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On the spherical magnetic trajectories

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Abstract. We consider spherical indicatrix magnetic trajectories of a magnetic field in Euclidean $3-$space. From classical formulation of Killing magnetic flow equations, we derive the differential equation systems for tangent spherical indicatrix magnetic trajectories in Euclidean $3-$space. Then we solve these equations by using Jacobi elliptic functions. Finally, we make similar calculations for curves whose principal normal and binormal spherical indicatrix are magnetic curves.

AMS Subject Classifications: 53A04, 53A05.

Keywords: Magnetic curve, spherical indicatrix, Killing magnetic flow.

Contents

1. Introduction

Any magnetic vector field is known divergence zero vector field in three dimensional spaces. A magnetic trajectory of a magnetic flow created by magnetic vector field are curves called as magnetic curves. Although the problem of investigating magnetic trajectories appears to be physical problem, recent studies show that the characterization of magnetic flow in a magnetic field have brought variational perspective in more geometrical manner [2, 8]. Let S be a surface in Euclidean $3-$space $\mathbb{R}^3$ and F denote a complete differential $2-$form in a open subset U of S. Then we can write $F = d\omega$ for some potential $1-$form ω. If we define Γ as smooth curves that connect two fixed point of U, the Lorentz force equation is known a minimizer of the functional $\mathcal{L} : \Gamma \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) : \frac{1}{2}\int_{\gamma} < \gamma', \gamma' > dt + \omega(\gamma')\, dt. \tag{1.1}$$

The Euler-Lagrange equation of the functional $\mathcal{L}$ is derived as

$$\phi(\gamma') = \nabla_{\gamma'}\gamma', \tag{1.2}$$

where ϕ is the skew-symmetric operator. The critical point of the functional $\mathcal{L}$ corresponds to the Lorentz force equation [2, 4].

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Any function defined from a space curve to a suitable sphere in Euclidean $3-$space is called the spherical indicatrix (spherical image) of the curve. The spherical indicatrix of a curve in Euclidean $3-$space emerges in three types: the tangent indicatrix (tangential indicatrix or tangent spherical indicatrix), the principle normal indicatrix and the binormal indicatrix of the curve. The spherical indicatrix is a nice way to envision the motion of the curve on a sphere by using components of the Frenet Frame. Furthermore, the movement of a spherical indicatrix describes the changes in the original direction of the curve [6, 7].

In this paper we consider the magnetic trajectories which are the tangent, principal and binormal indicatrices, separately. We first investigate the tangent indicatrix magnetic trajectories and we derive the Killing magnetic flow equations for tangent indicatrix magnetic vector field. Then we solve these equations by using elliptic functions. Then we apply this method the other imagine types of curves by using same calculations. But we do not dwell on variational and differential calculations of the problem of finding curves whose principle normal and binormal indicatrix are magnetic since the same procedure would repeat.

2. Preliminaries

We consider a regular curve γ in Euclidean $3-$space $\mathbb{R}^3$, parametrized by arc length s, $0 \leq s \leq \ell$. Let $T = \gamma'(s)$ denote the unit tangent vector field, $N(s)$ the unit principle normal vector field and $B = T \times N$ binormal vector field at point $\gamma(s)$. Then we have the Frenet frame $\{T, N, B\}$ along the curve γ and Frenet equations given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where $\kappa > 0$ and τ are respectively curvature and torsion of γ [6].

If the Frenet frame of the tangent indicatrix $\gamma_t = T$ of a space curve γ is $\{T_t, N_t, B_t\}$, then we have the following Frenet equations

$$\begin{pmatrix} T_t'(s_t) \\ N_t'(s_t) \\ B_t'(s_t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_t & 0 \\ -\kappa_t & 0 & \tau_t \\ 0 & -\tau_t & 0 \end{pmatrix} \begin{pmatrix} T_t \\ N_t \\ B_t \end{pmatrix}, \tag{2.2}$$

where

$$T_t = N,\ N_t = \frac{-T + fB}{\sqrt{1+f^2}},\ B_t = \frac{fT + B}{\sqrt{1+f^2}} \tag{2.3}$$

and

$$s_t = \int \kappa(s)\, ds,\ \kappa_t = \sqrt{1+f^2},\ \tau_t = \sigma\sqrt{1+f^2}, \tag{2.4}$$

where

$$f(s) = \frac{\tau(s)}{\kappa(s)} \text{ and } \sigma = \frac{f'(s)}{\kappa(s)\left(1+f^2\right)^{3/2}} = \frac{\tau_t}{\kappa_t}. \tag{2.5}$$

σ is the geodesic curvature of the principal image of the principal normal indicatrix of the curve γ, s_t is natural representation of the tangent indicatrix of the curve γ and equal the total curvature of the curve γ and κ_t and τ_t are the curvature and torsion of γ_t.

If the Frenet frame of the normal indicatrix $\gamma_n = N$ of a space curve γ is $\{T_n, N_n, B_n\}$, then we have the following Frenet equations

$$\begin{pmatrix} T_n'(s_n) \\ N_n'(s_n) \\ B_n'(s_n) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & 0 \\ -\kappa_n & 0 & \tau_n \\ 0 & -\tau_n & 0 \end{pmatrix} \begin{pmatrix} T_n \\ N_n \\ B_n \end{pmatrix}, \tag{2.6}$$

where

$$T_n = \frac{-T + fB}{\sqrt{1+f^2}},\; N_t = \frac{\sigma}{\sqrt{1+\sigma^2}}\left[\frac{-T + fB}{\sqrt{1+f^2}} - \frac{N}{\sigma}\right],$$
$$B_t = \frac{1}{\sqrt{1+\sigma^2}}\left[\frac{fT + B}{\sqrt{1+f^2}} + \sigma N\right].$$

and

$$s_n = \int \kappa(s)\left(\sqrt{1+f^2(s)}\right)ds,\; \kappa_n = \sqrt{1+\sigma^2},\; \tau_t = \Gamma\sqrt{1+\sigma^2},$$

where

$$\Gamma = \frac{\sigma'(s)}{\kappa(s)\sqrt{(1+f^2)}(1+\sigma^2)^{3/2}} = \frac{\tau_n}{\kappa_n},$$

where s_n is natural representation of the principal normal indicatrix of the curve γ and κ_n and τ_n are the curvature and torsion of γ_n.

If the Frenet frame of the binormal indicatrix $\gamma_b = N$ of a space curve γ is $\{T_b, N_b, B_b\}$, then we have Frenet formula:

$$\begin{pmatrix} T_b'(s_b) \\ N_b'(s_b) \\ B_b'(s_b) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_b & 0 \\ -\kappa_b & 0 & \tau_b \\ 0 & -\tau_b & 0 \end{pmatrix}\begin{pmatrix} T_b \\ N_b \\ B_b \end{pmatrix}, \tag{2.7}$$

where

$$T_b = -N,\; N_t = \frac{T - fB}{\sqrt{1+f^2}},\; B_t = \frac{fT + B}{\sqrt{1+f^2}}.$$

and

$$s_b = \int \tau(s)\,ds,\; \kappa_b = \frac{\sqrt{1+f^2}}{f},\; \tau_b = -\sigma\frac{\sqrt{1+f^2}}{f},$$

where

$$\sigma = \frac{\tau_b}{\kappa_b},$$

where s_b is the natural representation of the binormal indicatrix of the curve γ and κ_b and τ_b are the curvature and torsion of γ_b [1].

3. Spherical indicatrix magnetic fields

Let V be a divergence-free vector field in Euclidean 3-space $\mathbb{R}^3$. Then it defines a magnetic vector field. Given a differential $2-$form F is a magnetic field on $\mathbb{R}^3$. The Lorentz force of F is defined to be the skew-symmetric operator ϕ given by

$$<\phi(X), Y> = F(X, Y) \tag{3.1}$$

for all vector field $X, Y \in \chi\left(\mathbb{R}^3\right)$. The associated magnetic trajectories are curves γ on $\mathbb{R}^3$ that satisfies the Lorentz force equation (1.2). On the other hand the Lorentz force ϕ can be write as follows

$$\phi(X) = V \times X, \tag{3.2}$$

that is, the Lorentz force ϕ of V is defined via cross product on $\mathbb{R}^3$. Combining (1.2) and (3.2), the Lorentz equation can be written by

$$\phi(\gamma') = \nabla_{\gamma'}\gamma' = V \times \gamma'$$

for a curve γ on $\mathbb{R}^3$.

By means of these structures defined on $\mathbb{R}^3$, the Killing magnetic flow equations corresponding to spherical indicatrix for a unit-speed curve γ on $\mathbb{R}^3$ will be found.

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a reparametrized curve in Euclidean $3-$space and $\{T_t, N_t, B_t\}$ is the Frenet frame along γ_t. Then the Lorentz force in the frame $\{T_t, N_t, B_t\}$ is written as

$$\phi(T_t) = \kappa_t N_t, \tag{3.3}$$
$$\phi(N_t) = -\kappa_t T_t + \omega_t B_t \tag{3.4}$$

and

$$\phi(B_t) = -\omega_t B_t, \tag{3.5}$$

where the function $\omega_t(s_t)$ associated with each tangent indicatrix magnetic curve is quasislope measured with respect to the magnetic field V_t.

Then we can give the following propositions.

Proposition 3.1. The tangential indicatrix γ_t is a magnetic trajectory of a magnetic field V_t if and only if V_t can be written along γ_t as

$$V_t = \omega_t T_t + \kappa_t B_t. \tag{3.6}$$

Proof. Assume that γ_t is a magnetic curve along a magnetic field V_t and the orthogonal frame along γ_t is given by $\{T_t, N_t, B_t\}$. Then, V_t can be written as

$$V_t =< V_t, T_t > T_t + < V_t, N_t > N_t + < V_t, B_t > B_t.$$

To find coefficient of V_t, we use the Lorentz force in orthogonal frame equations (3.3), (3.4) and (3.5):

$$\omega_t =< \phi(N_t), B_t >=< V_t \times N_t, B_t >=< V_t, T_t >,$$
$$0 =< \phi(T_t), B_t >=< V_t \times T_t, B_t >= - < V_t, B_t >$$

and

$$\kappa_t =< \phi(T_t), N_t >=< V_t \times T_t, N_t >=< V_t, B_t > .$$

Proposition 3.2. The principle indicatrix γ_n is a magnetic trajectory of a magnetic field V_n if and only if V_n can be written along γ_n as

$$V_n = \omega_n T_n + \kappa_n B_n.$$

Proposition 3.3. The binormal indicatrix γ_b is a magnetic trajectory of a magnetic field V_b if and only if V_b can be written along γ_b as

$$V_b = \omega_b T_b + \kappa_b B_b.$$

4. Killing magnetic flow equations for spherical indicatrix magnetic curves

Let γ_t be a tangential indicatrix of γ in $\mathbb{R}^3$ and V_t be a vector field along that curve. One can take a variation of γ_t in the direction of V_t, say a map

$$\begin{aligned} \Gamma : [0,1] \times (-\varepsilon, \varepsilon) &\to \mathbb{S}^2 \\ (s, w) &\to \Gamma(s, w) \end{aligned}$$

which satisfies

$$\Gamma(s, 0) = \gamma_t(s), \ \left(\frac{\partial \Gamma(s,w)}{\partial w}\right)_{w=0} = V_t(s),$$

and

$$\left(\frac{\partial \Gamma(s,w)}{\partial s}\right)_{w=0} = \gamma_t'(s).$$

One can write the speed function $\upsilon_t(s, w) = \left\| \frac{\partial \Gamma(s,w)}{\partial s} \right\|$, the curvature function $\kappa_t(s, w)$ and the torsion function $\tau_t(s, w)$ [2, 5].

Lemma 4.1 (see [2, 3]). Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$, γ_t denote the tangent indicatrices and V_t be a vector field along the curve γ_t. Then we have the following equalities

$$V_t(\upsilon_t) = \left(\frac{\partial \upsilon_t(s,w)}{\partial w}\right)_{w=0} =< \nabla_{T_t} V_t, T_t > \upsilon_t, \tag{4.1}$$

$$V_t(\kappa_t) = \left(\frac{\partial \kappa_t(s,w)}{\partial w}\right)_{w=0} = \frac{1}{\kappa_t} < \nabla^2_{T_t} V_t, \nabla_{T_t} T_t > -2\kappa_t < \nabla_{T_t} V_t, T_t > \tag{4.2}$$

and

$$V_t(\tau_t) = \left(\frac{\partial \tau_t(s,w)}{\partial w}\right)_{w=0} = \left(\frac{1}{\kappa_t^2} < \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t >\right)' \tag{4.3}$$
$$+\tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > .$$

Proposition 4.1. Let V_t be the restriction to the tangent indicatrix γ_t of a Killing vector field, say V_t of $\mathbb{R}^3$; then

$$V_t(\upsilon_t) = V_t(\kappa_t) = V_t(\tau_t) = 0. \tag{4.4}$$

Then we can give the Killing magnetic flow equations of the tangential indicatrix.

Theorem 4.1. Let γ_t be the tangential indicatrix of a regular curve γ. Suppose that $V_t = \omega_t T_t + \kappa_t B_t$ is a Killing vector field along γ_t. Then the tangential indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_t^2 \left(\frac{1}{2}\omega_t - \tau_t\right) = A_1. \tag{4.5}$$

and

$$\kappa_t'' + \kappa_t \tau_t (\omega_t - \tau_t) + C\kappa_t + \frac{1}{2}\kappa_t^3 - A_2 \kappa_t = 0, \tag{4.6}$$

where A_1, A_2 and C are undetermined constants.

Proof. Assume that V_t is a Killing vector field along γ_t on S^2. Along any spherical magnetic trajectory γ_t, we have $V_t = \omega_t T_t + \kappa_t B_t$. If V_t is Killing vector field, we calculate

$$\omega_t' = 0,$$

that is ω_t is a constant, and

$$\nabla_{T_t} V_t = \kappa_t (\omega_t - \tau_t) N_t + \kappa_t' B_t. \tag{4.7}$$

By using the first derivative of (4.7), (4.2) and (4.4), we get

$$\left(\kappa_t^2 \left(\frac{1}{2}\omega_t - \tau_t\right)\right)' = 0.$$

Similarly, from (4.2) and (4.4), we find to $V(\tau)$ as follows

$$V_t(\tau_t) = \left(\frac{\partial \tau_t(s,w)}{\partial w}\right)_{w=0} = \left(\frac{1}{\kappa_t^2} < \nabla^2_{T_t} V_t, T_t \times \nabla_{T_t} T_t >\right)'$$
$$+\tau_t < \nabla_{T_t} V_t, T_t > + < \nabla_{T_t} V_t, T_t \times \nabla_{T_t} T_t > .$$

Definition 4.1. Any tangent indicatrix of a Euclidean curve is called the tangent indicatrix magnetic trajectory of a magnetic field V_t if it satisfies the differential equation system (4.5) and (4.6).

We can combine Eqs. (4.5) and (4.6) as follows

$$\kappa_t'' + \frac{1}{2}\kappa_t^3 + \left(C - A_2 + \frac{1}{4}\omega_t^2\right)\kappa_t - \frac{A_1^2}{\kappa_t^3} = 0.$$

This equation admits an obvious first integral. In fact, just multiply by $2\kappa_t'$ and integrate to get

$$\left(\kappa_t'\right)^2+\frac{1}{4}\kappa_t^4+\left(C-A_2+\frac{1}{4}\omega_t^2\right)\kappa_t^2-\frac{A_1^2}{\kappa_t^2}=A_3.$$

Since this equation is of the type $(u')^2=P\left(h\right)$, where P is a polynomial of degree 3 in u, it can be solved using elliptic functions as follows

$$\begin{aligned}\kappa_t\left(s\right)&=\sqrt{a_3\left(1-q^2sn\left(rs,p\right)\right)},\\ \tau_t\left(s\right)&=\frac{1}{2}\omega_t-\frac{A_1}{\kappa_t^2},\end{aligned}$$

when $\kappa_t\neq const.$, where

$$\left(u_t'\right)^2+\left(u-a_1\right)\left(u-a_2\right)\left(u-a_3\right)=0,\quad u=\kappa_t^2,$$

$$p=\frac{a_3-a_2}{a_3-a_1},\ q^2=\frac{a_3-a_2}{a_3}\text{ and }r=\frac{1}{2}\sqrt{a_3-a_1}.$$

So, the curvature and the curvature of γ must satisfy the equations

$$\frac{\tau}{\kappa}=\sqrt{\sqrt{a_3\left(1-q^2sn\left(rs,p\right)\right)}-1},$$

and

$$\left(\frac{\tau}{\kappa}\right)'=\kappa_t\left(1+\frac{\tau^2}{\kappa^2}\right)\left(1+\omega_t-\frac{1}{\sqrt{a_3\left(1-q^2sn\left(rs,p\right)\right)}}\right).$$

Making similar calculations we can give the Killing magnetic flow equations of the principle normal and binormal indicatrix.

Theorem 4.2. Let γ_n be the normal indicatrix of a regular curve γ. Suppose that $V_n=\omega_nT_n+\kappa_nB_n$ is a Killing vector field along γ_n. Then the normal indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_n^2\left(\frac{1}{2}\omega_n-\tau_n\right)=A_4. \tag{4.8}$$

and

$$\kappa_n''+\kappa_n\tau_n\left(\omega_n-\tau_n\right)+C_1\kappa_n+\frac{1}{2}\kappa_n^3-A_5\kappa_n=0, \tag{4.9}$$

where A_4, A_5 and C_1 are undetermined constants.

Theorem 4.3. Let γ_b be the binormal indicatrix of a regular curve γ. Suppose that $V_b=\omega_bT_b+\kappa_bB_b$ is a Killing vector field along γ_b. Then the binormal indicatrix magnetic trajectories are curves on $\mathbb{S}^2$ satisfying following differential equations

$$\kappa_b^2\left(\frac{1}{2}\omega_b-\tau_b\right)=A_6. \tag{4.10}$$

and

$$\kappa_b''+\kappa_b\tau_b\left(\omega_b-\tau_b\right)+C_2\kappa_b+\frac{1}{2}\kappa_b^3-A_7\kappa_b=0, \tag{4.11}$$

where A_6, A_7 and C_2 are undetermined constants.

Definition 4.1. Any principle normal (binormal) indicatrix of a Euclidean curve is called the principle (binormal) indicatrix magnetic trajectory of a magnetic field V_n (V_b) if it satisfies the differential equation system (4.8) and (4.9) (resp., (4.10) and (4.11)).

Example 4.1. We consider a unit-speed circular helix $\beta\left(s\right)=\left(\cos\frac{s}{\sqrt{2}},\sin\frac{s}{\sqrt{2}},-\frac{s}{\sqrt{2}}\right)$ [2]. The curve $\beta\left(s\right)$ can be seen on Fig. 1.

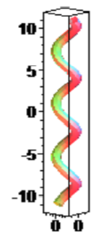

Figure 1: The helix $\beta\left(s\right)$

Curvature and torsion of $\beta\left(s\right)$ are found as $\kappa=-\tau=\frac{1}{2}$. Then, tangent indicatrix of the circular helix is

$$\beta_t \approx \beta'\left(s\right)=\left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right).$$

β_t is a circle cut from unit sphere by the plane $z=-\frac{1}{\sqrt{2}}$. The curvature and the torsion of the tangent indicatrix of the circular helix are found as $\kappa_t=\sqrt{2}$, $\tau_t=0$. We can see from (4.5) and (4.6), the tangent indicatrix β_t of β is a tangent indicatrix magnetic trajectory with $A_2=C+1$ of the Killing magnetic field $V_t=A_1T_t+\sqrt{2}B_t$. The principal normal indicatrix of the circular helix is

$$\beta_n \approx N\left(s\right)=\left(-\cos\frac{s}{\sqrt{2}},-\sin\frac{s}{\sqrt{2}},0\right),$$

that is, β_n lies on the great circle lines on the sphere with $\kappa_n=1$ and $\tau_n=0$. From (4.8) and (4.9), we show that β_n is a principle normal indicatrix magnetic trajectory with $A_5=C_1+\frac{1}{2}$ of the Killing magnetic field $V_t=2A_4T_n+B_n$. Finally, the binormal indicatrix of the circular helix is

$$\beta_b \approx B\left(s\right)=\left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\right),$$

that is, β_b is a circle cut from unit sphere by the plane $y=\frac{1}{\sqrt{2}}$. From (4.10) and (4.11), β_b is a binormal indicatrix magnetic trajectory with $A_7=C_2+1$ of the Killing magnetic field $V_b=A_6T_b+\sqrt{2}B_b$. The graphs of β_t, β_n and β_b are given as follows.

Fig. 2. The magnetic trajectory β_t

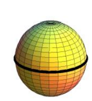

Fig. 3. The magnetic trajectory β_n

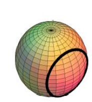

Fig. 4. The magnetic trajectory β_b

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