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On the rational difference equation $x_{n+1} = \frac{x_n \cdot (\overline{a} x_{n-k} + a x_{n-k+1})}{b x_{n-k+1} + c x_{n-k}}$

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Abstract. This work studies an explicit and a constructive solution for the difference equation

$$x_{n+1} = \frac{x_n \cdot (\overline{a} x_{n-k} + a x_{n-k+1})}{b x_{n-k+1} + c x_{n-k}}, \quad n = 0, 1, \ldots,$$

where $\overline{a} \geq 0, a > 0, b > 0, c > 0$ and $k \geq 1$ is an integer, with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$. We also will determine the global behavior of this solution. For the case when $\overline{a} = 0$, the method presented here gives us the particular solution obtained by Gümüş and Abo-Zeid that establishes an inductive type of proof.

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Contents

1. Introduction

The study of rational difference equations currently represents a fruitful area of study that attracts many mathematical researchers. Many difference equations have been successfully used for modeling real phenomena [3, 5, 7].

In 2019 Abo-Zeid [1] published a study on the global behavior of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{\pm b x_{n-1} + c x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers, and obtained its general solution. Similarly, Abo-Zeid [2] also studied the solutions to

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_{n-2} + b x_{n-3}}, \quad n = 0, 1 \ldots,$$

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for a, b positive constants. Motivated by these results, in 2020 Gümüş and Abo-Zeid [4] found an explicit solution and studied the global behavior of the equation

$$x_{n+1}=\frac{ax_nx_{n-k+1}}{bx_{n-k+1}+cx_{n-k}},\quad n=0,1\ldots,$$

where a, b, c are positive constants and $k \geq 1$ is an integer.

In this work we will generalize the result found by Gümüş and Abo-Zeid by explicitly solving

$$x_{n+1}=\frac{x_n\cdot(\overline{a}x_{n-k}+ax_{n-k+1})}{(bx_{n-k+1}+cx_{n-k})},\quad n=0,1,\ldots, \tag{1.1}$$

where $\overline{a}\geq 0, a>0, b>0, c>0$ and $k\geq 1$ is an integer, with the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$.

2. Preliminaries

The Riccati difference equation is defined by

$$R_nR_{n-1}+A(n)R_n+B(n)R_{n-1}=C(n). \tag{2.1}$$

Following the ideas found in the book by Mickens [6, Chapter 6], we make the change of variable

$$R_n=\frac{Q_n-B(n)Q_{n+1}}{Q_{n+1}},$$

which transforms (2.1) into a linear second order equation of the form

$$(A(n)B(n)+C(n))Q_{n+1}+(B(n-1)-A(n))Q_n-Q_{n-1}=0.$$

In order to solve (1.1), the first step is to transform it into a Riccati equation. Indeed, (1.1) is equivalent to

$$bx_{n+1}x_{n-k+1}+cx_{n+1}x_{n-k}=\overline{a}x_nx_{n-k}+ax_nx_{n-k+1},$$

or

$$b\frac{x_{n+1}}{x_n}\cdot\frac{x_{n-k+1}}{x_{n-k}}+c\frac{x_{n+1}}{x_n}=\overline{a}+a\frac{x_{n-k+1}}{x_{n-k}}.$$

Upon applying the change of variable

$$y_n=\frac{x_{n+1}}{x_n}, \tag{2.2}$$

we have

$$y_ny_{n-k}+\frac{c}{b}y_n-\frac{a}{b}y_{n-k}=\frac{\overline{a}}{b}. \tag{2.3}$$

We can see here that the solution for y_n depends exclusively on what happens to y_{n-k} (that is, k steps before). Therefore, we can solve the Riccati equation

$$z_mz_{m-1}+\frac{c}{b}z_m-\frac{a}{b}z_{m-1}=\frac{\overline{a}}{b}, \tag{2.4}$$

with initial condition $z_{-1}:=y_{-k+i}$, where $y_{-k+i}=\frac{x_{-k+i+1}}{x_{-k+i}}$ for some $i=0,1,\ldots,k-1$ fixed (z_{-1} depends on i). It is evident that the solutions to (2.3) and (2.4) are related by

$$z_m=y_{mk+i}. \tag{2.5}$$

By making the change of variable

$$z_m = \frac{w_m + (a/b)w_{m+1}}{w_{m+1}},$$

equation (2.4) transforms into the homogeneous linear second order equation with constant coefficients

$$(\overline{a}b - ac)w_{m+1} - (a+c)bw_m - b^2 w_{m-1} = 0.$$

The roots of the characteristic polynomial associated to this last equation are given by

$$r_{2,1} := \frac{(a+c)b \pm b\sqrt{(a-c)^2 + 4\overline{a}b}}{2(\overline{a}b - ac)}. \tag{2.6}$$

Hence, the general solution of (2.4) is given by

$$z_m = \frac{(C_1 r_1^m + C_2 r_2^m) + (a/b)(C_1 r_1^{m+1} + C_2 r_2^{m+1})}{C_1 r_1^{m+1} + C_2 r_2^{m+1}}.$$

Making the change of variable $\overline{C}_i := C_2/C_1$, this becomes

$$z_m = \frac{(1 + \frac{a}{b} r_1)(\frac{r_1}{r_2})^m + \overline{C}_i (1 + \frac{a}{b} r_2)}{r_1 (\frac{r_1}{r_2})^m + \overline{C}_i r_2}. \tag{2.7}$$

With the initial condition z_{-1}, we obtain

$$\overline{C}_i = -\frac{r_2}{r_1} \cdot \frac{(b + ar_1 - br_1 z_{-1})}{(b + ar_2 - br_2 z_{-1})}. \tag{2.8}$$

Therefore, by means of recursive backward application of the changes of variable previously done, we obtain the explicit solution to (1.1), as shown in Theorem 3.1 below.

Remark 2.1. *In the particular case when $\overline{a} = 0$, we get $r_1 = -\frac{b}{c}$ and $r_2 = -\frac{b}{a}$, and thus we have*

$$z_m = \frac{1}{\frac{b}{a-c} + \overline{C} \cdot (\frac{c}{a})^m},$$

with $\overline{C} = \frac{c}{a}\left(\frac{a-c-bz_{-1}}{(a-c)z_{-1}}\right)$. By recursive backward application of the changes of variables previously done, we get

$$y_{mk+i} = \frac{a-c}{\left(\frac{a-c-bz_{-1}}{z_{-1}}\right)(\frac{c}{a})^{m+1} + b},$$

which implies that

$$x_{mk+i+1} = x_{mk+i} \cdot \left(\frac{a-c}{\frac{a-c-by_{-k+i}}{y_{-k+i}}(\frac{c}{a})^{m+1} + b}\right),$$

from which we can deduce the Gümüş and Abo-Zeid result in [4].

3. Solution to equation (1.1)

Since the case $\overline{a} = 0$ was already solved by Gümüs and Abo-Zeid [4], we can focus on the case $\overline{a} \neq 0$ and normalize this coefficient to obtain

$$x_{n+1} = \frac{x_n \cdot (x_{n-k} + ax_{n-k+1})}{bx_{n-k+1} + cx_{n-k}}, \quad n = 0, 1, \ldots \tag{3.1}$$

We also can assume that $b \neq ac$. Indeed, if $b = ac$, then (3.1) reduces to

$$x_{n+1} = \frac{x_n}{c},$$

which represents a simple case.

Observe that under these conditions, the roots r_1, r_2 in (2.6) are equal to

$$r_{2,1} = \frac{(a+c)b \pm b\sqrt{(a-c)^2 + 4b}}{2(b - ac)}. \tag{3.2}$$

Moreover, since $\left|(a+c) - \sqrt{(a-c)^2 + 4b}\right| < \left|(a+c) + \sqrt{(a-c)^2 + 4b}\right|$, these roots satisfy

$$\left|\frac{r_1}{r_2}\right| < 1.$$

We also note that $r_1 \neq 0, r_2 \neq 0$.

In order for the solution of (3.1) to be well defined, it is necessary to assume that the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ satisfy the following conditions:

$$(\mathrm{H}) : \begin{cases} 1)\ \ x_{-k}, \ldots, x_{-1} \text{ are non-zero.} \\ 2)\ \ b + ar_2 \neq br_2\left(\frac{x_{-k+i+1}}{x_{-k+i}}\right), \text{ for every } i = 0, 1, \ldots, k-1, \text{ where } r_2 \\ \ \text{is defined as in (3.2), and } b \neq ac. \\ 3)\ \ \left(\frac{r_1}{r_2}\right)^{j+1} \neq -\overline{C}_i \text{ for every integer } j \geq 0 \text{ and for every } i = 0, 1, \ldots, k-1, \\ \ \text{where } \overline{C}_i \text{ is defined as in (2.8), and } z_{-1} = \frac{x_{-k+i+1}}{x_{-k+i}}. \end{cases}$$

Theorem 3.1. *Consider the difference equation*

$$x_{n+1} = \frac{x_n \cdot (x_{n-k} + ax_{n-k+1})}{bx_{n-k+1} + cx_{n-k}},$$

with $a, b, c > 0$ *such that* $b \neq ac$*, and initial conditions* $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ *satisfying (H). Let* r_1 *and* r_2 *be defined as in* (3.2)*. Let us define the functions*

$$\beta_i(j) = \frac{(1 + \frac{a}{b}r_1)(\frac{r_1}{r_2})^j + \overline{C}_i(1 + \frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^j + \overline{C}_i r_2}, \tag{3.3}$$

with $\overline{C}_i$ *as in* (2.8)*. Then the solution to this equation is given by*

$$\begin{cases} x_{mk} = x_0 \displaystyle\prod_{j=0}^{m-1} \prod_{i=0}^{k-1} \beta_i(j) \\ x_{mk+1} = \beta_0(m) \cdot x_{mk} \\ x_{mk+2} = \beta_0(m)\beta_1(m) \cdot x_{mk} \\ \qquad \vdots \\ x_{mk+(k-1)} = \beta_0(m) \cdots \beta_{k-2}(m) \cdot x_{mk}. \end{cases}$$

for $m = 0, 1, 2, 3, \ldots$

Proof. From (2.5) and (2.7), we obtain

$$y_{mk+i} = \frac{(1 + \frac{a}{b}r_1)(\frac{r_1}{r_2})^m + \overline{C}_i(1 + \frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^m + \overline{C}_i r_2}.$$

Since we defined $y_n = \frac{x_{n+1}}{x_n}$ in (2.2), then

$$x_{mk+i+1} = x_{mk+i} \cdot \left(\frac{(1+\frac{a}{b}r_1)(\frac{r_1}{r_2})^m + \overline{C}_i(1+\frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^m + \overline{C}_i r_2} \right).$$

By applying this equality recursively for all non-negative integers m and k, and for $i = 0, 1, 2, 3, \ldots, k-1$, we immediately obtain the Theorem's result. ■

4. Asymptotic behavior of the solution to equation (3.1)

For the analysis of the global behavior of (3.1), let us consider the following additional conditions:

$$(\mathrm{H}_1) : \begin{cases} b + ar_1 \neq br_1 \left(\frac{x_{-k+i+1}}{x_{-k+i}} \right) \text{ for every } i = 0, 1, \ldots, k-1, \text{ where } r_1 \\ \text{is defined as in (3.2), and } b \neq ac. \end{cases}$$

$$(\mathrm{H}_2) : \begin{cases} \overline{C}_i \neq \frac{(1+\frac{a}{b}r_1)}{(1+\frac{a}{b}r_2)} \left(\frac{r_1}{r_2} \right)^j \text{ for all } i \text{ and for all } j \geq 0. \end{cases}$$

We can see that r_2, as given in (3.2) with $b \neq ac$, satisfies

$$\begin{aligned} \frac{1}{r_2} + \frac{a}{b} &= \frac{2(b-ac)}{b((a+c)+\sqrt{(a-c)^2+4b})} + \frac{a}{b} \\ &= \frac{\sqrt{(a-c)^2+4b}-(a+c)}{2b} + \frac{a}{b} = \frac{\sqrt{(a-c)^2+4b}+(a-c)}{2b}. \end{aligned}$$

We also see that $\frac{1}{r_2} + \frac{a}{b} > 0$. Moreover,

$$\begin{aligned} &\frac{1}{r_2} + \frac{a}{b} < 1 \Leftrightarrow \sqrt{(a-c)^2+4b} < 2b-(a-c) \Leftrightarrow 2b-(a-c) > 0 \quad \text{and} \\ &(2b-(a-c))^2 > (a-c)^2+4b \Leftrightarrow 2b-(a-c) > 0 \quad \text{and} \quad b-(a-c) > 1 \\ &\Leftrightarrow b-(a-c) > 0. \end{aligned}$$

From this, and in the same manner for the remaining cases, we have

a) $\frac{1}{r_2} + \frac{a}{b} < 1 \iff b-(a-c) > 1.$

b) $\frac{1}{r_2} + \frac{a}{b} > 1 \iff b-(a-c) < 1.$

c) $\frac{1}{r_2} + \frac{a}{b} = 1 \iff b-(a-c) = 1.$

Theorem 4.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be the solution to (3.1) such that the initial conditions $x_{-k}, \ldots, x_0$ satisfy (H) and (H$_1$). Then,*

1. *If $b-(a-c) > 1$, then $\{x_n\}_{n=-k}^{\infty}$ converges to 0.*
2. *If $b-(a-c) < 1$ and the initial conditions satisfy (H$_2$) as well, then $\{x_n\}_{n=-k}^{\infty}$ is unbounded.*
3. *If $b-(a-c) = 1$, then $\{x_n\}_{n=-k}^{\infty}$ converges to a finite limit.*

Proof. From conditions, we have $\overline{C}_i \neq 0$ for all i. On the other hand, since $|r_1/r_2| < 1$, it follows for all i that $\beta_i(j) \to \frac{1}{r_2} + \frac{a}{b}$ if $j \to \infty$, where $\beta_i(j)$ is as defined in (3.3).

1. If $b-(a-c)>1$, then $\frac{1}{r_2}+\frac{a}{b}<1$. Hence, there exist $0<\varepsilon<1$ and $j_0 \in \mathbb{N}$ such that $|\beta_i(j)|<\varepsilon$ for all $j \geq j_0$ and for all i. Then, for large enough values of m, we have

$$\begin{aligned} |x_{mk}| &= |x_0| \left| \prod_{j=0}^{j_0-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \left| \prod_{j=j_0}^{m-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \\ &< |x_0| \left| \prod_{j=0}^{j_0-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \cdot \varepsilon^{k(m-j_0)}. \end{aligned}$$

We conclude that as m tends to infinity, then x_{km} converges to 0. Moreover, for $i \in \{1,2,\ldots,k-1\}$, we have

$$x_{mk+i} = x_{mk} \cdot \left| \prod_{l=0}^{i-1} \beta_l(m) \right|.$$

Therefore, $\{x_n\}_{n=-k}^{\infty}$ tends to 0.

2. If $b-(a-c)<1$, then $\frac{1}{r_2}+\frac{a}{b}>1$. Hence, there exist $1<\varepsilon_1<\frac{1}{r_2}+\frac{a}{b}$ and $j_1 \in \mathbb{N}$ such that $\beta_i(j)>\varepsilon_1>1$ for all $j \geq j_1$ and for all i. Moreover, by condition (H_2), we have $\beta_i(j) \neq 0$ for all i and for all j. Then, for large enough values of m, we have

$$\begin{aligned} |x_{mk}| &= |x_0| \left| \prod_{j=0}^{j_1-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \left| \prod_{j=j_1}^{m-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \\ &> |x_0| \left| \prod_{j=0}^{j_1-1} \prod_{i=0}^{k-1} \beta_i(j) \right| \cdot \varepsilon_1^{k(m-j_1)}. \end{aligned}$$

We conclude that $|x_{km}| \to \infty$ when $m \to \infty$. Moreover, for $i \in \{1,2,\ldots,k-1\}$, we have

$$x_{mk+i} = x_{mk} \cdot \prod_{l=0}^{i-1} \beta_l(m).$$

Therefore, the solution set $\{x_n\}_{n=-k}^{\infty}$ is unbounded.

3. If $b-(a-c)=1$, then $\frac{1}{r_2}+\frac{a}{b}=1$. Hence, there exists $j_2 \in \mathbb{N}$ such that $\beta_i(j)>0$ for $j \geq j_2$ and for all i. Then, we have

$$\begin{aligned} x_{km} &= x_0 \left(\prod_{j=0}^{j_2-1} \prod_{i=0}^{k-1} \beta_i(j) \right) \left(\prod_{j=j_2}^{m-1} \prod_{i=0}^{k-1} \beta_i(j) \right) \\ &= x_0 \left(\prod_{j=0}^{j_2-1} \prod_{i=0}^{k-1} \beta_i(j) \right) \exp\left(\sum_{j=j_2}^{m-1} \sum_{i=0}^{k-1} \ln(\beta_i(j)) \right). \end{aligned}$$

Let us define

$$
\begin{aligned}
a_j &:= \sum_{i=0}^{k-1} \ln(\beta_i(j)) = \sum_{i=0}^{k-1} \ln\left(\frac{1+\frac{1}{\overline{C}_i r_2}\left(1+\frac{a}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j}{1+\frac{r_1}{\overline{C}_i r_2}\left(\frac{r_1}{r_2}\right)^j}\right) \\
&= \sum_{i=0}^{k-1}\left(\ln\left(1+\frac{1}{\overline{C}_i r_2}(1+\frac{a}{b}r_1)(\frac{r_1}{r_2})^j\right) - \ln\left(1+\frac{r_1}{\overline{C}_i r_2}(\frac{r_1}{r_2})^j\right)\right) \\
&= \sum_{i=0}^{k-1}\left(\left(\frac{1}{\overline{C}_i r_2}\left(1+\frac{a}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right)\right)\right. \\
&\left. - \left(\frac{r_1}{\overline{C}_i r_2}\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right)\right)\right) \\
&= \sum_{i=0}^{k-1}\left(\frac{1}{\overline{C}_i r_2}\left(1+\frac{a-b}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}((r_1/r_2)^{2j})\right) \\
&= \frac{1}{r_2}\left(1+\frac{a-b}{b}r_1\right)\left(\sum_{i=0}^{k-1}\frac{1}{\overline{C}_i}\right)\cdot\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right).
\end{aligned}
$$

Then, we have

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \left|\frac{r_1}{r_2}\right| < 1.$$

By D'Alembert's ratio test, the series $\sum_{j=j_2}^{\infty}\sum_{i=0}^{k-1}\ln(\beta_i(j))$ converges. Hence, there exists $v\in\mathbb{R}$ such that

$$\lim_{m\to\infty} x_{km} = v.$$

In the same way, for $i\in\{1,\dots,k-1\}$, we have

$$x_{mk+i} = x_{mk}\cdot\prod_{l=0}^{i-1}\beta_l(m)\to v \quad \text{when} \quad m\to\infty.$$

Therefore, the solution set $\{x_n\}_{n=-k}^{\infty}$ converges to a finite limit.

■

5. Numerical Results

Numerical simulations performed with MATLAB for the three cases stated in Theorem 4.1 are shown in the following examples.

Example 1. Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-4}+7.3x_{n-3})}{3.5x_{n-3}+5.8x_{n-4}}.$$

In this case we have $a=7.3$, $b=3.5$, $c=5.8$ and $k=4$. Also, we can see that $b-a+c>1$. Table 1 shows convergence to zero.

Table 1: Numerical results for Example 1.

n	x_n	n	x_n
-4	2.1	10	-0.083543908285124
-3	1	20	0.012701017754877
-2	8.5	50	0.003594428250519
-1	-3.3	100	$2.862071648505816 \times 10^{-8}$
0	-1.7	200	$1.691446180919758 \times 10^{-18}$
1	-1.019132653061225	500	$3.491237927069944 \times 10^{-49}$
2	-1.807491245443325	999	$3.185169006739856 \times 10^{-100}$

Example 2 Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-3} + 0.8x_{n-2})}{0.2x_{n-2} + 0.1x_{n-3}}.$$

In this case, we have $a = 0.8, b = 0.2, c = 0.1, k = 3$. Also, we can see that $b - a + c < 1$. Table 2 shows the solution set is unbounded.

Table 2: Numerical results for Example 2.

n	x_n	n	x_n
-3	2.8	5	$3.976943951329059 \times 10^{3}$
-2	7.5	10	$7.870828852071307 \times 10^{6}$
-1	1.3	20	$3.259245595490367 \times 10^{13}$
0	0.7	50	$2.322461318837964 \times 10^{33}$
1	3.460674157303371	100	$2.844463544208173 \times 10^{66}$
2	29.261541884525528	150	$3.483792297723871 \times 10^{99}$
3	$2.015795107600647 \times 10^{2}$	200	$4.266818183833936 \times 10^{132}$

Example 3 Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-5} + 1.5x_{n-4})}{1.7x_{n-4} + 0.8x_{n-5}}.$$

In this case we have $a = 1.5$, $b = 1.7$, $c = 0.8$, $k = 5$. Also, we can see that $b - a + c = 1$. Table 3 shows convergence to a finite limit approximately equal to 2.804367096028192.

Table 3: Numerical results for Example 3.

n	x_n	n	x_n
-5	3.1	2	2.824563238832514
-4	2.1	20	2.804362901181129
-3	1.8	50	2.804367096027094
-2	6.5	100	2.804367096028192
-1	3.3	200	2.804367096028192
0	2.7	500	2.804367096028192
1	2.789256198347107	999	2.804367096028192

6. Conclusion

In Theorem 3.1 we found an explicit solution for equation (1.1) when $\overline{a} \geq 0$, $a > 0$, $b > 0$, $c > 0$, and $k \geq 1$ is an integer. The idea behind the construction of such a solution was to transform the given equation into a Riccati difference equation, which can be easily transformed into a linear difference equation with constant coefficients.

Similarly, in Theorem 4.1 we obtained results concerning the asymptotic behaviour of the solutions to (1.1). We determined that solutions can be convergent or divergent, depending on whether the value of $b - a + c$ is greater than, less than or equal to 1, when $\overline{a} = 1$. We also performed some numerical experiments in order to verify such behaviours for different values of a, b, c and k.

The author considers that similar techniques can be used to obtain explicit solutions, or at least results about the global behavious of such solutions, for the case when $\overline{a}$, a, b and/or c are negatives, or when these coefficients are linear on n. The author conjectures that the first case could give rise to periodical solutions, while the second case can be dealt with by converting the resulting Riccati equation into a Cauchy-Euler equation.

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