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Fixed points of multiplicative closed graph operators on b-multiplicative metric spaces

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Abstract. This article discusses fixed point iterations of some multiplicative contraction mappings with variations in b-multiplicative metric space domains. A map that satisfies the multiplicative contraction condition has been proposed for an increasing sequence of subsets of a b-multiplicative metric space, where one element of the sequence is mapped into the following member of the sequence. Additionally, several fixed point conclusions are established for various multiplicative contraction mappings with multiplicative closed graphs.

AMS Subject Classifications: Primary 54E40; Secondary 54H25.

Keywords: Multiplicative contraction; b-Multiplicative metric space; Multiplicative closed graph, exponential transformation.

Contents

1. Introduction

W. A. Kirk et al. [10] proposed using a cycle of domains to derive various fixed point theorems for metric spaces. C. G. Moorthy and P. X. Raj considered an increasing sequence of subsets $\Xi_1 \subseteq \Xi_2 \subseteq ...$ of a metric space (Ξ, d), and a map $G : \Xi \to \Xi$ satisfying a contraction condition such that $G(\Xi_i) \subseteq \Xi_{i+1}, \forall i$, and $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$ in [11]. Also, the fixed point results of [11] are generalized in some articles [14–16].

A.E. Bashirov et al. introduced multiplicative metric space (also known as MMS) in [5]. M. Ozavsar and A. C. Cevikel [13] developed topological features of multiplicative metric spaces (or MMSs) and established fixed point findings in MMSs. There are numerous papers [1–3, 7–9, 12, 17] for fixed point theory in MMSs.

b-Metric space, a generalisation of a metric space, was first introduced by Czerwik [6]. b-MMS was introduced by M. U. Ali et al in [4]. There are some topological properties and fixed point results in b-MMSs.

By variations in b-MMS domains, we prove some more fixed point theorems for different types of multiplicative contraction mappings with multiplicative closed graphs. Also, we generalize a main result of [11] and we derive that result by using exponential transformation.

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2. b-Multiplicative metric spaces

Let us give some preliminary and known results in this section. See [4] for further information.

Definition 2.1. *[4] Assuming that* $\Xi \neq \emptyset$ *is a set and* $s \in \mathbb{R}$ *with* $s \geq 1$. *A multiplicative metric is a mapping* $d : \Xi \times \Xi \to \mathbb{R}^+ = [0, \infty)$ *satisfying the next four axioms.*

(i) $d(\kappa, \iota) \geq 1$, $\forall\, \kappa, \iota \in \Xi$,

(ii) $d(\kappa, \iota) = 1$ *if and only if(or, iff)* $\kappa = \iota$ *in* Ξ,

(iii) $d(\kappa, \iota) = d(\iota, \kappa)$, $\forall\, \kappa, \iota \in \Xi$,

(iv) $d(\kappa, \iota) \leq [d(\kappa, \rho)d(\rho, \iota)]^s$, $\forall\, \kappa, \iota, \rho \in \Xi$.

The triple (Ξ, d, s) *is then referred to as a b-MMS.*

Definition 2.2. *[4] Assuming that* (Ξ, d, s) *is a b-MMS,* $\{\kappa_n\}$ *is a sequence in* Ξ, *and* $\kappa \in \Xi$. *Then* $\{\kappa_n\}$ *is called multiplicative converging to* κ, *if for every multiplicative open ball* $B_\epsilon(\kappa) = \{\iota : d(\kappa, \iota) < \epsilon\}$, $\epsilon > 1$, *there exists* $N \in \mathbb{N}$ *such that* $\kappa_n \in B_\epsilon(\kappa)$, $\forall\, n > N$. *It is denoted by* $\kappa_n \to \kappa(n \to \infty)$.

Lemma 2.3. *[4] Assuming that* (Ξ, d, s) *is a b-MMS,* $\{\kappa_n\}$ *is a sequence in* Ξ *and* $\kappa \in \Xi$. *Then* $\kappa_n \to \kappa(n \to \infty)$ *iff* $d(\kappa_n, \kappa) \to 1(n \to \infty)$.

Lemma 2.4. *[4] Assuming that* (Ξ, d, s) *is a b-MMS, and* $\{\kappa_n\}$ *is a sequence in* Ξ. *Then every multiplicative convergent sequence* $\{\kappa_n\}$ *has an unique multiplicative limit point.*

Definition 2.5. *[4] Assuming that* (Ξ, d, s) *is a b-MMS. The sequence* $\{\kappa_n\} \in \Xi$ *is called a multiplicative Cauchy sequence(or, MCS) if for every* $\epsilon > 1$, *there exists* $N \in \mathbb{N}$ *such that* $d(\kappa_n, \kappa_m) < \epsilon, \forall m, n \geq N$.

Lemma 2.6. *[4] Assuming that* (Ξ, d, s) *is a b-MMS and* $\{\kappa_n\}$ *is a sequence in* Ξ. *Then* $\{\kappa_n\}$ *is a MCS iff* $d(\kappa_n, \kappa_m) \to 1(m, n \to \infty)$.

Definition 2.7. *[4] Assuming that* (Ξ, d, s) *is a b-MMS. Then* (Ξ, d, s) *is said to be multiplicative complete, if every MCS is multiplicative convergent in* Ξ.

Theorem 2.8. *[4] Assuming that* (Ξ, d, s) *is a b-MMS. Let* $\{\kappa_n\}$ *and* $\{\iota_n\}$ *be two sequences in* Ξ *such that* $\kappa_n \to \kappa$, $\iota_n \to \iota$ $(n \to \infty)$, $\kappa, \iota \in \Xi$. *Then* $d(\kappa_n, \iota_n) \to d(\kappa, \iota)(n \to \infty)$.

Definition 2.9. *Assuming that* $G : (\Xi, d, s) \to (\Xi, d, s)$ *is a self mapping on a b-MMS* (Ξ, d, s). *If whenever* $\kappa_n \to \kappa_0$ *and* $G\kappa_n \to \iota_0$ *for some sequence* $\{\kappa_n\}$ *in* Ξ *and some* κ_0, ι_0 *in* Ξ, *we have* $\iota_0 = G\kappa_0$, *then* G *is said to have a multiplicative closed graph(or, MCG).*

3. Main results

Let's prove some fixed point theorems for various multiplicative contractions on b-MMSs in this section.

Theorem 3.1. *Assuming that* (Ξ, d, s) *is a complete b-MMS, and* $G : \Xi \to \Xi$ *have a MCG. Let* $\Xi_1 \subseteq \Xi_2 \subseteq ...$ *be subsets of* Ξ *such that* $\Xi = \bigcup\limits_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, *and* $d(Gt, Gz) \leq d(t, z)^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, *where* $\xi_i \in (0, \infty)$ *are real positive constants such that* $\sum\limits_{n=1}^{\infty} s^n \xi_1 \xi_2 ... \xi_n < \infty$. *Then, for any fixed* $t_1 \in \Xi$, $\{G^n t_1\}$ *multiplicative converges to a fixed point.*
Moreover, if $\xi_i \in (0, 1)$, $\forall i$, *then* G *has a unique fixed point(or, UFP) in* Ξ.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1, 2, 3, \ldots$. Then we have,

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} \\ &\leq d(Gt_1, t_1)^{\xi_{n+1}\xi_n\xi_{n-1}\ldots\xi_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have,

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} \ldots d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum\limits_{i=n}^{m-1} s^i \xi_2\xi_3\ldots\xi_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1$ $(m, n \to \infty)$. Since Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
Assuming additionally that $\xi_i \in (0, 1)$, $\forall i$.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq d(t^*, z^*)^{\xi_n}.$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.2. *Assuming that (Ξ, D) is a complete metric space, and $G : \Xi \to \Xi$ have a closed graph. Let $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $\Xi = \bigcup\limits_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $D(Gt, Gz) \leq \xi_i D(t, z)$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum\limits_{n=1}^{\infty} \xi_1\xi_2\ldots\xi_n < \infty$.*
Then, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ converges to a fixed point. Also, if $\xi_i \in (0, 1)$, $\forall i$, then G has a UFP in Ξ.

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(t, z))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum\limits_{n=1}^{\infty} \xi_1\xi_2\ldots\xi_n < \infty$. Theorem 3.1 now leads to Corollary 3.2. ■

The above Corollary is Theorem 2.1 of [11]

Example 3.3. *Let $\Xi = \left[\frac{1}{4}, \infty\right)$. Assuming that $d(t, z) = \max\{tz^{-1}, zt^{-1}\}$, $\forall t, z \in \Xi$.*
Then (Ξ, d, s) is a complete b-MMS with $s = 1$.
Assuming that $\Xi_n = [\frac{1}{4}, n]$, and $\xi_n = \frac{n^2}{(n+1)^2} \in [\frac{1}{4}, 1)$, for $n = 1, 2, 3, \ldots$. Then $\sum\limits_{n=1}^{\infty} s^n \xi_1\xi_2\ldots\xi_n < \infty$.
Define $G : \Xi \to \Xi$ by $Gt = t^{\frac{1}{4}}$, if $t \in \Xi_n$, for $n \in \mathbb{N}$.
For $t, z \in \Xi_n$, we get

$$\begin{aligned} d(Gt, Gz) &= \max\left\{\left(\frac{t}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{t}\right)^{\frac{1}{4}}\right\} \\ &= (d(t, z))^{\frac{1}{4}} \\ &\leq d(t, z)^{\xi_n},\ \forall n \in \mathbb{N}. \end{aligned}$$

Theorem 3.1's hypotheses are then fulfilled. Moreover, the UFP is 1.

Theorem 3.4. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Let $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq (d(Gt, t)d(Gz, z))^{\xi_i}$, $\forall t, z \in \Xi_i$,$\forall i$, where $\xi_i \in (0, 1)$ are real positive constants such that $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Then G has a UFP in Ξ.*
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to the UFP.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\ n = 1, 2, \ldots$. Then we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq (d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{\xi_{n+1}} \\ &= d(G^{n+1}t_1, G^n t_1)^{\xi_{n+1}} d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}}. \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\frac{\xi_{n+1}}{1-\xi_{n+1}}} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n\vartheta_{n-1}\ldots\vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} \ldots d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2 \vartheta_3 \ldots \vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to w^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq (d(Gt^*, t^*)d(Gz^*, z^*))^{\xi_n} = 1.$$

Therefore $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.5. *Assuming that (Ξ, D) be a complete metric space, and $G : \Xi \to \Xi$ have a closed graph. Let $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $D(Gt, Gz) \leq \xi_i(D(Gt, t) + D(Gz, z))$, $\forall t, z \in \Xi_i$,$\forall i$, where $\xi_i \in (0, 1)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Then G has a UFP in Ξ. Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ converges to the UFP.*

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(Gt, t)d(Gz, z))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1 \vartheta_2 \ldots \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Theorem 3.4 now leads to Corollary 3.5. ■

Theorem 3.6. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Let $\Xi_1 \subseteq \Xi_2 \subseteq \ldots$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq (d(Gt, z)d(Gz, t))^{\xi_i}$, $\forall t, z \in \Xi_i$,$\forall i$,*

where $\xi_i \in (0, \frac{1}{2})$ *are real positive constants such that* $\sum_{n=1}^{\infty} s^n \vartheta_1\vartheta_2...\vartheta_n < \infty$*, where* $\vartheta_i = \frac{s\xi_i}{1-s\xi_i}$*,* $\forall i$*. Then* G *has a UFP in* Ξ*.*
Moreover, for any fixed $t_1 \in \Xi$*,* $\{G^n t_1\}$ *multiplicative converges to the UFP.*

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\ n = 1, 2, 3, ...$ Then we have

$$d(G^{n+1}t_1, G^n t_1) \leq (d(G^{n+1}t_1, G^{n-1}t_1)d(G^n t_1, G^n t_1))^{\xi_{n+1}}$$

Since $d(G^n t_1, G^n t_1) = 1$, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^{n+1}t_1, G^{n-1}t_1)^{\xi_{n+1}} \\ &\leq (d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{s\xi_{n+1}} \\ &= d(G^{n+1}t_1, G^n t_1)^{s\xi_{n+1}} d(G^n t_1, G^{n-1}t_1)^{s\xi_{n+1}}. \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{s\xi_{n+1}}{1-s\xi_{n+1}}\right)} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n\vartheta_{n-1}...\vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} ... d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2\vartheta_3...\vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1$ $(m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to w^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also a MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq (d(Gt^*, z^*)d(Gz^*, t^*))^{\xi_n} \\ &= d(t^*, z^*)^{2\xi_n}. \end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(2\xi_n)^m}$, $\forall\ m \in \mathbb{N}$. Since $(2\xi_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Corollary 3.7. *Assuming that* (Ξ, D) *be a complete metric space, and* $G : \Xi \to \Xi$ *have a closed graph. Let* $\Xi_1 \subseteq \Xi_2 \subseteq ...$ *be subsets of* Ξ *such that* $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$*,* $G(\Xi_i) \subseteq \Xi_{i+1}$*,* $\forall i$*, and* $D(Gt, Gz) \leq \xi_i(D(Gt, z) + D(Gz, t))$*,* $\forall t, z \in \Xi_i, \forall i$*, where* $\xi_i \in (0, 1)$ *are real positive constants such that* $\sum_{n=1}^{\infty} \vartheta_1\vartheta_2...\vartheta_n < \infty$*, where* $\vartheta_i = \frac{\xi_i}{1-\xi_i}$*,* $\forall i$*. Then* G *has a UFP in* Ξ*. Moreover, for any fixed* $t_1 \in \Xi$*,* $\{G^n t_1\}$ *converges to the UFP.*

Proof. Let $d = \exp D$. That is $d(t, z) = \exp D(t, z)$, $\forall t, z \in \Xi$. Then (Ξ, d) is a complete b-MMS with $s = 1$. Also, $d(Gt, Gz) \leq (d(Gt, z)d(Gz, t))^{\xi_i}$, $\forall t, z \in \Xi_i$, $\forall i$, where $\xi_i \in (0, \infty)$ are real positive constants such that $\sum_{n=1}^{\infty} \vartheta_1\vartheta_2...\vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i}{1-\xi_i}$, $\forall i$. Theorem 3.6 now leads to Corollary 3.7. ■

Theorem 3.8. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq d(t, z)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i$,$\forall i$, where $\xi_i, \mu_i \in (0, 1)$ are real positive constants such that $\xi_i + \mu_i < 1$, $\forall i$, and $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i + s\mu_i}{1 - s\mu_i}$, $\forall i$. Then G has a UFP in Ξ.*
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to the UFP.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1, 2, 3, ...$ Then we have

$$\begin{aligned}
d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} d(G^{n-1}t_1, G^{n+1}t_1)^{\mu_{n+1}} \\
&\leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}} d(G^{n-1}t_1, G^n t_1)^{s\mu_{n+1}} d(G^n t_1, G^{n+1}t_1)^{s\mu_{n+1}} \\
&= d(G^n t_1, G^{n-1}t_1)^{(\xi_{n+1}+s\mu_{n+1})} d(G^n t_1, G^{n+1}t_1)^{s\mu_{n+1}}
\end{aligned}$$

Now, we get

$$\begin{aligned}
d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{\xi_{n+1}+s\mu_{n+1}}{1-s\mu_{n+1}}\right)} \\
&= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\
&\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n\vartheta_{n-1}...\vartheta_2}.
\end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned}
d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} ... d(G^{n+1}t_1, G^n t_1)^{s^n} \\
&\leq d(Gt_1, t_1)^{\left(\sum_{i=n}^{m-1} s^i \vartheta_2\vartheta_3...\vartheta_{i+1}\right)}.
\end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m, n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*, Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned}
1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n} d(z^*, Gt^*)^{\mu_n} \\
&\leq d(t^*, z^*)^{(\xi_n+\mu_n)}.
\end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n+\mu_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n + \mu_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ∎

Remark 3.9. *In Theorem 3.8, replacement of the condition $d(Gt, Gz) \leq d(t, z)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i$,$\forall i$, where $\xi_i, \mu_i \in (0, 1)$ are real positive constants such that $\xi_i + \mu_i < 1$, $\forall i$, and $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i + s\mu_i}{1 - s\mu_i}$, $\forall i$ by the condition $d(Gt, Gz) \leq d(Gt, Gz)^{\xi_i} d(z, Gt)^{\mu_i}$, $\forall t, z \in \Xi_i$,$\forall i$, where $\xi_i, \mu_i \in (0, 1)$ are real positive constants such that $\xi_i + \mu_i < 1$, $\forall i$, and $\sum_{n=1}^{\infty} s^n \vartheta_1 \vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{s\mu_{i+1}}{1 - s\mu_{i+1} - \xi_{i+1}}$, $\forall i$ gives a UFP.*

Theorem 3.10. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and*

$d(Gt,Gz) \leq d(t,z)^{\xi_i}(d(Gt,t)d(Gz,z))^{\mu_i}(d(Gt,z)d(Gz,t))^{\nu_i}$, $\forall t,z \in \Xi_i, \forall i$, where $\xi_i, \mu_i, \nu_i \in (0,\frac{1}{2})$ are real positive constants such that $\xi_i + 2\nu_i < 1$, $\forall i$, and $\sum\limits_{n=1}^{\infty} s^n \vartheta_1\vartheta_2 ... \vartheta_n < \infty$, where $\vartheta_i = \frac{\xi_i+\mu_i+s\nu_i}{1-\mu_i-s\nu_i}$, $\forall i$. Then G has a UFP in Ξ.
Moreover, for any fixed $t_1 \in \Xi$, $\{G^n t_1\}$ multiplicative converges to the UFP.

Proof. Fix $t_1 \in \Xi_1$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n = 1,2,3,....$ Then we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) \leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}}(d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{\mu_{n+1}} \\ (d(G^{n+1}t_1, G^{n-1}t_1)d(G^n t_1, G^n t_1))^{\nu_{n+1}}. \end{aligned}$$

Since $d(G^n t_1, G^n t_1) = 1$, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) \leq d(G^n t_1, G^{n-1}t_1)^{\xi_{n+1}}(d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{\mu_{n+1}} \\ (d(G^{n+1}t_1, G^n t_1)d(G^n t_1, G^{n-1}t_1))^{s\nu_{n+1}}. \end{aligned}$$

Now, we get

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\left(\frac{\xi_{n+1}+\mu_{n+1}+s\nu_{n+1}}{1-\mu_{n+1}-s\nu_{n+1}}\right)} \\ &= d(G^n t_1, G^{n-1}t_1)^{\vartheta_{n+1}}, \\ &\leq d(Gt_1, t_1)^{\vartheta_{n+1}\vartheta_n\vartheta_{n-1}...\vartheta_2}. \end{aligned}$$

Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(G^m t_1, G^n t_1) &\leq d(G^m t_1, G^{m-1}t_1)^{s^{m-1}} d(G^{m-1}t_1, G^{m-2}t_1)^{s^{m-2}} ... d(G^{n+1}t_1, G^n t_1)^{s^n} \\ &\leq d(Gt_1, t_1)^{\left(\sum\limits_{i=n}^{m-1} s^i \vartheta_2\vartheta_3...\vartheta_{i+1}\right)}. \end{aligned}$$

Therefore, $d(G^m t_1, G^n t_1) \to 1 (m,n \to \infty)$. By Lemma 2.6, $\{G^m t_1\}_{m=1}^{\infty}$ is an MCS in Ξ. Let $\{G^m t_1\}_{m=1}^{\infty}$ multiplicative converge to t^* in Ξ, which is multiplicative complete. Remember that $\{G^{m+1}t_1\}_{m=1}^{\infty}$ is also an MCS and it multiplicative converges to t^* in Ξ. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.
These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.
If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n}(d(Gt^*, t^*)d(Gz^*, z^*))^{\mu_n}(d(Gt^*, z^*)d(Gz^*, t^*))^{\nu_n} \\ &\leq d(t^*, z^*)^{(\xi_n+2\nu_n)}. \end{aligned}$$

Then, $d(t^*, z^*) \leq d(t^*, z^*)^{(\xi_n+2\nu_n)^m}$, $\forall\, m \in \mathbb{N}$. Since $(\xi_n + 2\nu_n)^m \to 0$ as $m \to \infty$, $d(t^*, z^*) = 1$ and $t^* = z^*$. Hence, G has a UFP. ■

Theorem 3.11. *Assuming that (Ξ, d, s) is a complete b-MMS. Let $G : \Xi \to \Xi$ have a MCG. Let $\xi_i \in (0,1)$, $\forall i$, such that for $1 \leq n \leq m$, $\frac{s^n \xi_1\xi_2...\xi_m}{1-s^n\xi_n} \to 0$ as $n \to \infty$, and let $\Xi_1 \subseteq \Xi_2 \subseteq ...$ be subsets of Ξ such that $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt,Gz) \leq d(t,z)^{\xi_i}$, $\forall t \in \Xi_i$, $\forall z \in \Xi, \forall i$. Suppose $t_1 \in \bigcup\limits_{j=1}^{\infty} \Xi_j$. Then $\{G^n t_1\}$ multiplicative converges to the fixed point of G in Ξ. If $\Xi = \bigcup\limits_{j=1}^{\infty} \Xi_j$, then G has a UFP in Ξ.*

Proof. Fix $t_1 \in \Xi$, and set $t_{n+1} = Gt_n = G^n t_1$, $\forall\, n \in \mathbb{N}$. Then for each n, we have

$$\begin{aligned} d(G^{n+1}t_1, G^n t_1) &\leq d(G^n t_1, G^{n-1}t_1)^{\xi_n} \\ &\leq d(Gt_1, t_1)^{\xi_n\xi_{n-1}...\xi_1}. \end{aligned}$$

Therefore, $d(t_{n+1}, t_n) \leq d(Gt_1, t_1)^{\xi_{n-1}\xi_{n-2}\dots\xi_1}$. Further, for $1 \leq n < m$, we have

$$\begin{aligned} d(t_n, t_m) &\leq d(t_n, t_{n+1})^{s^n} d(t_{n+1}, t_{m+1})^{s^{n+1}} d(t_m, t_{m+1})^{s^m} \\ &\leq d(Gt_1, t_1)^{s^n \xi_1\xi_2\dots\xi_{n-1}} d(t_n, t_m)^{s^{n+1}\xi_{n+1}} d(Gt_1, t_1)^{s^m \xi_1\xi_2\dots\xi_{m-1}} \end{aligned}$$

so that,

$$d(t_n, t_m) \leq d(Gt_1, t_1)^{\frac{(s^n\xi_1\xi_2\dots\xi_{n-1})+(s^m\xi_1\xi_2\dots\xi_{m-1})}{(1-s^{n+1}\xi_{n+1})}}.$$

Therefore, $d(t_n, t_m) \to 1$ $(n, m \to \infty)$. Since Lemma 2.6, $\{t_n\}$ is a MCS. By Ξ is multiplicative complete, $\{t_n\} \to w^*$, for some t^* in Ξ. Then $\{Gt_n\} \to t^*$ and $Gt^* = t^*$, because G has a MCG. Hence, we obtained a fixed point t^* of G.

These processes can be extended to the general case: $t_1 \in \Xi_n$, for some n.

Assuming now $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$\begin{aligned} 1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) &\leq d(t^*, z^*)^{\xi_n} \\ &\leq d(t^*, z^*)^{(\xi_n)^m}, \ \forall\, m > 1. \end{aligned}$$

Therefore, $d(t^*, z^*) = 1$, because $(\xi_n)^m \to 0$ as $m \to \infty$. Hence, G has a UFP, when $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. ■

Theorem 3.12. *Assuming that (Ξ, d, s) is a complete b-MMS, and $G : \Xi \to \Xi$ have a MCG. Suppose $d(t, z) \leq \alpha, \forall t, z \in \Xi$ and for some $\alpha \in [1, \infty)$. Let $\xi_i \in (0, 1)$, $\forall i$, be such that $\xi_1\xi_2\dots\xi_n \to 0$ as $n \to \infty$. Suppose $\Xi_1 \subseteq \Xi_2 \subseteq \dots$ be subsets of Ξ such that $G(\Xi_i) \subseteq \Xi_{i+1}$, $\forall i$, and $d(Gt, Gz) \leq d(t, z)^{\xi_i}$, $\forall t \in \Xi_i$, $\forall z \in \bigcup_{j=1}^{\infty} \Xi_j$, $\forall i$. Let $t_1 \in \bigcup_{j=1}^{\infty} \Xi_j$. Then the sequence $\{G^n t_1\}$ multiplicative converges to a unique fixed point G in Ξ. If $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$, then G has a UFP in Ξ.*

Proof. Fix $t_1, z_1 \in \Xi_1$. Set $t_{n+1} = Gt_n = G^n t_1$, and $z_{n+1} = Gz_n = G^n z_1$, $\forall\, n \in \mathbb{N}$. For $m < n$, we have

$$\begin{aligned} d(t_n, z_m) = d(Gt_{n-1}, Gz_{m-1}) &\leq d(t_{n-1}, z_{m-1})^{\xi_{m-1}} \\ &\leq d(t_{n-m+1}, z_1)^{\xi_{m-1}\xi_{m-2}\dots\xi_2\xi_1} \\ &\leq \alpha^{\xi_{m-1}\xi_{m-2}\dots\xi_2\xi_1}, \end{aligned}$$

because $d(t, z) \leq \alpha, \forall t, z \in \Xi$. Hence, $d(t_n, z_m) \to 1$ as $m, n \to \infty$. Also $d(t_n, t_m) \to 1$, and $d(z_n, z_m) \to 1$ as $m, n \to \infty$. So, $\{t_n\}$ and $\{z_n\}$ are multiplicative Cauchy sequences in Ξ, because of Lemma 2.6. By (Ξ, d) is multiplicative complete, $\{t_n\}$ and $\{z_n\}$ multiplicative converges to a unique point t^* in Ξ, because of Lemma 2.8. Since $\{t_n\} \to t^*$, we have $\{Gt_n\} \to t^*$. Also, MCG of G gives $Gt^* = t^*$. Hence, we obtained a fixed point t^* of G.

These processes can be extended to the general case: $t_1, z_1 \in \Xi_n$, for some n.

Assuming now $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. If $Gt^* = t^*$, $Gz^* = z^*$ in G, then let $t^*, z^* \in \Xi_n$, for some n, so we have

$$1 \leq d(t^*, z^*) = d(Gt^*, Gz^*) \leq d(t^*, z^*)^{\xi_n} \leq d(t^*, z^*)^{(\xi_n)^m}, \ \forall\, m > 1.$$

So, $d(t^*, z^*) = 1$, because $(\xi_n)^m \to 0$ as $m \to \infty$. Therefore, G has a UFP, when $\Xi = \bigcup_{j=1}^{\infty} \Xi_j$. ■

4. Conclusion

If $s = 1$ in a b-multiplicative metric space (Ξ, d, s), then it becomes a multiplicative metric space. All fixed point results can be converted from b-multiplicative metric spaces to metric spaces through exponential transformation. It has been illustrated in Corollary 3.2, Corollary 3.5, and Corollary 3.7. As a result, studies of fixed points of multiplicative contractions in b-multiplicative metric spaces are important.

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