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Existence and controllability of impulsive stochastic integro-differential equations with state-dependent delay

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Abstract. This study aims to investigate the existence of mild solutions for a class of impulsive stochastic integrodifferential equations with state-dependent delay in a real separable Hilbert space, as well as the controllability of these solutions. We offer Sufficient conditions for the existence and controllability results using the fixed point techniques combined with the theory of resolvent operator in Grimmer and analysis stochastic. Finally, we provide an example to illustrate the obtained results.

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Contents

1. Introduction

Integrodifferential equations represent a wide range of natural phenomena, including biological models, chemical kinetics, electronics, and fluid dynamics. The theory of the integrodifferential equations was generalized to a stochastic functional integrodifferential equations by considering disturbances. As a result, many mathematicians have studied the theory of integrodifferential equations with resolution operators in recent decades (see [22, 23, 35, 39] and the references therein). The resolvent operator is analogous to the semigroup

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for differential equations in a Banach space. However, it will not be a semigroup since it does not satisfy the semigroup properties. The existence, uniqueness, stability, controllability, and other quantitative and qualitative aspects of stochastic integrodifferential equations have recently attracted much attention (see [6, 12–14, 37]). In many cases, deterministic models will fluctuate due to random or seemingly random environmental noise. As a result, we have to move from deterministic to stochastic situations

Delay differential equations are an essential branch of nonlinear analysis with numerous applications in almost every field. Usually, the deviation of the arguments depends only on time (see [4, 15, 16]); however, when the deviation of the arguments depends on both the state variable x and the time t, this form of the equation is known as self-reference or state-dependent equations. Equations with state-dependent delays have piqued the interest of specialists because they have numerous application models, such as the two-body problem of classical electrodynamics, and they also have numerous applications within the class of problems with memories, such as in hereditary phenomena, see [42, 43]. Several articles (see [1, 18, 30, 31]) investigated this equation.

Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences can change state abruptly or be perturbed quickly. We can see these disruptions as impulses in the system. In addition to communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology, impulsive issues emerge in various other areas. The monographs by Benchohra et al. [7], Graef et al. [20, 21], Laskshmikantham et al. [3], and Samoilenko and Perestyuk [40] provide a thorough introduction to basic theory. On the other hand, Milman and Myshkis [34] studied differential equations with impulses for the first time, followed by a period of active research culminating in the monograph by Halanay and Wexler [24].

In the field of mathematical control theory, the idea of controllability plays an essential role. It makes it possible to use a control that is admissible to guide the system from its initial state to its final state within a specified amount of time. The concept of controllability is crucial to the study of finite-dimensional control theory. Therefore, it is only natural to attempt to generalize it to an infinite number of dimensions. The controllability of nonlinear systems with different types of nonlinearities has been studied using fixed point concepts[5]. Several authors have investigated the controllability of semilinear and nonlinear systems, represented by differential and integrodifferential equations in finite or infinite dimensional Banach spaces, respectively[9, 36, 44, 46].

Recently, Ma and Liu [32] studied the exact controllability and continuous dependence of fractional neutral integrodifferential equations with state-dependent delay in Banach spaces. Slama and Boudaoui [41], authors proved a new set of sufficient conditions for a class of fractional nonlinear stochastic differential inclusions using fractional calculus, stochastic analysis theory, semigroup theory, and Bohnenblust-Karlin's fixed point theorem.To the best of our knowledge, the literature related to stochastic impulsive integrodifferential remains limited.

Inspired by the works mentioned above, the main objective of this manuscript is to investigate the existence and controllability results for the following model

$$\begin{cases} dz(t) = \Big[Az(t) + \displaystyle\int_0^t \Gamma(t-s)z(s)ds + g\left(t, z_t, \int_0^t h(t,s,z_s)ds\right)\Big]dt \\ \qquad\qquad +\xi(t, z_{\sigma(t,z_t)})dw(t),\ t \in J = [0,a], t \neq t_i, \\ \Delta z(t_i) = I_i(z_{t_i}), i = 1, \dots, m \\ z_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where the state $z(\cdot)$ takes values in a real separable Hilbert space $\mathbb{X}$, the operator A is an infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathbb{X}$, $(\Gamma(t))_{t\geq 0}$ is a family of closed linear operators on $\mathbb{X}$ with domain $D(\Gamma(t)) \supset D(A)$, the history $z_t : (-\infty, 0] \longrightarrow \mathbb{X}, z_t(\alpha) = z(t+\alpha)$, for $t \leq 0$, belongs to the phase space $\mathcal{B}$, which will be described axiomatically later, the mappings $g : J \times \mathcal{B} \times \mathbb{X} \longrightarrow \mathbb{X}, h : J \times J \times \mathcal{B} \longrightarrow \mathbb{X}, \xi :$ $J \times \mathcal{B} \longrightarrow \mathcal{L}_Q(\mathbb{Y}, \mathbb{X}), \sigma : J \times \mathcal{B} \times (-\infty, a]$ are appropriate functions that will be specified later, for $i = 1, \dots, m$, the functions $I_i : \mathcal{B} \longrightarrow \mathbb{X}$ represent the impulses, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$ be prefixed points, and $\Delta z(t_i)$ represents the jump of the function z at t_i, which is defined by $\Delta z(t_i) = z(t_i^+) - x(t_z^-)$, where $z(t_i^+)$ and $z(t_i^-)$ denote the right and left limits of $z(t)$ at $t = t_i$, respectively. It should be emphasized that the existence and controllability results for impulsive stochastic evolution integral differential equations with state dependent delay in the form (1.1) have not yet been explored.

Here, we discuss the existence and controllability of the system(1.1) by using resolvent operator in the sense of Grimmer and stochastic analysis tools combined with fixed point theory.

We will proceed as follows: Definitions and Lemma, which are necessary to derive the main results, are outlined in Section 2. In Section 3, we prove the existence results using the Krasnoselskii-Schaefer fixed point theorem's implication. Section 4 is devoted to controllability. As a concluding point, an example is provided in Section 5 to illustrate the theoretical outcomes.

2. Preliminaries

In this section, we briefly review some basic definitions and notations that will be used in the subsequent sections.

2.1. Brownian motion

Let $(\mathbb{X}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{Y}, \|\|_{\mathbb{Y}}, \langle\cdot,\cdot\rangle_{\mathbb{Y}})$ be two real separable Hilbert spaces, $\{\eta_k\}_{k=1}^{\infty}$ denote a complete orthonormal basis of $\mathbb{Y}$ and $\{w(t) : t \geq 0\}$ be a cylindrical $\mathbb{Y}-$value $Q-$Wiener process in which Q is a finite nuclear covariance operator. Denote $Tr(Q) = \sum_{k=1}^{\infty} \gamma_k$, which satisfies $Q\eta_k = \gamma_k \eta_k$, $(\gamma_k \geq 0, k = 1, 2, \ldots)$. Set

$$w(t) = \sum_{k=1}^{\infty} \sqrt{\gamma_k} \beta_k(t) \eta_k, t \geq 0,$$

where $\{\beta_k(t)\}_{k=1}^{\infty}$ is a sequence of real-values independant one-dimensional standards Brownian motions over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

It is assumed that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the $\sigma-$ algebra generated by w and $\mathcal{F}_t = \{\mathcal{F}_t\}_{t\geq 0}$. Let $\psi \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ and define

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\gamma_k} \psi \eta_k\|^2,$$

where ψ^* is the adjoint of the operator ψ, and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes the space of all bounded linear operators from K into H endowed with the same norm $\|\cdot\|$. if $\|\psi\|_Q^2 < \infty$, the ψ is called a $Q-$Hilbert- Schmidt operator. The completion $\mathcal{L}_Q(\mathbb{Y}, \mathbb{X})$ of $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ with respect to the topology induced by the norm $\|\cdot\|_Q$, is a Hilbert space with the above norm topology, where $\|\psi\|_Q = \langle\psi, \psi\rangle^{\frac{1}{2}}$. The collection of all strongly measurable, square-integrable, $\mathbb{X}-$ valued random variables, denoted by $\mathcal{L}_2(\Omega, \mathbb{X})$, is a Banach space equipped with the norm $\|z\|_{\mathcal{L}_2} = \left(\mathbb{E}\|z\|^2\right)^{\frac{1}{2}}$, where the expectation E is defined by $\mathbb{E}z = \int_{\Omega} z(w) d\mathbb{P}$.

Let $C(J, \mathcal{L}_2(\Omega, \mathbb{X}))$ be the Banach space of all continuous maps from J into $\mathcal{L}_2(\Omega, \mathbb{X})$ satisfying the condition $\sup_{0\leq t\leq a} \mathbb{E}\|z(t)\|^2 < \infty$. An important subspace $\mathcal{L}_2^0(\Omega, \mathbb{X})$ of $\mathcal{L}_2(\Omega, \mathbb{X})$ is given by

$$\mathcal{L}_2^0(\Omega, \mathbb{X}) = \{z \in \mathcal{L}_2(\Omega, \mathbb{X}) : z\, is\, \mathcal{F}_0\text{-measurable}\}.$$

For more details, we refer the reader to Da Patro and Zabczyk, LesZek Gawarecki and Vidyadhar Mandrekar [19].

2.2. Integrodifferential equations in Banach spaces

Here we recall some knowledge on partial integrodifferential equations and the related resolvent operators. Let $\mathcal{H}$ be the Banach space $D(A)$ equipped with the graph norm defined by

$$\|\theta\|_{\mathcal{H}} := \|A\theta\| + \|\theta\| \text{ for } \theta \in \mathcal{H}.$$

We denote by $\mathcal{C}(\mathbb{R}^+, \mathcal{D})$, the space of all functions from $\mathbb{R}^+$ into $\mathcal{D}$ which are continuous. Let us consider the following system:

$$\begin{cases} \theta'(t) = A\theta(t) + \displaystyle\int_0^t \Gamma(t-s)\theta(s)ds \quad \text{for } \ t \in [0,a], \\ \theta(0) = \theta_0 \in \mathcal{D}, \end{cases} \tag{2.1}$$

where A and $\Gamma(t)$ are closed linear operators on a Banach space $\mathcal{D}$.

Definition 2.1 ([22]). *A resolvent operator for Eq.* (2.1) *is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathcal{D})$ for $t \in [0,a]$, having the following properties :*

(i) *$R(0) = I$ (the identity map of $\mathcal{D}$) and $\|R(t)\| \leq Ne^{\beta t}$ for some constants $N > 0$ and $\beta \in \mathbb{R}$.*

(ii) *For each $\theta \in \mathcal{D}$, $R(t)\theta$ is strongly continuous for $t \in [0,a]$.*

(iii) *For $\theta \in \mathcal{H}$, $R(\cdot)\theta \in \mathcal{C}^1(\mathbb{R}^+;\mathcal{D}) \cap \mathcal{C}(\mathbb{R}^+;\mathcal{H})$ and*

$$\begin{aligned} R'(t)\theta &= AR(t)\theta + \int_0^t \Gamma(t-s)R(s)\theta ds \\ &= R(t)A\theta + \int_0^t R(t-s)\Gamma(s)\theta ds, \quad \text{for } t \in [0,a]. \end{aligned}$$

In what follows, we make the following assumptions.

$(\mathbf{R_1})$ The operator A is the infinitesimal generator of a strongly continuous semigroup $(\mathsf{T}(t))_{t\geq 0}$ on $\mathcal{D}$.

$(\mathbf{R_2})$ $(\Gamma(t))_{t\geq 0}$ is is a family of linear operators on $\mathcal{D}$ such that $\Gamma(t)$ is continuous when regarded as a linear map from $\mathcal{H}$ into $\mathcal{D}$ for almost all $t \geq 0$. For any $\theta \in \mathcal{D}$, the map $t \mapsto \Gamma(t)\theta$ is bounded, differentiable and the derivative $t \mapsto \Gamma'(t)\theta$ is bounded and uniformly continuous for $t \geq 0$.

Theorem 2.2. [22] *Assume that* $(\mathbf{R_1})$-$(\mathbf{R_2})$ *hold. Then there exists a unique resolvent operator to the Cauchy problem* (2.1).

We have the following useful results.

Lemma 2.3. *[11] Let the assumptions* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *be satisfied. Then, there exists a constant $\Delta = \Delta(a)$ such that*

$$\|R(t+\epsilon) - R(\epsilon)R(t)\|_{\mathcal{L}(\mathcal{D})} \leq \Delta\epsilon, \qquad \forall\, 0 < \epsilon \leq t \leq a.$$

Theorem 2.4 (Theorem 6, [17]). *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $(\boldsymbol{\Gamma}(t))_{t\geq 0}$ satisfy* $(\mathbf{R_2})$. *Then the resolvent operator $(R(t))_{t\geq 0}$ for Eq.* (2.1) *is compact for $t > 0$ if and only if $(T(t))_{t\geq 0}$ is compact for $t > 0$.*

In the sequel, we recall some results on the existence of solutions for the following integro-differential equation:

$$\begin{cases} \theta'(t) = A\theta(t) + \displaystyle\int_0^t \Gamma(t-s)\theta(s)ds + l(t) \quad \text{for } \ t \in [0,a], \\ \theta(0) = \theta_0 \in \mathcal{D}. \end{cases} \tag{2.2}$$

where l is a continuou function.

Definition 2.5. [23] *A continuous function $\theta : \ [0,+\infty) \to \mathcal{D}$ is said to be a strict solution of Eq.* (2.2) *if*

1. *$\theta \in \mathcal{C}^1([0,+\infty),\mathcal{D}) \cap \mathcal{C}([0,+\infty),\mathcal{H})$,*
2. *θ satisfies Eq.* (2.2) *for $t \geq 0$.*

Theorem 2.6. [23] *Assume that* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *hold. If* θ *is a strict solution of Eq.* (2.2), *then the following variation of constant formula holds*

$$\theta(t) = R(t)\theta_0 + \int_0^t R(t-s)l(s)ds \quad for\ t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definition.

Definition 2.7. *[23] A function* $\theta : [0,\infty) \longrightarrow \mathcal{D}$ *is said a mild solution of Eq.* (2.2) *for* $\theta_0 \in \mathcal{D}$, *if* θ *satisfies the variation of constants formula* (2.3).

In what follows, we say a function $z : [b,c] \longrightarrow \mathbb{X}$ is a normalized piecewise continuous function on $[b,c]$ if z is piecewise continuous and left continuous on $(b,c]$. We denote by $\mathcal{PC}([b,c],\mathbb{X})$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable process from $[b,c]$ into $\mathbb{X}$. Particularly, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{X}$ valued stochastic process $\{z(t) : t \in [0,a]\}$ such that z is continuous at $t \neq t_i$, $z(t_i^-) = z(t_i)$ and $z(t_i^+)$ exists, for $i = 1,2,\cdots,m$. Then $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space with norm given by

$$\|z\|_{\mathcal{PC}} = \sup_{s\in J}\left(E\|z(s)\|^2\right)^{\frac{1}{2}}.$$

For $z \in \mathcal{PC}$, we denote $\tilde{z}_i, i = 1,2,\ldots,m$, the function $\tilde{z}_i \in C\big([t_i,t_{i+1}];\mathcal{L}_2(\Omega,\mathbb{X})\big)$ given by

$$\tilde{z}_i(t) = \begin{cases} z(t), & for\, t \in (t_i,t_{i+1}], \\ z(t_i^+), & if\, t = t_i. \end{cases}$$

Moreover, for $\mathcal{E} \subseteq \mathcal{PC}$, we denote by $\tilde{\mathcal{E}}_i, i = 0,1,2,\ldots,m$, the set $\tilde{\mathcal{E}}_i = \{\tilde{z}_i : z \in \mathcal{E}\}$.

Lemma 2.8 ([26], [28]). .
A set $\mathcal{E} \subseteq \mathcal{PC}$, *is relatively compact in* $\mathcal{PC}$, *if and only if the set* $\tilde{\mathcal{E}}_i$ *is relatively compact in* $C([t_i,t_{i+1}];\mathcal{L}_2(\Omega,\mathbb{X}))$ *for every* $i = 0,1,\ldots,m$.

In order to deal with the infinite delay, we will consider the phase space $\mathcal{B}$ which was described by Hale and Kato in [25]. More precisely, $\mathcal{B}$ will be a seminormed linear space of $\mathcal{F}_0-$measurable functions defined from $(-\infty,0]$ into $\mathbb{X}$, and satisfying the following axioms:
A: If $z : (-\infty,\rho+a] \longrightarrow \mathbb{X}, a > 0$ is such that $z_\rho \in \mathcal{B}$ and $x|_{[\rho,\rho+a]} \in \mathcal{PC}([\rho,\rho+a],\mathbb{X})$, then, for every $t \in [\rho,\rho+a)$, : the following conditions hold:

(i) $z_t \in \mathcal{B}$,

(ii) $\mathbb{E}\|z(t)\| \leq H\|z_t\|_{\mathcal{B}}$,

(iii) $\|z_t\|_{\mathcal{B}} \leq K_1(t-\rho)\sup\limits_{\rho\leq s\leq t}\mathbb{E}\|z(s)\| + K_2(t-\rho)\|z_\rho\|_{\mathcal{B}}$,
where $H > 0$ is a constant, $K_1(\cdot), K_2(\cdot) : [0,+\infty) \longrightarrow [1,+\infty), K_1(\cdot)$ is continuous, $K_2(\cdot)$ is locally bounded, and $H, K_1(\cdot), K_2(\cdot)$ are independent of $z(\cdot)$.

B: The space $\mathcal{B}$ is complete.
The following results will be required in computation.

Lemma 2.9. *[45]. Let* $z : (-\infty,a] \longrightarrow \mathbb{X}$ *be can an* $\mathcal{F}_0$*-adapted process* $z_0 = \varphi(t) \in \mathcal{L}_2^0(\Omega,\mathcal{B})$ *and* $z|_J \in \mathcal{PC}(J,\mathbb{X})$, *then*

$$\|z_s\|_{\mathcal{B}} \leq \tilde{K}_2\|\varphi\|_{\mathcal{B}} + \tilde{K}_1\sup_{0\leq s\leq a}\mathbb{E}\|x(s)\|,$$

where $\tilde{K}_1 = \sup\limits_{t\in J} K_1(t)$ *and* $\tilde{K}_2 = \sup\limits_{t\in J} K_2(t)$.

In order to handle the delay function σ, the next result is a very useful.

Lemma 2.10. [27] *Let $z : (-\infty, a] \longrightarrow \mathbb{X}$ be a function such that $z_0 = \varphi$, and $z|_J \in \mathcal{PC}$. Then*

$$\|z_s\|_{\mathcal{B}} \leq (\tilde{K}_2 + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + \tilde{K}_1 \sup \mathbb{E}\{\|z(\theta)\| : \theta \in [0, \max\{0, s\}]\}, s \in \mathcal{Z}(\sigma^-) \cup J,$$

where $J_0^{\varphi} = \sup\{J^{\varphi}(t) : t \in \mathcal{Z}(\sigma^-)\}$.

Now, we give two important fixed point theorems and Burkholder- Davis-Gundy's inequality which are used in the proof of the main results.

Lemma 2.11. *[38]. Let Ψ be a condensing operator on a Banach space H, i.e., Ψ is continuous and takes bounded sets into bounded sets, and $\nu(\Psi(C)) \leq (C)$ for every bounded set C of $\mathbb{H}$ with $\nu(C) > 0$, where $\nu(\cdot)$ denotes the Kuratowskii measure of noncompactness. If $\Psi(F) \subset F$ for a convex, closed and bounded set F of $\mathbb{H}$, then Ψ has a fixed point in $\mathbb{H}$.*

Lemma 2.12. *[8] . Let Ψ_1 and Ψ_2 be two operators of a Banach space $\mathbb{H}$ such that*

- (a) *Ψ_1 is a contraction, and*
- (b) *Ψ_2 is completely continuous.*

Then, either

- (i) *the operator equation $\Psi_1 z + \Psi_2 z = z$ has a solution, or*
- (ii) *the set $\mathbb{M} = \{z \in \mathbb{H} : \alpha\Psi_1(\frac{z}{\alpha}) + \alpha\Psi_2(z) = z\}$ is unbounded for $\alpha \in (0, 1)$.*

Lemma 2.13. *[10] For any $p \geq 1$ and for arbitrary $L_Q(\mathbb{Y}, \mathbb{X})$-valued predictable process $\Psi(\cdot)$,*

$$\sup_{s\in[0,t]} \mathbb{E}\Big\| \int_0^s \Psi(r)dw(r)\Big\|_{\mathbb{X}}^{2p} \leq (p(2p-1))^p \Big(\int_0^t (\mathbb{E}\|\Psi(s)\|_Q^{2p})^{\frac{1}{p}} \Big)^p. \tag{2.4}$$

We now end this part by stating the definition of mild solution for Eq. (1.1).

Definition 2.14. *An $\mathcal{F}_t-$adapted stochastic process $z : (-\infty, a] \longrightarrow \mathbb{X}$ is said to be a mild solution of Eq.* (1.1) *if $z_0 = \varphi \in \mathcal{B}$, $z_\sigma(s, z_s) \in \mathcal{B}$ satisfying $z_0 \in \mathcal{L}_2^0(\Omega, \mathbb{X}), z|_J \in \mathcal{PC}$. The function $R(t-s)g(s, z_s, \int_0^s h(s, \tau, z_\tau)d\tau)$ is integrable for each $s \in [0, a]$ and the following conditions hold:*

- (i) *$\{z_t : t \in J\}$ is $\mathcal{B}-$valued and the restriction of $z(\cdot)$ to the interval $(t_i, t_{i+1}], i = 1, 2, \ldots, m$ is continuous ;*
- (ii) *$\Delta z(t_i) = I_i(z_{t_i}), i = 1, 2, \ldots, m;$*
- (iii) *for each $t \in J, z(t)$ satisfies the following integral equation*

$$z(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s, z_s, \int_0^s h(s, \tau, z_\tau)d\tau)ds$$
$$+ \int_0^t R(t-s)\xi(s, z_{\sigma(s,z_s)})dw(s) + \sum_{0<t_i<t} R(t-t_i)I_i(z_{t_i}).$$

3. Existence results

This section is devoted to the study of existence of mild solutions for Eq. (1.1). Throughout this work, we assume that $\sigma : J\times\mathcal{B} \longrightarrow (-\infty, a]$ is continuous and $M = \sup_{t\in J}\|R(t)\|$. In the following, we firstly introduce the subsequent hypotheses:

(A_1) Let $\mathcal{Z}(\sigma^-) = \{\sigma(s,\varphi) \leq 0, \sigma(s,\varphi) : (s,\varphi) \in J\times\mathcal{B}\}$. The function $t \longrightarrow \varphi_t$ is well defined from $\mathcal{Z}(\sigma^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^{\varphi} : \mathcal{Z}(\sigma^-) \longrightarrow (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^{\varphi}(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{Z}(\sigma^-)$.

(A_2) The resolvent operator $\big(R(t)\big)_{t\geq 0}$ is compact for $t > 0$.

(A_3) The function $\xi : J\times\mathcal{B} \longrightarrow \mathcal{L}_Q(\mathbb{Y},\mathbb{X})$ satisfies the following properties:

- (i) The function $\xi(\cdot, z) : J \longrightarrow \mathcal{L}_Q(\mathbb{Y},\mathbb{X})$ is strongly measurable for every $z \in \mathcal{B}$,
- (ii) The function $\xi(t,\cdot) : \mathcal{B} \longrightarrow \mathcal{L}_Q(\mathbb{Y},\mathbb{X})$ is continuous on $\mathcal{Z}(\sigma^-)\cup J$,
- (iii) There exist an integrable function $l : J \longrightarrow [0,\infty)$ and a non-decreasing function $\mu_l \in \mathcal{C}([0,\infty);(0,\infty))$ such that, for every $(t,z) \in J\times\mathcal{B}$,

$$\mathbb{E}\|\xi(t,z)\|^2 \leq l(t)\mu_l(\|z\|^2_{\mathcal{B}}),\ \lim_{\delta\longrightarrow\infty}\inf\frac{\mu_l(\delta)}{\delta} = \Theta < \infty.$$

(A_4) There exist constants $d_1 > 0$ and $d_1^* > 0$ for all $\varphi,\psi \in \mathcal{B}, t, s \in J$, such that

$$\mathbb{E}\left\|\int_0^t [h(t,s,\varphi) - h(t,s,\psi)]ds\right\|^2 \leq d_1\|\varphi-\psi\|^2$$

and $d_1^* = a\sup_{0\leq s\leq t\leq a}\mathbb{E}\|h(t,s,0)\|^2$.

(A_5) The function $g : J\times\mathcal{B}\times\mathbb{X} \longrightarrow \mathbb{X}$ is continuous and there exist constants $d_2 > 0$ and $d_2^* > 0$ for all $t \in J, \varphi_1, \varphi_2 \in \mathcal{B}, z_1, z_2 \in \mathbb{X}$ such that

$$\mathbb{E}\|g(t,\varphi_1,z_1) - g(t,\varphi_2,z_2)\|^2 \leq d_2\big(\|\varphi_1-\varphi_2\|^2_{\mathcal{B}} + \mathbb{E}\|z_1-z_2\|^2_{\mathbb{X}}\big)$$

and $d_2^* = \sup_{t\in J} E\|g(t,0,0)\|^2$.

(A_6) The maps I_i, are completely continuous and there exist positive constant $\lambda_i, i = 1,2,\ldots,m$, such that $\mathbb{E}\|I_i(z)\|^2 \leq \lambda_i\|z\|^2_{\mathcal{B}}$ for all $z \in \mathcal{B}$.

Remark 3.1. *Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t+\theta)$. Consequently, if the function $z(\cdot)$ in the Axiom A is such that $z_0 = \varphi$, then $z_t = \varphi_t$. We observe that φ_t is well defined for $t < 0$, since the domain of φ is $(-\infty, 0]$.*

Theorem 3.2. *Assume that $(\mathbf{R_1})$-$(\mathbf{R_2})$, $(A_1)-(A_6)$ are satisfied and $z_0 \in \mathcal{L}_2^0(\Omega,\mathbb{X}), \varphi \in \mathbb{X}$. If*

$$T_0 = 1 - 4\left(4M^2a^2d_2\tilde{K}_1^2 + 8d_2d_1M^2a^2\tilde{K}_1^2 + 2M^2m\tilde{K}_1^2\sum_{i=1}^{m}\lambda_i\right) > 0, \tag{3.1}$$

and

$$\frac{8\tilde{K}_1^2M^2Tr(Q)}{T_0}\int_0^a l(s)ds \leq \int_{T^*}^{\infty}\frac{ds}{\mu_l(s)},$$

where

$$T^* = C + \frac{8\tilde{K}_1^2}{T_0}\left[]M^2Hd^2\|\varphi\|^2_{\mathcal{B}} + 2M^2a^2\left(d_2C + 2d_2d_1C + 2d_2d_1^* + d_2^*\right) + M^2m\sum_{i=1}^{m}\lambda_iC\right]$$

with $C = 2(\tilde{K}_2 + J_0^{\varphi})^2\|\varphi\|^2_{\mathcal{B}}$, then there exists a mild solution of Eq (1.1).

Proof. Let $\mathbb{F} = \{z \in \mathcal{PC} : z(0) = \varphi(0)\}$ be a space endowed with the uniform convergence topology. We define the operator $\Psi : \mathbb{F} \longrightarrow \mathbb{F}$ by

$$(\Psi z)(t) = \begin{cases} 0, \quad t \in (-\infty, 0] \\ R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \\ + \displaystyle\int_0^t R(t-s)\xi(s, z_{\sigma(s,\tilde{z}_s)})dw(s) + \sum_{0<t_i<t} R(t-t_i)I_i(\tilde{z}_{t_i}), \quad t \in J, \end{cases}$$

where $\tilde{z} : (-\infty, a] \longrightarrow \mathbb{X}$ is such that $\tilde{z}_0 = \varphi$ and $\tilde{z} = z$ on J. In view of hypotheses (A_2),(A_4) and (A_5), we have the following inequality

$$\begin{aligned} &\mathbb{E}\left\| \int_0^t R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \right\|^2 \\ &\quad \leq 2M^2 a \int_0^t \left(d_2[\|\tilde{z}_s\|_{\mathcal{B}}^2 + 2d_1\|\tilde{z}_s\|_{\mathcal{B}}^2 + 2d_1^*] + d_2^* \right) ds. \end{aligned}$$

Then from the Bochner theorem [33], it follows that $R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)$ is integrable on $[0, t)$, which allows us to conclude that Ψ is a well-defined operator from $\mathbb{F}$ into $\mathbb{F}$. We prove that Ψ has a fixed point, which is a mild solution of the Eq.(1.1). Now, we decompose Ψ as $\Psi_1 + \Psi_2$, where

$$\begin{aligned} (\Psi_1 z)(t) &= R(t)\varphi(0) + \int_0^t R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \\ (\Psi_2 z)(t) &= \int_0^t R(t-s)\xi(s, z_{\sigma(s,\tilde{z}_s)})dw(s) + \sum_{0<t_i<t} R(t-t_i)I_i(\tilde{z}_{t_i}), \ t \in J. \end{aligned}$$

Firstly, we show that Ψ_1 is a contraction. Next, we prove that Ψ_2 is a completely continuous. In order to apply Lemma 2.12 we give the proof in several steps.

Step 1: We will show the set $\mathbb{S} = \{z \in \mathbb{F} : \epsilon\Psi_1(\frac{z}{\epsilon}) + \epsilon\Psi_2(z) = z\}$ is bounded on J for some $\epsilon \in (0, 1)$. Consider the following nonlinear operator equation

$$z(t) = \epsilon\Psi z(t), \quad 0 < \epsilon < 1, \tag{3.2}$$

where the operator Ψ has already been defined. Next we give a priori estimate for the solutions of the above equation. Let $z \in \mathbb{F}$ be a possible solution of $z(t) = \epsilon\Psi z(t)$ for some $0 < \epsilon < 1$, we have

$$\begin{aligned} z(t) &= \epsilon R(t)\varphi(0) + \epsilon \int_0^t R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \\ &\quad + \epsilon \int_0^t R(t-s)\xi(s, z_{\sigma(s,\tilde{z}_s)})dw(s) + \epsilon \sum_{0<t_i<t} R(t-t_i)I_i(\tilde{z}_{t_i}), \quad t \in J. \end{aligned} \tag{3.3}$$

Using (3.3), hypotheses $(A_2)-(A_6)$, Hölder and Burkholder-Davis-Gundy's inequalities, we have

$$\begin{aligned}
&\mathbb{E}\|z(t)\|^2 \qquad (3.4)\\
&\quad\le 4\mathbb{E}\|\epsilon R(t)\varphi(0\|^2 + 4\mathbb{E}\|\epsilon\int_0^t R(t-s)g(s,\tilde{z}_s,\int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds\|^2\\
&\qquad + 4\|\epsilon\int_0^t R(t-s)\xi(s,z_{\sigma(s,\tilde{z}_s)})dw(s)\|^2 + 4\|\epsilon\sum_{0<t_i<t}R(t-t_i)I_i(\tilde{z}_{t_i})\|^2\\
&\quad\le 4M^2H^2\|\varphi\|^2_{\mathcal{B}} + 8M^2a\int_0^t\Bigg\{d_2\Bigg(2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\\
&\qquad + 2d_1\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\right] + 2d_1^*\Bigg) + d_2^*\Bigg\}ds\\
&\qquad + 4M^2Tr(Q)\int_0^t l(s)\mu_l\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le \tau\le a}\mathbb{E}\|z(\tau)\|^2\right]ds\\
&\qquad + 4M^2m\sum_{i=1}^m\lambda_i\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\right]\\
&\quad\le 4M^2H^2\|\varphi\|^2_{\mathcal{B}} + 8M^2a^2\Bigg[d_2\Bigg(2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\\
&\qquad + 2d_1\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\right] + 2d_1^*\Bigg) + d_2^*\Bigg]\\
&\qquad + 4M^2Tr(Q)\int_0^t l(s)\mu_l\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le s\le a}\mathbb{E}\|z(s)\|^2\right]ds\\
&\qquad + 4M^2m\sum_{i=1}^m\lambda_i\left[2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\tilde{K}_1^2\sup_{0\le t\le a}\mathbb{E}\|z(t)\|^2\right]. \qquad (3.5)
\end{aligned}$$

Let $\vartheta(s)=\sup_{0\le s\le a}\mathbb{E}\|z(s)\|^2$ and $C=2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}}$. From (3.4), we have

$$\begin{aligned}
&\vartheta(t)\\
&\quad\le 4\Bigg\{M^2H^2\|\varphi\|^2_{\mathcal{B}} + 2M^2a^2\Bigg\{d_2\left[C+2\tilde{K}_1^2\vartheta(t)+2d_1\left[C+2\tilde{K}_1^2\vartheta(t)\right]+2d_1^*\right]+d_2^*\Bigg\}\\
&\qquad + M^2Tr(Q)\int_0^t l(s)\mu_l\left[C+2\tilde{K}_1^2\vartheta(s)\right]ds + M^2m\sum_{i=1}^m\lambda_i\left[C+2\tilde{K}_1^2\vartheta(t)\right]\Bigg\}\\
&\quad\le 4\Bigg\{M^2H^2\|\varphi\|^2_{\mathcal{B}} + 2M^2a^2\left[d_2C+2d_2d_1C+2d_2d_1^*+d_2^*\right]\\
&\qquad + M^2Tr(Q)\int_0^t l(s)\mu_l\left[C+2\tilde{K}_1^2\vartheta(s)\right]ds + M^2m\sum_{i=1}^m\lambda_iC\\
&\qquad + \vartheta(t)\left[4M^2a^2d_2\tilde{K}_1^2+8d_2d_1M^2a^2\tilde{K}_1^2+2M^2m\tilde{K}_1^2\sum_{i=1}^m\lambda_i\right]\Bigg\}.
\end{aligned}$$

It follows that

$$\vartheta(t) \leq \frac{4}{T_0}\Big\{M^2H^2\|\varphi\|^2_{\mathcal{B}} + 2M^2a^2\left[d_2C + 2d_2d_1C + 2d_2d_1^* + d_2^*\right] + M^2Tr(Q)\int_0^t l(s)\mu_l\left[C + 2\tilde{K}_1^2\vartheta(s)\right]ds + M^2m\sum_{i=1}^m \lambda_i C\Big\}.$$

Let $\omega(t) = C + 2\tilde{K}_1^2\vartheta(t)$. Since $\sigma(s;\tilde{z}_s) \leq s$ for every $s \in [0,a]$, we have

$$\omega(t) \leq T^* + \frac{8\tilde{K}_1^2M^2Tr(Q)}{T_0}\int_0^t l(s)\mu_l(\omega(s))ds.$$

Denoting by $\nu(t)$ the right-hand side of the last inequality, we find that

$$\nu^{'}(t) \leq \frac{8\tilde{K}_1^2M^2Tr(Q)}{T_0} l(t)\mu_l(\nu(t))$$

and

$$\int_{T^*}^{\nu(t)}\frac{ds}{\mu_l(s)} \leq \frac{8\tilde{K}_1^2M^2Tr(Q)}{T_0}\int_0^a l(s)ds \leq \int_{T^*}^{\infty}\frac{ds}{\mu_l(s)}.$$

Consequently, we see that $\nu(t)$ is bounded, which proves that z is bounded in $\mathbb{F}$ for any $z \in \mathbb{S}$. Hence $\mathbb{S}$ is bounded on J for $\epsilon \in (0,1)$.

Step 2: Ψ_1 is a contraction operator on $\mathbb{F}$.

Let $t \in J$ and $z, y \in \mathbb{F}$. Then, by assumptions (A_2),(A_4) and (A_5) and Lemma 2.10 , we have

$$\begin{aligned}
&\mathbb{E}\|(\Psi_1 z)(t) - (\Psi_1 y)\|^2 \\
&\leq \mathbb{E}\left\|\int_0^t R(t-s)\big[g\left(s,\tilde{z}_s,\int_0^s h(s,\tau,\tilde{z}_\tau)d\tau\right) - g\left(s,\tilde{y}_s,\int_0^s h(s,\tau,\tilde{y}_\tau)d\tau\right)\big]ds\right\|^2 \\
&\leq M^2a\int_0^t \mathbb{E}\left\|g\left(s,\tilde{z}_s,\int_0^s h(s,\tau,\tilde{z}_\tau)d\tau\right) - g\left(s,\tilde{y}_s,\int_0^s h(s,\tau,\tilde{y}_\tau)d\tau\right)\right\|^2 ds \\
&\leq M^2a^2(d_2 + d_1d_2)\|\tilde{z}_s - \tilde{y}_s\|^2_{\mathcal{B}} \\
&\leq M^2a^2(d_2 + d_1d_2)\tilde{K}_1^2 \sup_{0\leq s\leq a}\|\tilde{z}(s) - \tilde{y}(s)\|^2 \\
&\leq M^2a^2(d_2 + d_1d_2)\tilde{K}_1^2\|\tilde{z} - \tilde{y}\|_{\mathcal{PC}} \\
&= L_0\|x - y\|^2_{\mathcal{PC}},
\end{aligned}$$

where $L_0 = M^2a^2(d_2 + d_1d_2)\tilde{K}_1^2$. By (3.1), we see that $L_0 < 1$. As a consequence Ψ_1 is a contraction operator on $\mathbb{F}$.

Step 3: Ψ_2 is a completely continuous operator on $\mathbb{F}$. We will do it into several steps.

(a) $\Psi_2 : \mathbb{F} \longrightarrow \mathbb{F}$ is continuous.
Let $\{z^n\}_{n=0}^{\infty} \subseteq \mathbb{F}$, with $z^n \longrightarrow z$ in $\mathbb{F}$. Then, there is a number $q > 0$ such that $\mathbb{E}\|z^n\|^2 \leq q$ for all n and a.e. $t \in J$, so $z^n \in B_q(0,\mathbb{F}) = \{z \in \mathbb{F} : \mathbb{E}\|z\|^2 \leq q\}$ and $z \in B_q(0,\mathbb{F})$. From Axiom A, it is not hard to see that $(\tilde{z}^n)_s \longrightarrow \tilde{z}_s$ uniformly for $s \in (-\infty, a]$ as $n \longrightarrow \infty$. By hypotheses (A_1) and (A_3), we obtain

$$\xi(s,\tilde{z}^n_{\sigma(s,(\tilde{z}^n)_s)}) \longrightarrow \xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)}) \quad \text{as} \quad n \longrightarrow \infty$$

for each $s \in [0,t]$, and

$\mathbb{E}\left\|\xi(s,\tilde{z}^n_{\sigma(s,(\tilde{z}^n)_s)})-\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})\right\|^2\leq 2l(t)\mu_l[2(\tilde{K}_2+J_0^{\varphi})^2\|\varphi\|^2_{\mathcal{B}}+2\tilde{K}_1^2q].$
Then, by the dominated convergence theorem and the continuity of $I_i,\ i=1,2,\ldots,m$, we get

$$\begin{aligned}
&\|\Psi_2z^n-\Psi_2z\|^2_{\mathcal{PC}}\\
&\quad\leq 2\mathbb{E}\left\|\int_0^t R(t-s)\left[\xi(s,\tilde{z}^n_{\sigma(s,(\tilde{z}^n)_s)})-\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})\right]dw(s)\right\|^2\\
&\qquad+2\mathbb{E}\left\|\sum_{0<t_i<t}R(t-t_i)[I_i(\tilde{z}^n_{t_i})-I_i(\tilde{z}_{t_i})]\right\|^2\\
&\quad\leq 2M^2Tr(Q)\int_0^t\mathbb{E}\left\|\xi(s,\tilde{z}^n_{\sigma(s,(\tilde{z}^n)_s)})-\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})\right\|^2\\
&\qquad+2M^2m\sum_{i=1}^m\mathbb{E}\left\|I_i(\tilde{z}^n_{t_i})-I_i(\tilde{z}_{t_i})\right\|^2\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Consequently, we get

$$\lim_{n\to\infty}\|\Psi_2z^n-\Psi_2z\|^2_{\mathcal{PC}}=0,$$

and this proves that Ψ_2 is continuous.

(b) Ψ_2 maps bounded sets into bounded sets in $\mathbb{F}$.
For each $q>0$, let $B_q(0,\mathbb{F})=\{z\in\mathbb{F}:\mathbb{E}\|z\|^2\leq q\}$. Then, $B_q(0,\mathbb{F})$ is a bounded closed convex subset of $\mathbb{F}$. In fact, it suffices to show that there is a positive constant N_0 such that $\mathbb{E}\|\Psi_2x\|^2\leq N_0$ for each $z\in B_q(0,\mathbb{F})$.
We set $q^*=2(\tilde{K}_2+J_0^{\varphi})^2\|\varphi\|^2_{\mathcal{B}}+2\tilde{K}_1^2q$. From Lemma 2.10, (A_3) and (A_6), we have

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2z)(t)\|^2\\
&\quad\leq 2\mathbb{E}\left\|\int_0^t R(t-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s)\right\|^2+2\mathbb{E}\left\|\sum_{0<t_i<t}R(t-t_i)I_i(\tilde{z}_{t_i})\right\|^2\\
&\quad\leq 2M^2Tr(Q)\int_0^a\mathbb{E}\|\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})\|ds+2M^2m\sum_{i=0}^m\mathbb{E}\|I_i(\tilde{z}_{t_i})\|^2\\
&\quad\leq 2M^2Tr(Q)\int_0^a l(s)\mu_l(q^*)ds+2M^2m\sum_{i=1}^m\lambda_iq^*\\
&\quad\leq N_0,
\end{aligned}$$

which gives that, for each $z\in B_q(0,\mathbb{F})$, $E\|\Psi_2z\|^2\leq N_0$.

Now it remains to show that $\Psi_2(B_q(0,\mathbb{F}))$ is equicontinuous and $\Psi_2(B_q(0,\mathbb{F}))(t)$ is precompact in $\mathbb{F}$. For this purpose, we decompose Ψ_2 as $\Upsilon_1+\Upsilon_2$, where Υ_1 and Υ_2 are the operators on $B_q(0,\mathbb{F})$ defined, respectively, by

$$(\Upsilon_1z)(t)=\int_0^t R(t-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s)$$

and

$$(\Upsilon_2z)(t)=\sum_{0<t_i<t}R(t-t_i)I_i(\tilde{z}_{t_i}).$$

(c) First, we show that $\Upsilon_1(B_q(0,\mathbb{F}))$ is equicontinuous and $\Upsilon_1(B_q(0,\mathbb{F}))(t)$ is relatively compact in $\mathbb{F}$. Let $0 < t_1 < t_2 \leq a$, for each $z \in B_q(0,\mathbb{F})$. Using (A_2) and (A_3), we obtain

$$\begin{aligned}
&\mathbb{E}\|(\Upsilon_1 z)(t_2) - (\Upsilon_1 z)(t_1)\|^2 \\
&\quad = \mathbb{E}\left\|\int_0^{t_2} R(t_2-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s) - \int_0^{t_1} R(t_1-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s)\right\|^2 \\
&\quad \leq 2\mathbb{E}\left\|\int_0^{t_1} \big[R(t_2-s) - R(t_1-s)\big]\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s)\right\|^2 \\
&\qquad + 2\mathbb{E}\left\|\int_{t_1}^{t_2} R(t_2-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s)\right\|^2 \\
&\quad \leq 2Tr(Q)\int_0^{t_1} l(s)\mu_l(q^*)\|R(t_2-s) - R(t_1-s)\|^2 ds \\
&\qquad + 2Tr(Q)\int_{t_1}^{t_2} l(s)\mu_l(q^*)\|R(t_2-s)\|^2 ds \\
&\quad \leq 2Tr(Q)\mu_l(q^*)\int_0^{a} l(s)\|R(t_2-s) - R(t_1-s)\|^2 ds \\
&\qquad + 2Tr(Q)M^2\mu_l(q^*)\int_{t_1}^{t_2} l(s)ds.
\end{aligned}$$

Since $R(t)$ is continuous in the uniform operator topology, it follows that the right-hand side of the above inequality tends to zero and hence $\mathbb{E}\|(\Upsilon_1 z)(t_2) - (\Upsilon_1 z)(t_1)\|^2$ converges to zero independent of $z \in B_q(0,\mathbb{F})$ as $t_2 - t_1 \longrightarrow 0$. Thus the set $\{\Upsilon_1 z : z \in B_q(0,\mathbb{F})\}$ is equicontinuous. The equicontinuity for the other cases $t_1 < t_2 < 0$ or $t_1 \leq 0 \leq t_2 \leq a$ are very simple.

Next, we show the precompactness of $\Upsilon_1(B_q(0,\mathbb{F}))(t)$ in $\mathbb{F}$. By the virtue of the compactness of the resolvent operator $R(t)$ for $t > 0$ and the continuity of ξ, we see that the set

$$\{R(t-s)\xi(s,\theta) : s \in [0,a], \|\theta\|_{\mathcal{B}}^2 \leq q^*\}$$

is relatively compact in $\mathbb{X}$. Then, applying the mean value theorem for the Bochner integral, we get

$$(\Upsilon_1 z)(t) \in t\overline{conv}(\{R(t-s)\xi(s,\theta) : s \in [0,a], \|\theta\|_{\mathcal{B}}^2 \leq q^*\}),$$

which implies that $\{(\Upsilon_1 z)(t) : z \in B_q(0,\mathbb{F})\}$ is relatively compact in $\mathbb{F}$.

(d) Υ_2 is completely continuous.

We prove that Υ_2 is completely continuous. We can conclude that Υ_2 is continuous based on the proof in Step 3 (a). From the definition of Υ_2, for $q > 0,\ t \in [t_i, t_{i+1}], i = 1,2,\ldots,m$, and $z \in B_q(0,\mathbb{F})$, we find that

$$\overline{\Upsilon_2} z(t) \in \begin{cases} \sum_{j=1}^{i} R(t-t_j)I_j(B_{q^*}(0,\mathbb{X})), & t \in (t_i, t_{i+1}), \\ \sum_{j=1}^{i} R(t_{i+1}-t_j)I_j(B_{q^*}(0,\mathbb{X})) & \text{if} \quad t = t_{i+1}, \\ \sum_{j=1}^{i-1} R(t_i-t_j)I_j(B_{q^*}(0,\mathbb{X})) + I_i(B_{q^*}(0,\mathbb{X})) & \text{if} \quad t = t_i, \end{cases}$$

where $q^* = 2(\tilde{K}_2 + J_0^{\varphi})^2\|\varphi\|_{\mathcal{B}}^2 + 2\tilde{K}_1^2 q$, which proves that $[\overline{\Upsilon_2(B_q)}]_i(t)$ is relatively compact in $\mathbb{F}$, for every $t \in [t_i, t_{i+1}]$, since the maps I_i are completely continuous for all $i = 1,2,\ldots,m.$ Moreover, using the compactness of the operators I_i and the assumption (A_2), we can prove that $[\overline{\Upsilon_2(B_q)}]_i$ is equicontinuous at t, for every $t \in [t_i, t_{i+1}]$. According to Lemma 2.8, we know that Υ_2 is completely continuous. As a result Ψ_2 is completely continuous. Hence, by Krasnoselskii-Schaefer fixed point theorem, we realize that Ψ has a fixed point on $\mathbb{F}$, which is a mild solution of Eq. (1.1). This completes the proof.

■

Instead of the assumption (A_6) discussed in Theorem 3.2, assume that the maps I_i satisfy some Lipschitz conditions. In this instance, we can also prove the existence of mild solutions. In addition, let us introduce the following condition:

(A_7) The maps I_i are completely continuous and there are positive constants b_i, c_i such that

$$\mathbb{E}\|I_i(z)-I_i(y)\|^2 \leq b_i\|z-y\|^2_{\mathcal{B}},$$

and $c_i=\sup_{t\in J}\mathbb{E}\|I_i(0)\|^2$, for $z,y\in\mathcal{B}, i=1,2,\ldots,m$.

Theorem 3.3. *Assume that* $(\mathbf{R_1})$*-*$(\mathbf{R_2})$*,*(A_1)*-*(A_5) *and* (A_7) *hold and* $z_0\in\mathcal{L}^0_2(\Omega,\mathbb{X})$*. Then there exists a mild solution of Eq.* (1.1) *provided that*

$$8M^2\tilde{K}_1^2\bigg(2a^2d_2+4a^2d_2d_1+2m\sum_{i=1}^m b_i+Tr(Q)\Theta\int_0^a l(s)ds\bigg)<1. \tag{3.6}$$

Proof. Let Ψ be the map defined as in the proof of Theorem 3.2. For better readability, we split the proof into two steps.

Step 1: $\Psi(B_q(0,\mathbb{F}))\subseteq B_q(0,\mathbb{F})$ for some $r>0$.
We affirm that there exist a positive constant $r>0$ such that $\Psi(B_q(0,\mathbb{F}))\subseteq B_q(0,\mathbb{F})$. We proceed by contradiction. Suppose that it is not true. Then for each $r>0$, there exists a function $z^q(t^q)\in B_q(0,\mathbb{F})$ such that $\Psi(z^q)\notin B_q(0,\mathbb{F})$, i.e., $q<\mathbb{E}\|(\Psi z^q)(t^q)\|^2$ for some $t^q\in J$. Thus, from the assumptions we have

$$\begin{aligned}
q &< \mathbb{E}\|(\Psi z^q)(t^q)\|^2\\
&\leq 4\mathbb{E}\|R(t^q)\varphi(0)\|^2+4\mathbb{E}\left\|\int_0^{t^q}R(t^q-s)g(s,\tilde{z}^q_s,\int_0^{t^q}h(s,\tau,\tilde{z}^q_\tau)d\tau)ds\right\|^2\\
&+4\mathbb{E}\left\|\int_0^{t^q}R(t^q-s)\xi(s,\tilde{z}^q_{\sigma(s,\tilde{z}^q_s)})dw(s)\right\|^2+4\mathbb{E}\left\|\sum_{0<t_i<t}R(t^q-s)I_i(\tilde{z}^q_{t_i})\right\|^2\\
&\leq 4M^2H^2\|\varphi\|^2+8M^2a\int_0^{t^q}\left\{d_2\left[\|\tilde{z}^q_s\|^2_{\mathcal{B}}+2d_1(\|\tilde{z}^q_s\|^2_{\mathcal{B}})+2d_1^*\right]+d_2^*\right\}\\
&+4M^2Tr(Q)\int_0^{t^q}\mathbb{E}\|\xi(s,\tilde{z}^q_{\sigma(s,\tilde{z}^q_s)})\|^2ds+4M^2m\sum_{i=1}^m\mathbb{E}\|I_i(\tilde{z}^q_{t_i})\|^2\\
&\leq 4M^2H^2\|\varphi\|^2+8M^2a^2\left\{d_2\left[(C+2\tilde{K}_1^2q)+2d_1(C+2\tilde{K}_1^2q)+2d_1^*\right]+d_2^*\right\}\\
&+4M^2Tr(Q)\int_0^a l(s)\mu_l(C+2\tilde{K}_1^2q)ds+4M^2m\sum_{i=1}^m\left\{2b_i(C+2\tilde{K}_1^2q)+2c_i\right\}
\end{aligned}$$

where $C=2(\tilde{K}_2+J_0^\varphi)^2\|\varphi\|^2$. Dividing both sides by q and taking the limit as $q\to\infty$, we obtain

$$1\leq 8M^2\tilde{K}_1^2\bigg(2a^2d_2+4a^2d_2d_1+2m\sum_{i=1}^m b_i+Tr(Q)\Theta\int_0^a l(s)ds\bigg)$$

which contradicts (3.6). Hence, for some positive number q, we have $\Psi(B_q(0,\mathbb{F}))\subseteq B_q(0,\mathbb{F})$.

Step 2: Ψ is a condensing map. Let $\Psi = \Psi_1 + \Psi_2$, where

$$(\Psi_1 x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s, \tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds + \sum_{0<t_i<t} R(t-t_i)I_i(\tilde{z}_{t_i})$$

$$(\Psi_2 z)(t) = \int_0^t R(t-s)\xi(s, \tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s),\ t \in J.$$

From the proof of Theorem 3.2, Ψ_2 is completely continuous on $B_q(0,\mathbb{F})$. Next, we have to show that Ψ_1 is a contraction map. Let $z, y \in B_q(0,\mathbb{F})$. Then, using hypotheses (A_2), (A_4), (A_5) and (A_7) we get

$$\begin{aligned}
&\mathbb{E}\|(\Psi_1 z)(t) - (\Psi_1 y)(t)\|^2 \\
&\leq \mathbb{E}\Big\| \int_0^t R(t-s)\big[g(s,\tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau) - g(s,\tilde{y}_s, \int_0^s h(s,\tau,\tilde{y}_\tau)d\tau)\big]ds \\
&\quad + \sum_{0<t_i<t} R(t-t_i)\big[I_i(\tilde{z}_{t_i}) - I_i(\tilde{y}_{t_i})\big]\Big\|^2 \\
&\leq 2M^2a^2\,(d_2 + d_1 d_2)\,\tilde{K}_1^2 \sup_{0\leq s\leq a} \mathbb{E}\|\tilde{z}(s) - \tilde{y}(s)\|^2 \\
&\quad + 2M^2 m \sum_{i=1}^m b_i \tilde{K}_1^2 \sup_{0\leq s\leq a} \mathbb{E}\|\tilde{z}(s) - \tilde{y}(s)\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\Psi_1 z - \Psi_1 y\|^2_{\mathcal{PC}} &\leq 2\tilde{K}_1^2\Big[M^2a^2(d_2 + d_1 d_2) + mM^2\sum_{i=1}^m b_i\Big]\|z - y\|_{\mathcal{PC}} \\
&= \kappa\|z - y\|_{\mathcal{PC}},
\end{aligned}$$

where $\kappa = 2\tilde{K}_1^2\left[M^2a^2(d_2 + d_1 d_2) + mM^2\sum_{i=1}^m b_i\right]$. By (3.6), we deduce that $\kappa < 1$, which yields that Ψ_1 is a contraction map. Considering Sadovskii's fixed point theorem, we conclude that there exists a fixed point for Ψ on $B_q(0,\mathbb{F})$, which is a mild solution for Eq. (1.1). ■

4. Controllability results

In this section, we examine the controllability of the following impulsive stochastic integro-differential equation with state-dependent delay:

$$\begin{cases}
dz(t) = \Big[Az(t) + \int_0^t \Gamma(t-s)z(s)ds + B\vartheta(t) + g\left(t, z_t, \int_0^t h(t,s,z_s)ds\right)\Big]dt \\
\qquad\qquad + \xi(t, z_{\sigma(t,z_t)})dw(t),\ t \in J = [0,a], t \neq t_i, \\
\Delta z(t_i) = I_i(z_{t_i}), i = 1, \ldots, m, \\
z_0 = \varphi \in \mathcal{B},
\end{cases} \tag{4.1}$$

where g, h, A, ξ, I_i, are the same as in the Eq. (1.1). The control function $\vartheta(\cdot)$ takes its values in $L_2(J,\mathcal{U})$ of admissible control functions for a separable Hilbert space $\mathcal{U}$, and B is a bounded linear operator from $\mathcal{U}$ into $\mathbb{X}$. First, we give the definitions of mild solution and controllability for the system (4.1).

Definition 4.1. *A $\mathcal{F}_t$−adapted stochastic process $z : (-\infty, a] \longrightarrow \mathbb{X}$ is called a mild solution of the system* (4.1) *if $z_0 = \varphi \in \mathcal{B}$, $z_{\sigma(s,z_s)} \in \mathcal{B}$ satisfying $z_0 \in \mathcal{L}_2^0(\Omega,\mathbb{X})$, $z|_J \in \mathcal{PC}$. The function $R(t-s)g(s, z_s, \int_0^s h(s,\tau,z_\tau)d\tau)$ is integrable for each $s \in [0,a]$ and the following conditions hold:*

(i) *$\{z_t : t \in J\}$ is $\mathcal{B}-$valued and the restriction of $z(\cdot)$ to the interval $(t_i, t_{i+1}], i = 1, 2, \ldots, m$ is continuous ;*

(ii) $\Delta z(t_i) = I_i(z_{t_i}), i = 1, 2, \ldots, m;$

(iii) *for each $t \in J$, $z(t)$ satisfies the following integral equation*

$$z(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s, z_s, \int_0^s h(s,\tau,z_\tau)d\tau)ds + \int_0^t R(t-s)B\vartheta(s)ds \\ + \int_0^t R(t-s)\xi(s, z_{\sigma(s,z_s)})dw(s) + \sum_{0<t_i<t} R(t-t_i)I_i(z_{t_i}).$$

Definition 4.2. *The system* (4.1) *is said to be controllable on the interval J, if for every initial function $z_0 = \varphi \in \mathcal{B}$, there exists a stochastic control $\vartheta \in L^2(J,\mathcal{U})$ that is adapted to the filtration $\{\mathcal{F}\}_{t\geq 0}$ such that the mild solution of the system* (4.1) *satisfies $z(a) = z_1$.*

We aim to transfer system (4.1) from $z(0)$ to $z(a) = z_1$. To achieve that purpose, we must assume:

(A_8) The linear operator $\mathcal{W} : L^2(J,\mathcal{U}) \longrightarrow \mathbb{X}$, defined by

$$\mathcal{W}_\vartheta = \int_0^a R(t-s)B\vartheta(s)ds,$$

has a bounded invertible operator $\mathcal{W}^{-1}$ which takes values in $L^2(J,\mathcal{U})/\ker\mathcal{W}$ and there exist positive constants M_1, M_2 such that $\|B\|^2 \leq M_1$ and $\|\mathcal{W}^{-1}\|^2 \leq M_2$.

(A_9) The function $\xi : J \times \mathcal{B} \longrightarrow \mathcal{L}_Q(\mathbb{Y},\mathbb{X})$ is continuous and there exists constant $M_\xi > 0, \tilde{M}_\xi > 0$ for $z, y \in \mathcal{B}$ such that

$$\mathbb{E}\|\xi(t,z) - \xi(t,y)\|^2 \leq M_\xi \|z-y\|^2_{\mathcal{B}}$$

and $\tilde{M}_\xi = \sup_{t\in J} \mathbb{E}\|\xi(t,0)\|^2$.

Theorem 4.3. *Assume that $(\mathbf{R_1}) - (\mathbf{R_2})$, $(A_4) - (A_5)$ and $(A_7) - (A_9)$ hold and $z_0 \in \mathcal{L}_2^0(\Omega,\mathbb{X})$. Then the system* (4.1) *is controllable provided that*

$$5G\tilde{K}_1^2\left(1 + 5M^2M_1M_2a^2\right) \leq 1 \tag{4.2}$$

where $G = 4M^2a^2d^2 + 8d_1d_2M^2a^2 + 4M^2aTr(Q)M_\xi + 4M^2m\sum_{i=1}^m b_i$ *and* $M = \sup_{0\leq t\leq a} \|R(t)\|$.

Proof. Define the control process with terminal state $z_1 = z(a)$.

$$\vartheta_z^a(t) = \mathcal{W}^{-1}\Big\{z_1 - R(a)\varphi(0) - \int_0^a R(a-s)g(s,\tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \\ - \int_0^a R(a-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s) - \sum_{i=1}^m R(a-t_i)I_i(\tilde{z}_{t_i})\Big\}(t).$$

Using this control, we define the operator $\varXi : \mathbb{F} \longrightarrow \mathbb{F}$ by

$$(\varXi z)(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s,\tilde{z}_s, \int_0^s h(s,\tau,\tilde{z}_\tau)d\tau)ds \\ + \int_0^t R(t-s)B\vartheta_z^a(s)ds + \int_0^t R(t-s)\xi(s,\tilde{z}_{\sigma(s,\tilde{z}_s)})dw(s) \\ + \sum_{0<t_i<t} R(t-t_i)I_i(\tilde{z}_{t_i}).$$

where $\tilde{z} : (-\infty, a] \longrightarrow \mathbb{X}$ is such that $\tilde{z}_0 = \varphi$ and $\tilde{z} = z$ on J. From the assumptions, we know that the map Ξ is well defined and continuous.

Now, we prove that the operator Ξ has a fixed point in $\mathbb{F}$, which is a mild system solution (4.1). Observe that $(\Xi z)(a) = z_1$. This means that the control ϑ_z^a steers the system from φ to z_1 in finite time a, implying that the system (4.1) is controllable.

Let $q^* = 2(\tilde{K}_2 + J_0^{\varphi})^2\|\varphi\|_{\mathcal{B}}^2 + 2\tilde{K}_1^2 q$ for each $q > 0$. For any $z \in B_q(0, \mathbb{F})$ and $q > 0$, from the assumptions $(A_4) - (A_5)$ and $(A_7) - (A_9)$, we have

$$
\begin{aligned}
&\mathbb{E}\|\vartheta_z^a(t)\|^2 \\
&\quad \leq 5\mathbb{E}\|\mathcal{W}^{-1}\| \Big\{ \mathbb{E}\|z_1\|^2 + \mathbb{E}\|R(t)\varphi(0)\|^2 \\
&\qquad + \mathbb{E}\Big\| \int_0^a R(a-s)g(s, \tilde{z}_s, \int_0^s h(s, \tau, \tilde{z}_\tau)d\tau)ds \Big\|^2 \\
&\qquad + \Big\| \int_0^a R(a-s)\xi(s, \tilde{z}_{\sigma(s, \tilde{z}_s)})dw(s) \Big\|^2 + \mathbb{E}\Big\| \sum_{0<t_i<t} R(a-t_i)I_i(\tilde{z}_{t_i}) \Big\|^2 \Big\} \\
&\quad \leq 5M_2 \Big\{ \mathbb{E}\|z_1\|^2 + M^2H^2\|\varphi\|_{\mathcal{B}}^2 + 2M^2a^2\left(d_2q^* + 2d_1d_2q^* + 2d_1^*d_2 + d_2^*\right) \\
&\qquad + M^2aTr(Q)\left(2M_\xi q^* + 2\tilde{M}_\xi\right) + M^2m\sum_{i=1}^m (2b_iq^* + 2c_i) \Big\} = \Omega.
\end{aligned}
$$

Furthermore, for any $z, y \in B_q(0, \mathbb{F})$, we obtain

$$
\begin{aligned}
&\mathbb{E}\|\vartheta_z^a(t) - \vartheta_y^a(t)\|^2 \\
&\quad \leq 3\tilde{K}_1^2 M_2 \Big\{ M^2a^2(d_2 + d_1d_2) + M^2aTr(Q)M_\xi + M^2m\sum_{i=1}^m b_i \Big\} \|z - y\|_{\mathcal{PC}}^2.
\end{aligned}
$$

For the sake of convenience, we break the proof into two steps.

Step 1: We show that Ξ maps $B_q(0, \mathbb{F})$ into itself.
It is enough to show that there exists a positive constant $q > 0$ such that $\Xi(B_q(0, \mathbb{F})) \subseteq B_q(0, \mathbb{F})$. Suppose that this assertion is false. Then for each > 0, there exists a function $z^q(t^q) \in B_q(0, \mathbb{F})$, such that $\Xi(z^q) \notin B_q(0, \mathbb{F})$, that is $q < \mathbb{E}\|(\Xi z^q)(t^q)\|^2$ for some $t^q \in J$. Thus, using hypotheses $(A_4) - (A_5)$ and $(A_7) - (A_9)$, we obtain

$$
\begin{aligned}
q &< \mathbb{E}\|(\Xi z^q)(t^q)\|^2 \\
&\leq 5M^2H^2\|\varphi\|_{\mathcal{B}}^2 + 10M^2a^2\left(d_2q^* + 2d_1d_2q^* + 2d_1^*d_2 + d_2^*\right) \\
&\quad + 5M^2a^2M_1\Omega + 10M^2aTr(Q)\left(M_\xi q^* + M_\xi^*\right) + 10M^2m\sum_{i=1}^m (b_iq^* + c_i).
\end{aligned}
$$

Dividing both sides by q and taking the limit as $q \to \infty$, we get

$$
1 < 5G\tilde{K}_1^2\left(1 + 5M^2M_1a^2M_2\right),
$$

where $G = 4M^2a^2d_2 + 8M^2a^2d_2d_1 + 4M^2aTr(Q)M_\xi + 4M^2m\sum_{i=1}^m b_i$, which is contrary to (4.2). Hence, for some positive number q, we have $\Xi(B_q(0, \mathbb{F})) \subseteq B_q(0, \mathbb{F})$.

Step 2: We prove that Ξ is a contraction operator. Let $z, y \in B_q(0, \mathbb{F})$, we obtain

$$
\begin{aligned}
&\mathbb{E}\|(\Xi z)(t) - (\Xi y)(t)\|^2 \\
&\quad \leq 4M^2a^2\left(d_2 + d_1d_2\right)\|z-y\|_{\mathcal{B}}^2 + 4M^2aTr(Q)M_\xi\|z-y\|_{\mathcal{B}}^2 \\
&\qquad + 4M^2m\sum_{i=1}^{m} b_i\|z-y\|_{\mathcal{B}}^2 \\
&\qquad + 12M^2M_1M_2a^2\big\{M^2a^2(d_2+d_1d_2) + M^2aTr(Q)M_\xi \\
&\qquad + M^2m\sum_{i=1}^{m} b_i\big\}\|z-y\|_{\mathcal{B}}^2 \\
&\quad \leq 4M^2a^2\tilde{K}_1^2\left(d_2 + d_1d_2\right)\|z-y\|_{\mathcal{PC}}^2 + 4M^2aTr(Q)\tilde{K}_1^2M_\xi\|z-y\|_{\mathcal{PC}}^2 \\
&\qquad + 4M^2m\tilde{K}_1^2\sum_{i=1}^{m} b_i\|z-y\|_{\mathcal{PC}}^2 \\
&\qquad + 12M^2M_1M_2a^2\tilde{K}_1^2\big\{M^2a^2(d_2+d_1d_2) + M^2aTr(Q)M_\xi \\
&\qquad + M^2m\sum_{i=1}^{m} b_i\big\}\|z-y\|_{\mathcal{PC}}^2 \\
&\qquad = 4G'\tilde{K}_1^2\left(1 + 3M_1M^2a^2M_2\right)\|z-y\|_{\mathcal{PC}}^2
\end{aligned}
$$

where $G' = M^2a^2(d_2 + d_1d_2) + M^2aTr(Q)M_\xi + M^2m\sum_{i=1}^{m} b_i$. Thanks to (4.2), we see that $4G'\tilde{K}_1^2\left(1 + 3M_1M^2a^2M_2\right) < 1$. Therefore Ξ is a contraction operator, and according to Banach's fixed point theorem it has a unique fixed point in $\mathbb{F}$, which is a mild solution Eq. (4.1). Thus, Eq. (4.1) is controllable. This completes the proof. ■

5. An example

To illustrate our obtained results, we consider the following impulsive stochastic integrodifferential equation with state-dependent delay of the form

$$
\begin{cases}
\dfrac{\partial}{\partial t}y(t,x) = \dfrac{\partial^2}{\partial x^2}y(t,x) + f\left(t, y(t-\tau,x), \displaystyle\int_0^t k(t,s,y(s-\tau,x))ds\right) \\
\qquad\qquad + \displaystyle\int_0^t u(t-s)\dfrac{\partial^2}{\partial x^2}y(s,x)ds \\
\qquad\qquad + \displaystyle\int_{-\infty}^t v(t,x,s-t)\mathcal{N}\left[y(s-\sigma_1(t)\sigma_2(\|y(t,x)\|),x)\right]dw(s), \\
0 \leq x \leq \pi, \tau > 0, t \in J = [0,a], \\
y(t,0) = y(t,\pi) = 0, t \in J, \\
y(\theta,x) = \varphi(\theta,x), \theta \in (-\infty,0], 0 \leq x \leq \pi, \\
\Delta y(t_i)(x) = \displaystyle\int_{-\infty}^{t_i} \alpha_i(t_i - s)y(s,x)ds, i = 1,2,\ldots,m, 0 \leq x \leq \pi,
\end{cases}
\tag{5.1}
$$

where $0 < t_1 < \cdots < t_m < a$ are prefixed numbers and $u : [0,\infty) \longrightarrow [0,\infty)$ is bounded and C^1-function such that u' is bounded and uniformly continuous,
$\sigma_1 : [0,\infty) \longrightarrow [0,\infty), \sigma_2 : [0,\infty) \longrightarrow [0,\infty)$ are continuous functions; $a > 0$; $f, k, v, \mathcal{N}$, $I_i, (i = 1,2,\ldots,m)$, and φ are appropriate functions, which will be specified later.

To study this system, we consider the space $\mathbb{X} = \mathbb{Y} = L^2([0, a], \mathbb{R})$ and define the operator $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ as

$$\begin{cases} D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi]) \\ A\kappa = \dfrac{\partial^2}{\partial x^2}\kappa. \end{cases}$$

Then, $A\kappa = -\sum_{n=1}^{\infty} n^2 \langle \kappa, s_n \rangle s_n,\ \kappa \in D(A)$, where $s_n(x) = \sqrt{\dfrac{2}{\pi}} \sin nx,\ n = 1, 2, \dots$ is the orthogonal basis of eigenvectors of A.
It is known that A is the infinitesimal generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathbb{X}$, which is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, s_n \rangle s_n \text{ for all } z \in \mathbb{X} \text{ and every } t \geq 0.$$

Therefore $(\mathbf{R_1})$ holds. In addition, the semigroup $\big(T(t)\big)_{t\geq 0}$ genered by A is compact for $t > 0$. Then by Theorem 2.4, the corresponding resolvent operator is also compact. Hence, (A_2) holds. Let $\varphi(t)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi], y(t)(x) = y(t, x)$. Assume that $q : (-\infty, 0] \longrightarrow (0, \infty)$ is a Lebesgue integrable function with $l = \int_{-\infty}^{0} \tilde{q}(t)dt < \infty$. For any $a > 0$, define

$$\mathcal{B} = \big\{\zeta : (-\infty, 0] \longrightarrow \mathbb{X} |\ (\mathbb{E}\|\zeta(\theta)\|^2)^{\frac{1}{2}} \quad \text{is a bounded and measurable function on } [-b, 0]$$
$$\text{and} \quad \int_{-\infty}^{0} q(s)(\mathbb{E}\|\zeta(s)\|^2)^{\frac{1}{2}} ds < \infty\big\}.$$

Now, we take $q(t) = e^{2t}, t < 0$, then we get $p = \int_{-\infty}^{0} q(t)dt = \frac{1}{2}$ and

$$\|\zeta\|_{\mathcal{B}} = \int_{-\infty}^{0} q(s) \sup_{s\leq\theta\leq 0} \left(\mathbb{E}\|\zeta(\theta)\|^2\right)^{\frac{1}{2}} ds.$$

It is easy to verify that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. In order to represent the system (5.1) to the abstract form (1.1), we define the functions $g : J \times \mathcal{B} \times \mathbb{X} \longrightarrow \mathbb{X}$, $\xi : J \times \mathcal{B} \longrightarrow \mathcal{L}_Q(\mathbb{Y}, \mathbb{X})$, $\sigma : J \times \mathcal{B} \longrightarrow (-\infty, 0]$, $I_k : \mathcal{B} \longrightarrow \mathbb{X}$ respectively by

$$\begin{aligned} g\left(t, \zeta, \int_0^t h(t, s, \zeta)ds\right)(x) &= f\left(t, \zeta(\theta, x), \int_0^t k(t, s, \zeta(\theta, x)ds\right) \\ &= \int_{-\infty}^{0} \gamma(\theta)\zeta(\theta)(x)d\theta + \int_0^t \int_{-\infty}^{0} \beta_1(s)\beta_2(\tau)\zeta(\tau, x)d\tau ds, \\ \xi(t, \zeta)(x) &= \int_{-\infty}^{0} v(t, x, \theta)\mathcal{N}(\zeta(\theta, x))d\theta, \\ \sigma(\theta, \zeta) &= \theta - \sigma_1(\theta)\sigma_2(\|\zeta(0)\|), \\ I_i(\zeta)(x) &= \int_{-\infty}^{0} \alpha_i(-\theta)\zeta(\theta, x)d\theta, i = 1, 2, \dots, m. \end{aligned}$$

On the other hand, let $\Gamma : D(A) \subset \mathbb{X} \to \mathbb{X}$ be the operator defined by

$$\Gamma(t)x = u(t)Ax, \text{ for } t \geq 0 \text{ and } x \in D(A).$$

Under these definition, system (5.1) is then rewriten in the following form

$$\begin{cases} dz(t) = \Big[Az(t) + \displaystyle\int_0^t \Gamma(t-s)z(s)ds + g\left(t, z_t, \int_0^t h(t, s, z_s)ds\right)\Big]dt \\ \qquad\qquad + \xi(t, z_{\sigma(t, z_t)})dw(t),\ t \in J = [0, a], t \neq t_i, \\ \Delta z(t_i) = I_i(z_{t_i}), i = 1, \dots, m \\ z_0 = \varphi \in \mathcal{B}. \end{cases} \tag{5.2}$$

Since u is bounded and C^1-function such that $u^{'}$ is bounded and uniformly continuous, ($\mathbf{R_2}$) is fulfilled. Hence, by Theorem 2.2, Eq. (2.1) has a unique resolvent operator $\big(R(t)\big)_{t\geq 0}$ on $\mathbb{X}$, which is also operator-norm continuous for $t \geq 0$ thanks to Theorem 2.4.

To establish the existence result for the mild solution of (5.1), we need the following conditions:

(i) The function $\gamma(\theta) \geq 0$ is continuous in $(-\infty, 0]$ satisfying

$$\int_{-\infty}^{0} \gamma^2(\theta)d\theta < \infty, \gamma_g = \left(\int_{-\infty}^{0} \frac{(\gamma(s))^2}{q(s)}ds\right)^{\frac{1}{2}} < \infty.$$

(ii) $\beta_1, \beta_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and

$$\gamma_g^* = \left(\int_{-\infty}^{0} \frac{(\beta_2(s))^2}{q(s)}ds\right)^{\frac{1}{2}} < \infty.$$

(iii) the functions $\alpha_i \in C(\mathbb{R}, \mathbb{R})$ and

$$c_i = \left(\int_{-\infty}^{0} \frac{\alpha_i^2(-\theta)}{q(s)}d\theta\right)^{\frac{1}{2}} < \infty, i = 1, 2, \ldots, m,$$

(iv) The function $v_2(t, x, \theta) \leq 0$ is continuous on $J \times [0, 2\pi] \times (-\infty, 0]$ and satisfies

$$\int_{-\infty}^{0} v(t, x, \theta)d\theta = \delta(t, x) < \infty.$$

(v) The function $\mathcal{N}(\cdot)$ is continuous and satisfies $0 \leq \mathcal{N}(y(\theta, x)) \leq \mu_l\left(\int_{-\infty}^{0} e^{2s}\|y(s, \cdot)\|_{\mathcal{L}_2}ds\right)$ for $(\theta, x) \in (-\infty, 0] \times [0, 2\pi]$, where $\mu_l(\cdot)$ is positive, continuous and nondecreasing in $[0, \infty)$.

Under the above assumptions, we obtain

$$\begin{aligned}
\|I_i(\zeta)\|_{\mathcal{L}_2} &= \left[\int_0^{\pi} \mathbb{E}\Big\|\int_{-\infty}^{0} \alpha_i(-\theta)\zeta(\theta, x)d\theta\Big\|^2 dx\right]^{\frac{1}{2}} \\
&= \left[\int_0^{\pi} \mathbb{E}\Big\|\int_{-\infty}^{0} \frac{\alpha_i(-\theta)}{q^{\frac{1}{2}}(\theta)} q^{\frac{1}{2}}(\theta)\zeta(\theta, x)d\theta\Big\|^2 dx\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^{\pi} \Big(\int_{-\infty}^{0} \frac{\alpha_i^2(-\theta)}{q(\theta)}d\theta\Big)\Big(\int_{-\infty}^{0} q(\theta)\mathbb{E}\|\zeta(\theta, x)\|^2 d\theta\Big)dx\right]^{\frac{1}{2}} \\
&= \Big(\int_{-\infty}^{0} \frac{\alpha_i^2(-\theta)}{q(\theta)}d\theta\Big)^{\frac{1}{2}}\Big[\int_0^{\pi}\int_{-\infty}^{0} q(\theta)\mathbb{E}\|\zeta(\theta, x)\|^2 d\theta dx\Big]^{\frac{1}{2}} \\
&\leq \Big(\int_{-\infty}^{0} \frac{\alpha_i^2(-\theta)}{q(\theta)}d\theta\Big)^{\frac{1}{2}}\Big[\int_{-\infty}^{0} q(\theta)\int_0^{\pi} \mathbb{E}\|\zeta(\theta, x)\|^2 dx d\theta\Big]^{\frac{1}{2}} \\
&\leq c_i\Big[\int_{-\infty}^{0} q(\theta)\sup_{s\leq\theta\leq 0} \mathbb{E}\|\zeta(\theta)\|^2 d\theta\Big]^{\frac{1}{2}} \\
&\leq c_i\|\zeta\|_{\mathcal{B}},
\end{aligned}$$

$$
\begin{aligned}
&\|g(t,\zeta,w)\|_{\mathcal{L}_2}\\
&\quad= \Big[\int_0^{\pi}\mathbb{E}\Big\|\int_{-\infty}^{0}\gamma(\theta)\zeta(\theta)(x)d\theta+\int_0^t\int_{-\infty}^0\beta_1(s)\beta_2(\tau)\zeta(\tau,x)d\tau ds\Big\|^2dx\Big]^{\frac{1}{2}}\\
&\quad\le \Big[\int_0^{\pi}\mathbb{E}\Big\|\int_{-\infty}^{0}\gamma(\theta)\zeta(\theta)(x)d\theta\Big\|^2dx\Big]^{\frac{1}{2}}\\
&\qquad+\Big[\int_0^{\pi}\mathbb{E}\Big\|\int_0^t\int_{-\infty}^0\beta_1(s)\beta_2(\tau)\zeta(\tau,x)d\tau ds\Big\|^2dx\Big]^{\frac{1}{2}}\\
&\quad\le \gamma_g\|\zeta\|_{\mathcal{B}}+a\Big[\int_0^{2\pi}\big(\int_0^t\beta_1^2(s)ds\big)\big(\mathbb{E}\Big\|\int_{-\infty}^0\beta_2(\tau)\zeta(\tau,x)d\tau\Big\|^2\big)dx\Big]^{\frac{1}{2}}\\
&\quad\le \gamma_g\|\zeta\|_{\mathcal{B}}+a\big(\int_0^t\beta_1^2(s)ds\big)^{\frac{1}{2}}\big[\int_0^{\pi}\mathbb{E}\Big\|\int_{-\infty}^0\beta_2(\tau)\zeta(\tau,x)d\tau\Big\|^2dx\big]^{\frac{1}{2}}\\
&\quad\le \gamma_g\|\zeta\|_{\mathcal{B}}+a\big(\int_0^t\beta_1^2(s)ds\big)^{\frac{1}{2}}\gamma_g^*\|\zeta\|_{\mathcal{B}}\\
&\quad= \big[\gamma_g+a\|\beta_1\|_\infty\gamma_g^*\big]\|\zeta\|_{\mathcal{B}},
\end{aligned}
$$

$$
\begin{aligned}
\|\xi(t,\zeta)\|_{\mathcal{L}_2} &= \Big[\int_0^{\pi}\mathbb{E}\Big\|\int_{-\infty}^0 v(t,x,\theta)\mathcal{N}(\zeta(\theta,x))d\theta\Big\|^2dx\Big]^{\frac{1}{2}}\\
&\le \Big[\int_0^{\pi}\mathbb{E}\Big\|\int_{-\infty}^0 v(t,x,\theta)\mu_l\left(\int_{-\infty}^0 e^{2s}\|\zeta(s)(\cdot)\|_{\mathcal{L}_2}ds\right)d\theta\Big\|^2dx\Big]^{\frac{1}{2}}\\
&\le \Big[\int_0^{\pi}\big(\int_{-\infty}^0 v(t,x,\theta)\mu_l\left(\int_{-\infty}^0 e^{2s}\sup\|\zeta(s)(\cdot)\|_{\mathcal{L}_2}ds\right)d\theta\big)^2dx\Big]^{\frac{1}{2}}\\
&\le \Big[\int_0^{\pi}\big(\int_{-\infty}^0 v(t,x,\theta)d\theta\big)^2dx\Big]^{\frac{1}{2}}\mu_l\left(\|\zeta\|_{\mathcal{B}}\right)\\
&= \Big[\int_0^{\pi}\delta^2(t,x)dx\Big]^{\frac{1}{2}}\mu_l(\|\zeta\|_{\mathcal{B}})\\
&= l(t)\mu_l(\|\zeta\|_{\mathcal{B}}),
\end{aligned}
$$

where $l(t)=\left(\int_0^{\pi}\delta^2(t,x)dx\right)^{\frac{1}{2}}$. Therefore, $g, I_i(i=1,2,\cdots,m)$ are bounded by $\mathbb{E}\|g\|_{\mathbb{X}}^2\le L_2, \mathbb{E}\|I_i\|_{\mathbb{X}}^2\le c_i^2$, where $L_2=[\gamma_g+a\|\beta_1\|_\infty\gamma_g^*]^2$. In addition, from the estimation of $\xi(t,\zeta)$, it is easy to see that the function ξ satisfy the hypothesis (A_3). Hence by Theorem 3.3, the system (5.1) has a mild solution on J.

Conclusion

This article focuses on a new kind of state-dependent delay neutrality of impulsive stochastic integrodifferential equations in a real separable Hilbert space. We obtained the existence and controllability of mild solutions using the fixed point theorems and resolvent operator theory in the sense of Grimmer. We provided an example to show the effectiveness of the main results.In addition, to obtain the immediate results discussed in this paper, the Krasnoselskii–Schaefer fixed point theorem, the Sadovskii fixed point theorem, and the Banach fixed point theorem were all successfully applied under a variety of distinct conditions. In upcoming research, we will investigate the controllability and stability of solutions for impulsive stochastic integrodifferential systems that either have jumps in their dynamics or are driven by the Rosenblatt process. This research will take place shortly.

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