(s,\tilde{x_s})}}\Big[M_be^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^t q_2(\theta)d\theta\Big)$$
$$+ M_be^{\omega a_{max}}\int_0^t q_1(t)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds\Big]$$
$$\le (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_bM_be^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^{\rho(s,\tilde{x_s})} q_2(\theta)d\theta\Big)$$
$$+ M_be^{\omega a_{max}}\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds$$
$$= P_1 + P_1\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)|\tilde{x}_{\rho(s,\tilde{x_s})}|_{\mathcal{B}}ds.$$

With $P_1 = (M_b+\zeta)\,|\phi|_{\mathcal{B}} + K_bM_be^{\omega a_{max}}\Big(\|\phi(0)\| + \int_0^{\rho(s,\tilde{x_s})} q_2(\theta)d\theta\Big)$ and $P_1 = M_be^{\omega a_{max}}$.
By Gronwall's Lemma, we deduce that

$$|\tilde{x}_{\rho(s,\tilde{x_s})}(\cdot,\phi)|_{\mathcal{B}} \le P_1e^{a_{max}P_2\int_0^{\rho(s,\tilde{x_s})} q_1(\theta)d\theta}.$$

Hence $\limsup_{t\to a_{max}^-}\|x(t,\varphi)\| < +\infty$. Therefore $a_{max} = +\infty$

■

5. Application

For illustration of our previous result, we propose to study the following model.

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \delta(t)\dfrac{\partial^2}{\partial x^2}v(t,x) + \beta(t)\displaystyle\int_{-\infty}^0 g\Big(\theta, v\Big(\theta+t-\rho_1(t)\rho_2\Big(\int_0^\pi w(s)|v(t,\theta)|^2ds\Big),x\Big)\Big)d\theta \\ \quad \text{for} \quad 0\le t\le b \text{ and } x\in[0,\pi], \\ v(t,0) = v(t,\pi) = 0 \quad \text{for} \quad 0\le t\le b, \\ v(\theta,x) = v_0(\theta,x) \quad \text{for} \quad \theta\le 0, 0\le x\le\pi, \end{cases} \tag{5.1}$$

where $\delta(\cdot)$ is a positive function in $C^1(\mathbb{R}^+,\mathbb{R}^+)$ with $\delta_0 := \inf_{t\geq 0}\delta(t) > 0$ and $\beta : [0,b] \to \mathbb{R}^+$ with $\beta \in L^1(J;[0,+\infty))$. $g : \mathbb{R}^- \times \mathcal{B} \to \mathbb{R}$ and $v_0 : (-\infty,0]\times[0,\pi] \to \mathbb{R}$ are functions. The functions $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$ are continuous and $w : \mathbb{R} \to \mathbb{R}$ is a positive continuous function. To rewrite Eq. (5.1) in the abstract form, we introduce the space $X := C([0,\pi],\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the uniform norm topology and we consider the operator $A : D \subset X \to X$ defined by:

$$\begin{cases} D = \{z \in C^2([0,\pi]) : z(0) = z(\pi) = 0\} \\ \\ Az(t,x) = \Delta z(t,x) \ \text{ with } \Delta := \dfrac{\partial^2}{\partial x^2}; \ \ t \in [0,b] \ \text{ and } \ x \in [0,\pi]. \end{cases}$$

Then it is well know that

$$\begin{cases} \overline{D} = \{z \in C([0,\pi] : \mathbb{R}) : z(0) = z(\pi) = 0\} \neq X, \\ \\ (0,+\infty) \subset \rho(A) \quad \text{and} \quad \|R(\lambda,A)\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0. \end{cases} \tag{5.2}$$

We choose the space of bounded uniformly continuous functions from $\mathbb{R}^-$ to X denoted by $BUC(\mathbb{R}^-,X)$ as a phase space $\mathcal{B} := BUC(\mathbb{R}^-,X)$ endowed with the following norm:

$$\|\psi\|_{\mathcal{B}} := \sup_{\theta\leq 0}\|\psi(\theta)\|.$$

Then, $\mathcal{B}$ satisfies Axioms $(\mathbf{B}_1) - (\mathbf{B})$.

By defining the operators $F : I \times \mathcal{B} \to X$ and $\tau : I \times \mathcal{B} \to \mathbb{R}$ by:

$y(t)(x) := v(t,x)$.

$\varphi(\theta)(x) := v_0(\theta,x)$ for $\theta \leq 0$.

$$F(t,\phi)(x) := \beta(t)\int_{-\infty}^{0} g\Big(\theta,\phi(\theta)(x)\Big)d\theta.$$

$$\tau(t,\phi) := t - \rho_1(t)\rho_2\Big(\int_0^{\pi} w(s)|\phi(0)(x)|^2 ds\Big).$$

Suppose that $\phi\mathcal{B}$ and let $(A(t))_{t\geq 0}$ be the family of operators defined by $A(t) := \delta(t)\dfrac{\partial^2}{\partial x^2}$. Then, Equation (5.1) takes the following abstract form :

$$\begin{cases} \dot{y}(t) = A(t)y(t) + F(t, y_{\tau(t,y_t)}) \quad \text{for} \quad t \in [0,b], \\ \\ y_0 = \varphi \in \mathcal{B}, \end{cases} \tag{5.3}$$

We have $D(A(t)) = D$ independent of t and for $\lambda > 0$,

$$\begin{aligned} R(\lambda, A(t)) &= (\lambda I - \delta(t)A)^{-1} \\ &= \frac{1}{\delta(t)}R\Big(\frac{\lambda}{\delta(t)}, A\Big). \end{aligned} \tag{5.4}$$

Using (5.2) and (5.4), we have for every $\lambda > 0$, $\lambda \in \rho(A(t))$ and $\|R(\lambda,A(t))\| \leq \frac{1}{\lambda}$.

Then $(0,+\infty) \subset \rho(A(t))$ and

$$\Big\|\prod_{i=1}^{k} R(\lambda, A(t_i))\Big\| \leq \frac{1}{\lambda^k}, \ \ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < +\infty.$$

Hence, the family of linear operators $(A(t))_{t\geq 0}$ on X satisfies the assumptions ($\mathbf{A}_1$) - ($\mathbf{A}_3$).
It is known from [10] that, the part Δ_0 of $\Delta = \dfrac{\partial^2}{\partial x^2}$ in $\overline{D(\Delta)}$ given by

$$\begin{cases} D(\Delta_0) = \Big\{ z \in D(\Delta) : \Delta z \in \overline{D(\Delta)} \Big\} \\ \\ \Delta_0 z = \Delta z, \end{cases} \tag{5.5}$$

generates a compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(\Delta)}$ such that

$$\|T_0(t)\| \leq e^{-t} \quad \text{for} \quad t \geq 0. \tag{5.6}$$

Thus, the part $A_0(.)$ of $A(.)$ in $\overline{D}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on $\overline{D}$ given by

$$U(t,s) = T_0\left(\int_s^t \delta(\tau)d\tau\right)$$

which is compact for $t > s$. By (5.6), one has

$$\|U(t,s)\| \leq e^{-\delta_0(t-s)}.$$

Hence ($\mathbf{C}_1$) is satisfies. We assume that:

1) $g : \mathbb{R}^- \times \mathcal{B} \to \mathbb{R}^+$ is nondecreasing integrable function which satisfies : $g(\theta, 0) = 0$ for $\theta \leq 0$.

2) v_0 is uniformly continuous and bounded with respect to $\theta \in \mathbb{R}^-$, uniformly with respect to $x[0,\pi]$.

Under the above conditions, we claim that $\varphi \in \mathcal{B}$. In fact,

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\leq 0}\|\varphi(\theta)\| = \sup_{\substack{\theta\leq 0 \\ x\in[0,\pi]}} \|v_0(\theta,x)\| < +\infty.$$

and

$$\begin{aligned} \|\varphi(\theta) - \varphi(\theta')\| &= \sup_{x\in[0,\pi]} \|\varphi(\theta)(x) - \varphi(\theta')(x)\| \\ &= \sup_{x\in[0,\pi]} \|v_0(\theta,x) - v_0(\theta',x)\| \to 0 \text{ as } \|\theta-\theta'\| \to 0. \end{aligned}$$

Therefore, $\varphi \in \mathcal{B}$ with $\varphi(0) \in \overline{D}$.

On the other hand, we have:

$$\|F(t,\phi)\| \leq \beta(t)\int_{-\infty}^0 \Big\|g\Big(\theta, \phi(\theta)\Big)\Big\| d\theta$$

for $\phi \in \mathcal{B}$. F satisfies ($\mathbf{C}_2$) with $p(t) = \beta(t)$ and $V(\|\phi\|_{\mathcal{B}}) = \displaystyle\int_{-\infty}^0 \Big\|g\Big(\theta, \phi(\theta)\Big)\Big\| d\theta.$

3) Let $\varphi \in \mathcal{B}$ such that $x_0 = \varphi$ and $t \longmapsto \varphi_t$ is a $\mathcal{B}$-valued. We assume that $\|\varphi_t\|_{\mathcal{B}} \leq \eta(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \tau^-$ where $\eta : \tau^- \to (0,\infty)$ is a continuous and bounded function with

$$\tau^- = \{\tau(s,\psi) : (s,\psi) \in J \times \mathcal{B}, \tau(s,\psi) \leq 0\}.$$

Hence ($\mathbf{C}_3$) is satisfies.

Then, the existence of mild solutions can be deduced from a direct application of Theorem 3.5 and we have the following result.

Theorem 5.1. *Assume* $\varphi(0) \in \overline{D}$ *and*

$$K_b e^{\delta_0 b} \|p\|_{L^1} < \int_N^{\infty} \frac{ds}{V(s)} \tag{5.7}$$

where $M = 1$, $\omega = \delta_0$, $N = (M_b + \zeta)\|\varphi\|_{\mathcal{B}} + K_b\|\varphi(0)\|$. *Then there exists at least one mild solution of Equation (5.1).*

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