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On lacunary $\mathcal{I}$-invariant arithmetic convergence

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Abstract. In this study, we investigate the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence for real sequences and examine relations between this new type convergence notion and the notions of lacunary invariant arithmetic summability, lacunary strongly q-invariant arithmetic summability and lacunary σ-statistical arithmetic convergence which are defined in this study. Finally, giving the notions of lacunary $\mathcal{I}_\sigma$ arithmetic statistically convergence, lacunary strongly $\mathcal{I}_\sigma$ arithmetic summability, we prove the inclusion relation between them.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

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1. Introduction and Background

The idea of arithmetic convergence was firstly originated by Ruckle [22]. Then, it was further investigated by many authors (for examples, see [9, 10, 34–38]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m,n \rangle}| < \varepsilon$, where the symbol $\langle m, n \rangle$ denotes the greatest common divisior of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC.

Statistical convergence of a real number sequence was firstly originated by Fast [2]. It became a notable topic in summability theory after the work of Fridy [3] and Šalát [23].

By a lacunary sequence, we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$. The idea of lacunary statistical convergence was investigated by Fridy and Orhan [4] and then studied by several authors (for examples, see [5, 6, 13, 17, 27]).

In the wake of the study of ideal convergence defined by Kostyrko et al. [11], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about $\mathcal{I}$-convergence of sequences after the work of [1, 7, 8, 12, 16, 24, 28–30, 32].

An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

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A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$, it is called the filter associated with the ideal.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_k) is $\mathcal{I}$-convergent to L, then we write $\mathcal{I} - \lim x = L$.

An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:

(1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;

(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.

The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.

The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{(x_k) \in l_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L\right\}$$

uniformly in n.

Several authors studied invariant mean and invariant convergent sequence (for examples, see [14, 15, 18–21, 25, 26, 31, 33]).

Savaş and Nuray [18] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [20] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence for real sequences and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Ulusu and Nuray [33] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Recently, the concept of strong σ-convergence was generalized by Savaş [25]. The concept of strongly σ-convergence was defined by Mursaleen [14].

Let θ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r := \min_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}$$
$$S_r := \max_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}.$$

If the following limits exist

$$\underline{V}_\theta(E) = \lim_{r \to \infty} \frac{s_r}{h_r}, \ \overline{V}_\theta(E) = \lim_{r \to \infty} \frac{S_r}{h_r},$$

then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set E, respectively. If $\underline{V}_\theta(E) = \overline{V}_\theta(E)$, then $V_\theta(E) = \underline{V}_\theta(E) = \overline{V}_\theta(E)$ is called the lacunary invariant uniform density of E.

The class of all $E \subseteq \mathbb{N}$ with $\underline{V}_\theta(E) = 0$ will be denoted by $\mathcal{I}_{\sigma\theta}$. Note that $\mathcal{I}_{\sigma\theta}$ is an admissible ideal.

A sequence (x_m) is lacunary $\mathcal{I}_\sigma$-convergent to L, if for each $\varepsilon > 0$,

$$E(\varepsilon) := \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_\theta(E(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_{\sigma\theta} - \lim x_m = L$.

The arithmetic statistically convergence and lacunary arithmetic statistically convergence was examined by Yaying and Hazarika [38].

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t\to\infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. We shall write $ASC - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

A sequence $x = (x_m)$ is said to be lacunary arithmetic statistically convergent if for $\varepsilon > 0$ there is an integer n such that

$$\lim_{r\to\infty} \frac{1}{h_r} | \left\{ m \in I_r : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon \right\} = 0.$$

We will use $ASC_\theta - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is lacunary arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

Kişi [9] investigated the concepts of invariant arithmetic convergence, strongly invariant arithmetic convergence, invariant arithmetic statistically convergence, lacunary invariant arithmetic statistical convergence and obtained interesting results.

In [10], arithmetic $\mathcal{I}$-statistically convergent sequence space and $\mathcal{I}$-lacunary arithmetic statistically convergent sequence space were given and established interesting results.

Kişi [10] examined $\mathcal{I}$-invariant arithmetic convergence, $\mathcal{I}^*$-invariant arithmetic convergence, q-strongly invariant arithmetic convergence of sequences.

A sequence $x = (x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} x_{\sigma^p(s)} = x_{\langle p,n\rangle}$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}(AV_\sigma)$ and the set of all invariant arithmetic convergent sequences will be demostrated by AV_σ.

A sequence $x = (x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| = 0$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}[AV_\sigma]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all invariant arithmetic convergent sequences will be demostrated by $[AV_\sigma]$.

A sequence $x = (x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{m\to\infty} \frac{1}{m} \left|\{p \leq m : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| = 0$$

uniformly in s. We shall use $AS_{\sigma}C$ to denote the set of all invariant arithmetic statistical convergent sequences. In this case we write $AS_{\sigma}C - \lim x_p = x_{\langle p,n\rangle}$ or $x_p \to x_{\langle p,n\rangle} (AS_{\sigma}C)$.

A sequence $x = (x_p)$ is said to be lacunary invariant arithmetic statistical convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}| = 0$$

uniformly in s. We shall use $AS_{\sigma\theta}C$ to denote the set of all lacunary invariant arithmetic statistical convergent sequences. In this case we write $AS_{\sigma\theta}C - \lim x_p = x_{\langle p,n\rangle}$.

The $\mathcal{I}$-invariant arithmetic convergence was defined by [10] as below:

A sequence $x = (x_p)$ is said to be $\mathcal{I}$-invariant arithmetic convergent if for every $\varepsilon > 0$, there is an integer η such that

$$\left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma}.$$

In this case we write $A\mathcal{I}_{\sigma}C - \lim x_p = x_{\langle p,\eta\rangle}$. We shall use $A\mathcal{I}_{\sigma}C$ to denote the set of all $\mathcal{I}$-invariant arithmetic convergent sequences.

2. Main Results

Definition 2.1. *A sequence $x = (x_p)$ is said to be lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p\in I_r} x_{\sigma^p(s)} = x_{\langle p,\eta\rangle},$$

uniformly in s, for an integer η.

Also, the set of lacunary strongly invariant arithmetic convergence sequences is defined as below:

$$[AV_{\sigma\theta}] = \left\{ x = (x_p) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{p\in I_r} \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| = 0 \right\}$$

uniformly in s. In this case, we write $x_p \to x_{\langle p,\eta\rangle} ([AV_{\sigma\theta}])$ to demonstrate the sequence (x_p) is lacunary strongly invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.

Definition 2.2. *A sequence $x = (x_p)$ is said to be lacunary strongly q-invariant arithmetic summable $(0 < q < \infty)$ to $x_{\langle p,\eta\rangle}$ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p\in I_r} \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|^q = 0,$$

uniformly in s and it is indicated by $x_p \to x_{\langle p,\eta\rangle} \left([AV_{\sigma\theta}]_q\right)$.

Definition 2.3. *A sequence $x = (x_p)$ is said to be lacunary σ-statistical arithmetic convergent to $x_{\langle p,\eta\rangle}$ if for every $\varepsilon > 0$, there is an integer η such that*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| = 0,$$

uniformly in s.

Definition 2.4. *A sequence $x = (x_p)$ is lacunary $\mathcal{I}_{\sigma}$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, if for each $\varepsilon > 0$, there is an integer η such that*

$$K(\varepsilon) := \left\{p \in \mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta},$$

i.e., $V_{\theta}(K(\varepsilon)) = 0$. In this case, we write $x_p \to x_{\langle p,\eta\rangle} (A\mathcal{I}_{\sigma\theta})$ or $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$.

Theorem 2.5. *Let (x_p) is bounded sequence. If (x_p) is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then (x_p) is lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. Let $s \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Also, we suppose that (x_p) is bounded sequence and (x_p) is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$. Now, we estimate

$$t_\theta(s) := \left| \frac{1}{h_r} \sum_{p \in I_r} x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|.$$

For every $s = 1, 2, ...$, we have

$$t_\theta(s) \le t_\theta^1(s) + t_\theta^2(s),$$

where

$$t_\theta^1(s) := \frac{1}{h_r} \sum_{p \in I_r,\, \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon} \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|$$

and

$$t_\theta^2(s) := \frac{1}{h_r} \sum_{p \in I_r,\, \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| < \varepsilon} \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|.$$

For every $s = 1, 2, ...$, it is obvious that $t_\theta^2(s) < \varepsilon$. Since (x_p) is bounded sequence, there is a $M > 0$ such that

$$\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \le M, \ (p \in I_r, s = 1, 2, ...)$$

and so we have

$$t_\theta^1(s) = \tfrac{1}{h_r} \sum_{p \in I_r,\, \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon} \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|$$

$$\le \tfrac{M}{h_r} \left| \left\{ p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon \right\} \right|$$

$$\le M \tfrac{\max_s \left| \left\{ p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \ge \varepsilon \right\} \right|}{h_r} = M \tfrac{S_r}{h_r}.$$

Hence, due to our assumption, (x_p) is lacunary invariant arithmetic summable to $x_{\langle p,\eta\rangle}$. ■

In general, the converse of the Theorem 2.5 does not hold. For example, let $x = (x_p)$ be the sequence defined as follows:

$$x_p := \begin{cases} 1, & \text{if } p_{r-1} < p < p_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } p \text{ is an even integer,} \\ 0, & \text{if } p_{r-1} < p < p_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } p \text{ is an odd integer.} \end{cases}$$

When $\sigma(s) = s + 1$, this sequence is lacunary invariant arithmetic summable to $\frac{1}{2}$ but it is not lacunary $\mathcal{I}_\sigma$ arithmetic convergent.

Now, we will give the following theorems which state relations between the notions of lacunary $\mathcal{I}_\sigma$ arithmetic convergence and lacunary strongly q-invariant arithmetic summability, and we will denote that these notions are equivalent for bounded sequences.

Theorem 2.6. *If a sequence $x = (x_p)$ is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$, then it is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $0 < q < \infty$. Suppose that $x_p \to x_{\langle p,\eta\rangle} \left([AV_{\sigma\theta}]_q \right)$ for an integer η. Then, for every $s = 1, 2, ...$ and $\varepsilon > 0$ we have

$$\begin{aligned} &\sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q \\ &\geq \sum_{p \in I_r, \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q \\ &\geq \varepsilon^q \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^q \max_s \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so

$$\frac{1}{h_r} \sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q \geq \varepsilon^q \frac{\max_s \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon \right\} \right|}{h_r} = \varepsilon^q \frac{S_r}{h_r}.$$

Hence, due to our assumption, $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. ■

Theorem 2.7. *Let (x_p) is bounded sequence. If $x = (x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then it is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. Assume that $(x_p) \in l_\infty$ and $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. Let $0 < q < \infty$ and $\varepsilon > 0$. The boundedness of (x_p) implies that there exists a $M > 0$ such that $\left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \leq M$, $(p \in I_r,\ s = 1, 2, ...)$. Therefore, we obtain

$$\begin{aligned} \frac{1}{h_r} \sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q &= \frac{1}{h_r} \sum_{\substack{p \in I_r \\ \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon}} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q + \frac{1}{h_r} \sum_{\substack{p \in I_r \\ \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| < \varepsilon}} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q \\ &\leq M \frac{\max_s \left| \left\{ p \in I_r : \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right| \geq \varepsilon \right\} \right|}{h_r} + \varepsilon^q \\ &= M \frac{S_r}{h_r} + \varepsilon^q. \end{aligned}$$

Therefore, we obtain

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{p \in I_r} \left| x_{\sigma^p(s)} - x_{\langle p,\eta\rangle} \right|^q = 0,$$

uniformly in s. Hence, we get $x_p \to x_{\langle p,\eta\rangle} \left([AV_{\sigma\theta}]_q \right)$. ■

Theorem 2.8. *A sequence $(x_p) \in l_\infty$. Then, $x = (x_p)$ to lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff it is lacunary strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$.*

Proof. This is an immediate consequence of Theorem 2.6 and Theorem 2.7. ■

Now, without proof, we will state a theorem that gives a relation between the notions of lacunary $\mathcal{I}_\sigma$ arithmetic convergence and lacunary σ-statistical arithmetic convergence.

Theorem 2.9. *A sequence $x = (x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff this sequence is lacunary σ-statistical arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Finally, introducing the notion of lacunary $\mathcal{I}_\sigma^*$ arithmetic convergence, we will give the relation between this notion and the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence.

Definition 2.10. *A sequence $x = (x_p)$ is said to be lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent or $A\mathcal{I}_{\sigma\theta}^*$-convergent to $x_{\langle p,\eta\rangle}$, if there exists a set $M = \{m_1 < m_2 < ... < m_p < ...\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ $(\mathbb{N} \setminus M = H \in \mathcal{I}_{\sigma\theta})$ and there is an integer η such that*

$$\lim_{p\to\infty} x_{m_p} = x_{\langle p,\eta\rangle}.$$

In this case, we write $A\mathcal{I}_{\sigma\theta}^* - \lim x_p = x_{\langle p,\eta\rangle}$ or $x_p \to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_{\sigma\theta}^*\right)$.

Theorem 2.11. *If a sequence $x = (x_p)$ is lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $\varepsilon > 0$. Since $A\mathcal{I}_{\sigma\theta}^* - \lim x_p = x_{\langle p,\eta\rangle}$, there exists a set $H \in \mathcal{I}_{\sigma\theta}$ such that for

$$M = \mathbb{N}\backslash H = \{m_1 < m_2 < ... < m_p < ...\}$$

and so there exists a $p_0 \in \mathbb{N}$ such that $|x_{m_p} - x_{\langle p,\eta\rangle}| < \varepsilon$ for every $p > p_0$. Then, for every $\varepsilon > 0$, we have

$$K(\varepsilon) = \{p \in \mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\}$$

$$\subset H \cup \{m_1 < m_2 < ... < m_p < ...\}.$$

Since $\mathcal{I}_{\sigma\theta}$ is admissible ideal,

$$H \cup \{m_1 < m_2 < ... < m_p < ...\} \in \mathcal{I}_{\sigma\theta}$$

and so we have $K(\varepsilon) \in \mathcal{I}_{\sigma\theta}$. Hence, we get $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. ■

The converse of the Theorem 2.11 holds if the ideal $\mathcal{I}_{\sigma\theta}$ has the property (AP).

Theorem 2.12. *Let the ideal $\mathcal{I}_{\sigma\theta}$ be with property (AP). If a sequence $x = (x_p)$ is lacunary $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is lacunary $\mathcal{I}_\sigma^*$ arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP) and $\varepsilon > 0$. Also, we suppose that $A\mathcal{I}_{\sigma\theta} - \lim x_p = x_{\langle p,\eta\rangle}$. Then, for every $\varepsilon > 0$ we have

$$K(\varepsilon) = \left\{p \in \mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta}.$$

Denote $K_1, K_2, ..., K_n$ as following

$$K_1 := \left\{p \in \mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq 1\right\}$$

and

$$K_n := \left\{p \in \mathbb{N} : \frac{1}{n} \leq \left|x_p - x_{\langle p,\eta\rangle}\right| < \frac{1}{n-1}\right\},$$

where $n \geq 2$ $(n \in \mathbb{N})$. Note that $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $K_i \in \mathcal{I}_{\sigma\theta}$ (for each $i \in \mathbb{N}$). Since $\mathcal{I}_{\sigma\theta}$ has the property (AP), there exists a set sequence $\{F_n\}_{n\in\mathbb{N}}$ such that the symmetric differences $K_i \Delta F_i$ are finite (for each $i \in \mathbb{N}$) and $F = \bigcup\limits_{j=1}^{\infty} F_j \in \mathcal{I}_{\sigma\theta}$. Now, to complete the proof, it is enough to prove that

$$\lim_{p\to\infty} x_p = x_{\langle p,\eta\rangle}, p \in M, \tag{2.1}$$

where $M = \mathbb{N} \setminus F$. Let $\gamma > 0$. Select $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \gamma$. Then, we get

$$\left\{p \in \mathbb{N} : \left|x_p - x_{\langle p,\eta\rangle}\right| \geq \gamma\right\} \subset \bigcup_{i=1}^{n+1} K_i.$$

Since the symmetric differences $K_i \Delta F_i$ $(i = 1, 2, ..., n+1)$ are finite, there exists a $p_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\left(\bigcup_{i=1}^{n+1} K_i\right) \cap \{p \in \mathbb{N} : p > p_0\} \\ &= \left(\bigcup_{i=1}^{n+1} F_i\right) \cap \{p \in \mathbb{N} : p > p_0\}. \end{aligned} \tag{2.2}$$

If $p > p_0$ and $p \notin F$, then

$$p \notin \bigcup_{i=1}^{n+1} F_i \text{ and by } (2.2) \ p \notin \bigcup_{i=1}^{n+1} K_i.$$

This give that

$$\left|x_p - x_{\langle p,\eta \rangle}\right| < \frac{1}{n+1} < \gamma$$

and so (2.1) holds. As a result, $A\mathcal{I}^*_{\sigma\theta} - \lim x_p = x_{\langle p,\eta \rangle}$. ■

Definition 2.13. *A sequence $x = (x_p)$ is said to be lacunary $\mathcal{I}$ invariant arithmetic statistically convergent to $x_{\langle p,\eta \rangle}$, for each $\varepsilon > 0$ and $\delta > 0$, there is an integer η such that*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{p \in I_r : \left|x_p - x_{\langle p,\eta \rangle}\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_{\sigma\theta}.$$

In this case, we write $x_p \to x_{\langle p,\eta \rangle}$ $(A\mathcal{I}_{\sigma\theta}(S))$.

Definition 2.14. *A sequence $x = (x_p)$ is said to be lacunary strongly $\mathcal{I}_\sigma$ arithmetic summable to $x_{\langle p,\eta \rangle}$ if for each $\varepsilon > 0$, there is an integer η such that*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{p \in I_r} \left|x_p - x_{\langle p,\eta \rangle}\right| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta}.$$

We will use $[A(\mathcal{I}_{\sigma\theta})] - \lim x_p = x_{\langle p,\eta \rangle}$ or $x_p \to x_{\langle p,\eta \rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$ to indicate the sequence (x_m) is lacunary strongly $\mathcal{I}_\sigma$ arithmetic convergent to $x_{\langle m,n \rangle}$.

Theorem 2.15. *Let $\theta = \{k_r\}$ be a lacunary sequence.*

(i) If $x_p \to x_{\langle p,\eta \rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$, then $x_p \to x_{\langle p,\eta \rangle}$ $(A\mathcal{I}_{\sigma\theta}(S))$.

(ii) If $x \in l_\infty$ and $x_p \to x_{\langle p,\eta \rangle}$ $(A\mathcal{I}_{\sigma\theta}(S))$, then $x_p \to x_{\langle p,\eta \rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$.

(iii) $(A\mathcal{I}_{\sigma\theta}(S)) \cap l_\infty = [A(\mathcal{I}_{\sigma\theta})] \cap l_\infty$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \to x_{\langle p,\eta \rangle}$ $([A(\mathcal{I}_{\sigma\theta})])$. Then, we can write

$$\begin{aligned} \tfrac{1}{h_r} \sum_{p \in I_r} \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| &\geq \tfrac{1}{h_r} \sum_{\substack{p \in I_r \\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| \geq \varepsilon}} \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| \\ &\geq \varepsilon . \tfrac{1}{h_r} \left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| \geq \varepsilon\right\}\right| \end{aligned}$$

for $s = 1, 2, ...$ So, for any $\delta > 0$,

$$\begin{aligned} &\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{p \in I_r} \left|x_{\sigma^p(s)} - x_{\langle p,\eta \rangle}\right| \geq \varepsilon . \delta\right\} \end{aligned}$$

uniformly in s. Since $x_p \to x_{\langle p,\eta\rangle}\left(\left[A\left(\mathcal{I}_{\sigma\theta}\right)\right]\right)$, the set on the right-hand side belongs to $\mathcal{I}_{\sigma\theta}$ and so we obtain $x_p \to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_{\sigma\theta}\left(S\right)\right)$.

(ii) Suppose that $x \in l_\infty$ and $x_p \to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_{\sigma\theta}\left(S\right)\right)$. Then, there exists a $M > 0$ such that

$$\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \leq M$$

for $s = 1, 2, ...$

Given $\varepsilon > 0$, we obtain

$$\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| = \frac{1}{h_r}\sum_{\substack{p\in I_r \\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| + \frac{1}{h_r}\sum_{\substack{p\in I_r \\ \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| < \varepsilon}}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right|$$

$$\leq \frac{M}{h_r}\left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| + \varepsilon$$

uniformly in s. Note that

$$A\left(\varepsilon\right) = \left\{r \in \mathbb{N} : \tfrac{1}{h_r}\left|\left\{p \in I_r : \left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq \varepsilon\right\}\right| \geq \frac{\varepsilon}{M}\right\}.$$

It is obvious that $A\left(\varepsilon\right) \in \mathcal{I}_{\sigma\theta}$. If $r \in \left(A\left(\varepsilon\right)\right)^c$ then

$$\frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| < 2\varepsilon.$$

Hence

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{p\in I_r}\left|x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}\right| \geq 2\varepsilon\right\} \subset A\left(\varepsilon\right)$$

and so belongs to $\mathcal{I}_{\sigma\theta}$. This shows that $x_p \to x_{\langle p,\eta\rangle}\left(\left[A\left(\mathcal{I}_{\sigma\theta}\right)\right]\right)$. This completes the proof.

(iii) This is an immediate consequence of (i) ve (ii). ■

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References

[1] P. Das, E. Savaş and S.K. Ghosal, On generalizations of certain summability methods using ideals, *Appl. Math. Lett.,* **24**(2011), 1509–1514.

[2] H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2**(1951), 241–244.

[3] J.A. Fridy, On statistical convergence, *Analysis*, **5**(1985), 301–313.

[4] J.A. Fridy and C. Orhan, Lacunary statistical convergence, *Pac J Math.*, **160(1)**(1993), 43–51.

[5] A.R. Freedman, J.J. Sember and M. Raphael, Some Cesàro-type summability spaces, *Proc. Lond. Math. Soc.*, **37**(1978), 508–520.

[6] J.A. Fridy and C. Orhan, Lacunary statistical summability, *J. Math. Anal. Appl.*, **173(2)**(1993), 497–504.

[7] M. GÜRDAL AND M.B HUBAN, On $\mathcal{I}$-convergence of double sequences in the Topology induced by random 2-norms, *Mat. Vesnik*, **66(1)**(2014), 73–83.

[8] M. GÜRDAL AND A. ŞAHINER, Extremal $\mathcal{I}$-limit points of double sequences, *Appl. Math. E-Notes,* **8**(2008), 131–137.

[9] Ö. KIŞI, On invariant arithmetic statistically convergence and lacunary invariant arithmetic statistically convergence, *Palest. J. Math.*, in press.

[10] Ö. KIŞI, On $\mathcal{I}$-lacunary arithmetic statistical convergence, *J. Appl. Math. Informatics*, in press.

[11] P. KOSTYRKO, M. MACAJ AND T. ŠALÁT, $\mathcal{I}$-convergence, *Real Anal. Exchange*, **26(2)**(2000), 669–686.

[12] P. KOSTYRKO, M. MACAJ, T. ŠALÁT AND M. SLEZIAK, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, *Math. Slovaca*, **55**(2005), 443–464.

[13] J. LI, Lacunary statistical convergence and inclusion properties between lacunary methods, *Int. J. Math. Math. Sci.*, **23(3)**(2000), 175–180.

[14] M. MURSALEEN, Matrix transformation between some new sequence spaces, *Houston J. Math.*, **9**(1983), 505–509.

[15] M. MURSALEEN, On finite matrices and invariant means, *Indian J. Pure Appl. Math.*, **10**(1979), 457–460.

[16] A. NABIEV, S. PEHLIVAN AND M. GÜRDAL, On $\mathcal{I}$-Cauchy sequences, *Taiwanese J. Math.*, **11**(2007), 569–566.

[17] F. NURAY, Lacunary statistical convergence of sequences of fuzzy numbers, *Fuzzy Sets and Systems*, **99(3)**(1998), 353–355.

[18] F. NURAY AND E. SAVAŞ, Invariant statistical convergence and A-invariant statistical convergence, *Indian J. Pure Appl. Math.,* **25(3)**(1994), 267–274.

[19] F. NURAY AND E. SAVAŞ, On σ statistically convergence and lacunary σ statistically convergence, *Math. Slovaca,* **43(3)**(1993), 309–315.

[20] F. NURAY AND H. GÖK AND U. ULUSU, $\mathcal{I}_\sigma$*-convergence*, *Math. Commun.*, **16**(2011), 531–538.

[21] R.A. RAIMI, Invariant means and invariant matrix methods of summability, *Duke Math. J.*, **30**(1963), 81–94.

[22] W.H. RUCKLE, Arithmetical summability, *J. Math. Anal. Appl.*, **396**(2012), 741–748.

[23] T. ŠALÁT, On statistically convergent sequences of real numbers, *Math. Slovaca*, **30**(1980), 139–150.

[24] T. ŠALÁT, B.C. TRIPATHY AND M. ZIMAN, On some properties of $\mathcal{I}$-convergence, *Tatra Mt. Math. Publ.,* **28**(2004), 279–286.

[25] E. SAVAŞ, Some sequence spaces involving invariant means, *Indian J. Math.*, **31**(1989), 1–8.

[26] E. SAVAŞ, Strong σ-convergent sequences, *Bull. Calcutta Math. Soc.*, **81**(1989), 295–300.

[27] E. SAVAŞ AND R.F. PATTERSON, Lacunary statistical convergence of multiple sequences, *Appl. Math. Lett.*, **19(6)**(2006), 527–534.

[28] E. SAVAŞ AND M. GÜRDAL, $\mathcal{I}$-statistical convergence in probabilistic normed space, *Sci. Bull. Series A Appl. Math. Physics,* **77(4)**(2015), 195–204.

[29] E. Savaş and M. Gürdal, Certain summability methods in intuitionistic fuzzy normed spaces, *J. Intell. Fuzzy Syst.*, **27(4)**(2014), 1621–1629.

[30] E. Savaş and M. Gürdal, A generalized statistical convergence in intuitionistic fuzzy normed spaces, *Science Asia,* **41**(2015), 289–294.

[31] P. Schaefer, Infinite matrices and invariant means, *Proc. Amer. Math. Soc.,* **36**(1972), 104–110.

[32] B.C. Tripathy and B. Hazarika, $\mathcal{I}$-monotonic and $\mathcal{I}$-convergent sequences, *Kyungpook Math. J.,* **51**(2011), 233–239.

[33] U. Ulusu and F. Nuray, Lacunary $\mathcal{I}$-invariant convergence, *Cumhuriyet Sci. J.*, **41(3)**(2020), 617–624.

[34] T. Yaying and B. Hazarika, On arithmetical summability and multiplier sequences, *Nat. Acad. Sci. Lett.*, **40(1)**(2017), 43–46.

[35] T. Yaying and B. Hazarika, On arithmetic continuity, *Bol. Soc. Parana Mater.*, **35(1)**(2017), 139–145.

[36] T. Yaying, B. Hazarika and H. Çakalli, New results in quasi cone metric spaces, *J. Math. Comput. Sci.*, **16**(2016), 435–444.

[37] T. Yaying and B. Hazarika, On arithmetic continuity in metric spaces, *Afr. Mat.*, **28**(2017), 985-989.

[38] T. Yaying and B. Hazarika, Lacunary Arithmetic Statistical Convergence, *Nat. Acad. Sci. Lett.*, **43(6)**(2020), 547–551.