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Existence of mild solutions of second-order impulsive differential equations in Banach spaces

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Abstract. We discuss the existence of solutions for second-order impulsive differential equation with nonlocal conditions in Banach spaces. Our approach is based on the generalization of Schauder fixed point principle that is Darbo fixed point theorem. An example is also presented for illustration.

AMS Subject Classifications: 34G20, 35R10.

Keywords: Second order differential equations, mild solution, impulse, nonlocal condition, Kuratowski measures of noncompactness.

Contents

1. Introduction

In the present paper we consider the abstract second-order nonlinear impulsive differential equation with non local condition

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)),\ t \in J = [0, T],\ t \neq t_i, i = 0, ..., p \\ x(0) = x_0 + g(x),\ x'(0) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i)),\ i = 0, ..., p \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., p. \end{cases} \tag{1.1}$$

Where A is a linear operator from a Banach space E into itself, $\triangle x(t_i) = x(t_i^+) - x(t_i^-)$, $\triangle x'(t_i) = x'(t_i^+) - x'(t_i^-)$, $0 < t_1 < t_2 < ... < t_p < T$ are the instants of impulse effect, $f : [0, T] \times E \times E \to E$, $I_i : E \to E$, $D_i : E \times E \to E$, $x_0, x_1 \in E$ and $g(x)$ is a function with values in E to be specified later.
For the basic theory on impulsive differential equations in infinite dimensional spaces, the reader is referred to the literature [2, 3]. The impulsive differential equations has become an important area of investigation by many authors because of their applications. For more details, we refer the reader to [3, 11, 15]. In [4], Peng and

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Xiang discuss the existence of optimal controls for a Lagrange problem of systems governed by the second-order nonlinear impulsive differential equations in infinite dimensional spaces:

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)) + B(t),\ t \in J = [0, T],\ t \neq t_i, i = 0, ..., p \\ x(0) = x_0,\ x'(0) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i), x'(t_i)),\ i = 0, ..., p \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., p. \end{cases}$$

They apply a direct approach to derive the maximum principle for the problem at hand. The authors [6] considered the existence of mild solutions for a class of abstract impulsive second-order neutral functional differential equations. In [10], the authors studied the abstract second-order nonlinear impulsive differential equation with nonlocal condition

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)), x(b_1(t)), x(b_2(t)), ..., \\ x(b_m(t)), x'(b_1(t)), ..., x'(b_m(t)))\ t \in J = [0, T], \\ x(0) = x_0,\ x'(0) + g(x) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i),),\ i = 0, ..., m \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., m. \end{cases}$$

In the present work, the existence of a mild solution for problem (1.1) is obtained by the cosine family theory, measure of non-compactness and the the well known Schauder fixed point principle. Its generalization, called the Darbo fixed point theorem. It should be pointed out that the restrictive condition on the impulsive term is removed. The work is organized as follows: In Section two, we recall some definitions and facts about the cosine family and facts concerning the Kuratowski measures of noncompactness in the Banach space $PC([0, T], E)$. In Section three, we give the existence of mild solutions to the problem (1.1). In Section four we present an example to illustrate our main result.

2. Preliminaries

We begin by giving some notation. Let E be a Banach space with the norm $\|\cdot\|$. We use θ to present the zero element in E. For any constant $T > 0$, denote $J = [0, T]$. Let $C(J, E)$ and be the Banach space of all continuous functions from J into E endowed with the supremum-norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$ for every $x \in C(J, E)$. From the associate literature, we consider the following space of piecewise continuous functions,

$$PC(J, E) = \big\{u : J \to E : x \text{ is continuous for } t \neq t_k, \\ \text{left continuous at } t = t_k \text{ and } x(t_k^+) \text{ exists for } k = 1, 2, \dots, m\big\}.$$

It easy to see that $PC(J, E)$ is a Banach space endowed with the PC-norm

$$\|x\|_{PC} = \max\Big\{\sup_{t \in J} \|x(t^+)\|,\ \sup_{t \in J} \|x(t^-)\|\Big\}, \quad x \in PC(J, E),$$

where $x(t^+)$ and $x(t^-)$ represent respectively the right and left limits of $x(t)$ at $t \in J$. Similarly, PC^1 will be the space of the functions $x(\cdot) \in PC$ such that $x(\dot)$ is continuously differentiable on J, t_i, $i = 1, 2, \dots, n$ and the derivatives

$$x_r'(t) = \lim_{s \to 0} \frac{x(t+s) - x(t^+)}{s}, \quad x_l'(t) = \lim_{s \to 0} \frac{x(t+s) - x(t^-)}{s}$$

are continuous on $[0, T[$ and $]0, T]$, respectively. Next, for $x \in PC^1$, we represent, by $x'(t)$, the left derivative at $t \in]0, T]$ and, by $x'(0)$, the right derivative at zero. It is easy to see that PC^1, provided with the norm

$$\|x\|_{PC^1} := \max\{\|x\|_{PC}, \|x'\|_{PC}\}$$

is a Banach space. For each finite constant $r > 0$, let

$$\Omega_r = \{u \in PC(J, E) : \|u(t)\| \le r,\ t \in J\},$$

then Ω_r is a bounded closed and convex set in $PC(J, E)$.

Let $\mathcal{L}(E)$ be the Banach space of all linear and bounded operators on E. Since the semigroup $T(t)(t \ge 0)$ generated by A is a C_0-semigroup in E, denoting

$$M := \sup_{t \in J} \|T(t)\|_{\mathcal{L}(E)}, \tag{2.1}$$

then $M \ge 1$ is a finite number.

Definition 2.1. A C_0-semigroup $T(t)(t \ge 0)$ in E is said to be equicontinuous if $T(t)$ is continuous by operator norm for every $t > 0$.

Now we introduce some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.2. *[1, 8] The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on a bounded set S of Banach space E is*

$$\alpha(S) := \inf\{\delta > 0 : S = \cup_{i=1}^{m} S_i \text{ with } \operatorname{diam}(S_i) \le \delta \text{ for } i = 1, 2, \ldots, m\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.3. *[1, 8] Let E be a Banach space and $S, U \subset E$ be bounded. The following properties are satisfied:*

(i) $\alpha(S) = 0$ *if and only if* $\overline{S}$ *is compact, where* $\overline{S}$ *means the closure hull of* S*;*

(ii) $\alpha(S) = \alpha(\overline{S}) = \alpha(\operatorname{conv} S)$*, where* $\operatorname{conv} S$ *means the convex hull of* S*;*

(iii) $\alpha(\lambda S) = |\lambda|\alpha(S)$ *for any* $\lambda \in \mathbb{R}$*;*

(iv) $S \subset U$ *implies* $\alpha(S) \le \alpha(U)$*;*

(v) $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$*;*

(vi) $\alpha(S + U) \le \alpha(S) + \alpha(U)$*, where* $S + U = \{x \mid x = y + z, y \in S, z \in U\}$*;*

(vii) If the map $Q : \mathcal{D}(Q) \subset E \to X$ *is Lipschitz continuous with constant* k*, then* $\alpha(Q(V)) \le k\alpha(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$*, where* X *is another Banach space.*

In this work, we denote by $\alpha(\cdot)$, $\alpha_c(\cdot)$, $\alpha_{pc}(\cdot)$ and $\alpha_{pc^1}(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of E, $C(J, E)$, $PC(J, E)$ and $PC^1(J, E)$, respectively.
In the following, let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], ..., J_{p-1} = (t_{p-1}, t_p]$ and $J_p = (t_p, 1]$, $t_{p+1} = 1$. For any $X \subset PC(J, E)$, we denote by $X' = \{x' : x \in X\} \subset PC(J, E)$ and by $X(t) = \{x(t) : x \in X\} \subset E$ and by $X'(t) = \{x(t) : x \in X\} \subset E$ for $t \in J$. To discuss the problem (1.1), we also need the following lemma [12].

Lemma 2.4. *[12] If $X \subset PC^1(J, E)$ is bounded and the elements of X' are equicontinuous on each $J_k, k = 0, 1, ..., p$ then*

$$\alpha_{pc^1}(X) = \max\{\sup_{t \in J} \alpha(X(t)), \sup_{t \in J} \alpha(X'(t))\} \tag{2.2}$$

Obviously the following formulated theorem constitutes the well known Schauder fixed point principle. Its generalization, called the Darbo fixed point theorem, is formulated below.

Lemma 2.5. *[8] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(T(X)) = k\mu(X)$ for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then T has a fixed point in the set Ω.*

Lemma 2.6. *[5, 16] Let E be a Banach space, and let $X \subset E$ be bounded. Then there exists a countable set $X_0 \subset X$, such that $\alpha(X) \leq 2\alpha(X_0)$.*

Lemma 2.7. *[13] Let E be a Banach space, and let $X = \{u_n : n = 0, 1, ...\} \subset PC([0, T], E)$ be a bounded and countable set for constants $-\infty < 0 < T < +\infty$. Then $\alpha(X(t))$ is Lebesgue integral on $[0, T]$, and*

$$\alpha\Big(\Big\{ \int_0^T u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\{\int_0^T \alpha(u_n(t))dt : n = 0, 1, ...\}.$$

Lemma 2.8. *[1] Let E be a Banach space, and let $X \subset C([0, T], E)$ be bounded and equicontinuous. Then $\alpha(X(t))$ is continuous on $[0, T]$, and*

$$\alpha_c(X) = \max_{t\in[0,T]} \alpha(X(t)).$$

Next, we shall need the following definitions [25].

Definition 2.9. A one parameter family $\{C(t), t \in J\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$, for all $s, t \in J$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is continuous in t on J, for each $x \in X$.

Define the associated sine family $S(t), t \in J$ by

$$S(t)x := \int_0^t C(s)xds, \quad x \in X,\ t \in J$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in J\}$ is the operator $A : X \to X$, defined by

$$Ax = \lim_{t\to 0} \frac{d^2}{dt^2}C(t)x, \quad x \in D(A),$$

where $D(A) := \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$.

Define $E := \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. We assume

(H_A) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in J\}$ of bounded linear operators in the Banach space X.

To establish our main theorem, we need the following lemmas.

Lemma 2.10. *Let (H_A) hold. Then*

(i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq Me^{\omega|t|}$ and

$$\|S(b) - S(a)\| \leq M|\int_b^a e^{\omega|s|}ds|, \quad \text{for } a, b \in J;$$

(ii) $S(t)X \subset E$ and $S(t)E \subset D(A)$, for $t \in J$;

(iii) $\frac{d}{dt}C(t)x = AS(t)x$, *for* $x \in E$ *and* $t \in J$;

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$, *for* $x \in D(A)$ *and* $t \in J$.

Further we denote by$\|C(t)\|$ and $\|S(t)\|$ the operators norm of $C(t), S(t)$ for $t \in [0,T]$ in the Banach space E, respectively. From assumption (H_A) it follows that there is a constant $M \geq 1$ such that

$$\|C(t)\| \leq M \text{ and } \|S(t)\| \leq M \text{ for } t \in [0,T].$$

Lemma 2.11. *[25] Let* (H_A) *hold and* $v : \mathcal{R} \to X$ *be such that* v *is continuous and let* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then* q *is twice continuously differentiable and, for* $t \in \mathcal{I}$: $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ *and*

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

3. Main results

We first give the following hypotheses:

(H_A) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X.

(H_f) (i) $(t,x,y) \mapsto f(t,x,y)$ satisfies the Carathéodory conditions, i.e. $f(.,x,y)$ is measurable for $x, y \in E$ and $f(t,.,.)$ is continuous for a.e. $t \in [0,T]$

(ii) There exist $m \in L^1([0,T], \mathbb{R}_+)$ such that $\|f(t,x,y)\| \leq m(t)(\|x\| + \|y\|)$ for a.e. $t \in [0,T]$ and all $x \in E$.

(iii) There exists a function $l \in L^1([0,T], \mathbb{R}_+)$ such that for any bounded subset $B, D \subset E, \alpha(f(t,B,D)) = l(t)\max\{\alpha(B), \alpha(D)\}$ for a.e. $t \in [0,T]$.

(H_g) (i) g is continuous.

(ii) There is nonnegative constant q such that $\alpha(g(D)) \leq q\alpha_{pc^1}(D)$ for any bounded set $D \subset PC^1([0,T], E)$.

(H) (i) I_i and D_i are continuous.

(ii) There exist nonnegative constants k_i^1 and k_i^2 such that $\alpha(I_i(B)) \leq k_i^1\alpha(B)$ and $\alpha(D_i(B,D)) \leq k_i^2 \max(\alpha(B), \alpha(D))$ for any nonempty and bounded subset $B, D \subset E$ and $i = 1, ..., p$.

(H_R) There exists a number $R > 0$ such that

$$\max(\eta_1(R), \eta_1(R)) \leq R,$$

where,

$$\eta_1(R) = M\big[\|x_0\| + \|x_1\| + C_1\big] + 2MR \sup_{t\in[0,T]} \big(\int_0^t m(s)ds\big) + Mp(C_2 + C_3)$$

and

$$\eta_2(R) = M\big[\|A\|(\|x_0\| + C_1) + \|x_1\|\big] + 2MR \sup_{t\in[0,T]} \big(\int_0^t m(s)ds\big) + Mp(\|A\|C_2 + C_3),$$

where

$$C_1 = \sup_{x \in B_{pc^1}(R)} g(\|x\|),$$

$$C_2 = \sup_{x \in B_{pc^1}(R)} \|I_i(x(t_i))\|$$

and

$$C_3 = \sup_{x \in B_{pc^1}(R)} \|D_i(x(t_i), x'(t_i))\|.$$

Next, let us start by defining what we mean by a solution of the problem (1.1)(see [6]).

Definition 3.1. *A function $x \in PC^1([0,T], E)$ is said to be a mild solution of the problem (1.1) if x satisfies the equation*

$$x(t) = C(t)[x_0 + g(x)] + S(t)x_1 + \int_0^t S(t-s)f(s, x(s), x'(s))ds$$
$$+ \sum_{0<t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} S(t-t_i)D_i(x(t_i), x'(t_i)),\ t \in [0,T]. \tag{3.1}$$

Remark 3.2. *Assumptions $(H_f)(i)$, $(H_g)(ii)$ and $(H)(ii)$ imply that mappings f, g, I_i and I_i are bounded on bounded subsets of $PC^1([0,T], E)$ and E, respectively.*

To simplify the writing and the calculation one poses

$$\widetilde{L} = \int_0^T l(s)ds,\ S_1 = \sum_{0<t_i<t} k_i^1 \text{ and } S_2 = \sum_{0<t_i<t} k_i^2$$

.

Theorem 3.3. *Let E be a separable Banach space. Assume that the assumptions (H_A), (H_f), (H_g), (H) and (H_R) are satisfied. If*

$$\max\{q + \widetilde{L} + S_1 + S_2; \|A\|q + \widetilde{L} + \|A\|S_1 + S_2\} < \frac{1}{M},$$

then for each $x_0 \in E$, the equation (1.1) has at least one mild solution x in $PC^1(J, E)$.

Proof. Consider the operator $Fx : PC^1([0,T], E) \to PC^1([0,T], E)$ define by

$$(Fx)(t) = C(t)[x_0 + g(x)] + S(t)x_1 + \int_0^t S(t-s)f(s, x(s), x'(s))ds$$
$$+ \sum_{0<t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} S(t-t_i)D_i(x(t_i), x'(t_i)),\ t \in [0,T]. \tag{3.2}$$

It easy to see that $(Fx) \in PC([0,T], E)$ for $x \in PC^1([0,T], E)$. Moreover,

$$(Fx)'(t) = \frac{\partial (Fx)}{\partial t}(t) = AS(t)[x_0 + g(x)] + C(t)x_1 + \int_0^t C(t-s)f(s, x(s), x'(s))ds$$
$$+ \sum_{0<t_i<t} AS(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} C(t-t_i)D_i(x(t_i), x'(t_i)),\ t \in [0,T]. \tag{3.3}$$

Then, we get that $(Fx)' \in PC([0,T], E)$ and therefore, $Fx \in PC^1([0,T], E)$. So, F maps the Banach space $PC^1([0,T], E)$ into itself. Next, Let R be a positive number satisfying the inequality from assumption (H_R). Taking an arbitrary function $x \in B_{pc^1}(R)$, we get

$$\begin{aligned}
\|Fx(t)\|_{pc} \leq\ & M\big[\|x_0\| + g(\|x\|)\big] + M\|x_1\| + M\int_0^t m(s)(\|x(s)\| + \|x'(s)\|)ds \\
& + M\sum_{0<t_i<t}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\|D_i(x(t_i), x'(t_i))\| \\
& \leq M\big[\|x_0\| + \sup_{x\in B_{pc^1}(R)} g(\|x\|)\big] + M\|x_1\| \\
& + 2M\sup_{t\in[0,T]}\big(\int_0^t m(s)\max\{\|x(s)\|, \|x'(s)\|\}ds\big) \\
& + M\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|D_i(x(t_i), x'(t_i))\| \\
& \leq \eta_1(R). \qquad (3.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|(Fx)'(t)\|_{pc} \leq\ & M\|A\|\big[\|x_0\| + g(\|x\|)\big] + M\|x_1\| + M\int_0^t m(s)(\|x(s)\| + \|x'(s)\|)ds \\
& + M\|A\|\sum_{0<t_i<t}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\|D_i(x(t_i), x'(t_i))\| \\
& \leq M\big[\|x_0\| + \sup_{x\in B_{pc^1}(R)} g(\|x\|)\big] + M\|x_1\| \\
& + 2M\sup_{t\in[0,T]}\big(\int_0^t m(s)\max\{\|x(s)\|, \|x'(s)\|\}ds\big) \\
& + M\|A\|\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|I_i(x(t_i))\| + M\|\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|D_i(x(t_i), x'(t_i))\| \\
& \leq \eta_2(R), \qquad (3.5)
\end{aligned}$$

and thus,

$$\begin{aligned}
\|(Fx)(t)\|_{pc^1} &= \max\Big\{\|(Fx)(t)\|_{pc}, \|(Fx)'(t)\|_{pc}\Big\} \\
&\leq \max\Big\{\eta_2(R), \eta_2(R)\Big\} = \eta(R) \leq R. \qquad (3.6)
\end{aligned}$$

The last inequality shows that $(Fx) \in B_{pc^1}(R)$ for $x \in B_{pc^1}(R)$, that is $F(B_{pc^1}(R)) \subset B_{pc^1}(R)$. Now, we prove that operator F is continuous in $B_{pc^1}(R)$. To do this, let us fix $x \in B_{pc^1}(R)$ and take an arbitrary sequence $(x_n) \in B_{pc^1}(R)$ such that $x_n \to x$ in $B_{pc^1}(R)$. It also implies that the family $\{Fx\ \ x \in B_{pc^1}(R)\}$ is equibounded. Next, we shall show that the family $\{Fx\ x \in B_{pc^1}(R)\}$ is equicontinuous on each interval of continuity J_k, $k = 0, 1, ..., p$. For this, let $x \in B_{pc^1}(R)$ and $0 \leq t_1 < t_2 \leq T$. Then we have

$$\begin{aligned}
& (Fx)'(t_2) - (Fx)'(t_1) = A[S(t_2) - S(t_1)][x_0 + g(x)] + [C(t_2) - C(t_2)]x_1 \\
& + \int_0^{t_1}[C(t_2 - s) - C(t_2 - s)]f(s, x(s), x'(s))ds + \int_{t_1}^{t_2} C(t_2 - s)f(s, x(s), x'(s))ds \\
& + \sum_{0<t_i<t_1} A[S(t_2 - t_i) - S(t_1 - t_i)]I_i(x(t_i)) + \sum_{t_1<t_i<t_2} A[S(t_2 - t_i)]I_i(x(t_i)) \\
& + \sum_{0<t_i<t_1}[C(t_2 - t_i) - C(t_1 - t_i)]D_i(x(t_i), x'(t_i)) + \sum_{t_1<t_i<t_2}[C(t_2 - t_i)]D_i(x(t_i), x'(t_i).
\end{aligned}$$

So,

$$\|(Fx)'(t_2)-(Fx)'(t_1)\| \leq \|A\|\|S(t_2)-S(t_1)\|[\|x_0\|+\|g(x)\|]+[\|C(t_2)-C(t_2)\|]\|x_1\|$$
$$+\int_0^{t_1}[\|C(t_2-s)-C(t_2-s)\|]\|f(s,x(s),x'(s))\|ds+\int_{t_1}^{t_2}\|C(t_2-s)\|\|f(s,x(s),x'(s))\|ds$$
$$+\sum_{0<t_i<t_1}\|A\|[\|S(t_2-t_i)-S(t_1-t_i)\|]\|I_i(x(t_i))\|+\sum_{t_1<t_i<t_2}\|A\|\|S(t_2-t_i)\|\|I_i(x(t_i))\|$$
$$+\sum_{0<t_i<t_1}\|C(t_2-t_i)-C(t_1-t_i)\|\|D_i(x(t_i),x'(t_i))\|+\sum_{t_1<t_i<t_2}\|C(t_2-t_i)\|\|D_i(x(t_i),x'(t_i))\|.$$

Then,

$$\|(Fx)'(t_2)-(Fx)'(t_1)\| \leq \|A\|\|S(t_2)-S(t_1)\|[\|x_0\|+C_1]+\|C(t_2)-C(t_2)\|\|x_1\|$$
$$+R\int_0^{t_1}[\|C(t_2-s)-C(t_2-s)\|]m(s)ds+MR\int_{t_1}^{t_2}m(s)ds$$
$$+\|A\|C_2\sum_{0<t_i<t_1}\|S(t_2-t_i)-S(t_1-t_i)\|+\|A\|MC_2i(t_1,t_2)$$
$$+C_3\sum_{0<t_i<t_1}\|C(t_2-t_i)-C(t_1-t_i)\|+Mi(t_1,t_2). \tag{3.7}$$

where, $i(t_1,t_2)$ is the number of instants of impulse effect in the interval $[t_1,t_2)$. First, notice that the right-hand side of inequality is independant of the choose of $x\in B_{pc^1}(R)$. Further, from the uniform continuity of $C(t)$ and $S(t)$ on J in the operator norm, all norm in the right-hand side converge to 0 as $t_1\to t_2$. Finally $i(t_1,t_2)$ is zero for t_1,t_2 both in one of the intervals of continuity $J_k, k=0,1,...,p$. This, prove that the family of functions $\{(Fx)' \,:\, x\in B_{pc^1}(R)\}$ is equicontinuous on each interval $J_k,\ k=0,1,...,p$. In what follows, we will show that F is a strict set contraction from $PC^1(J,E)$ into itself. Let Q be a bounded set of $PC^1(J,E)$. Then $F(Q)\subset PC^1(J,E)$ is bounded and by (3.7) the elements of $(F(Q))'$ are equicontinuous on each interval $J_k, k=0,1,...,p$. Hence by lemma 2.4, we get

$$\alpha_{pc^1}(FQ)=\max\{\sup_{t\in J}\alpha((FQ)(t)),\sup_{t\in J}\alpha((FQ)'(t))\}. \tag{3.8}$$

Firstly,

$$\alpha((FQ)(t))\leq M\alpha(g(Q))+M\int_0^t\alpha(f(s,Q(s),Q'(s)))ds$$
$$+M\sum_{0<t_i<t}\alpha(I_i(Q(t_i)))+M\sum_{0<t_i<t}\alpha(D_i(Q(t_i),Q'(t_i))$$
$$\leq Mq\alpha_{pc^1}(Q)+M\alpha_{pc^1}(Q)\int_0^t l(s)ds$$
$$+M\sum_{0<t_i<t}k_i^1\alpha(Q(t_i)))+M\alpha_{pc^1}(Q)\sum_{0<t_i<t}k_i^2$$
$$\leq M(q+\widetilde{L}+S_1+S_2)\alpha_{pc^1}(Q). \tag{3.9}$$

Similarly,

$$\alpha((FQ)(t))\leq M(\|A\|q+\widetilde{L}+\|A\|S_1+S_2)\alpha_{pc^1}(Q). \tag{3.10}$$

Finally, inequalities (3.8), (3.9) and (3.10) imply that

$$\alpha_{pc^1}((FQ))\leq MK\alpha_{pc^1}(Q),$$

where $K=\max(\|A\|q+\widetilde{L}+\|A\|S_1+S_2, q+\widetilde{L}+S_1+S_2)$

By lemma 2.5 the theorem (3.3) is proved.

4. Application

Consider the following impulse scalar second order differential equation with nonlocal conditions

$$\begin{cases} x''(t) = x(t) + \frac{\arctan(t)}{18+t^2}\left[x(t) + x'(t)\right],\ t \in J = (0,1]\backslash\{\beta_1, \beta_2, ..., \beta_5\} \\ x(0) = x_0 + \frac{1}{9}\sum_{j=1}^{3} 2^{-j} x(t_j),\ x'(0) = x_1 \\ \triangle(x(\frac{1}{4})) = \frac{1}{30} x(\frac{1}{4}),\ i = 0, ..., 5 \\ \triangle(x'(\frac{1}{4})) = \frac{1}{100}(x(\frac{1}{4}) + x'(\frac{1}{4})), \end{cases} \tag{4.1}$$

where $0 < \beta_1 < \beta_2 < ... < \beta_5 < 1$ and $t_j \in (0,1], j = 1, 2, ..., p$. Here $E = \mathbb{R}$, $C(t) = \cosh(t)$, $S(t) = \sinh(t)$, $\max\limits_{t\in[0,1]} \cosh(t) = \cosh(1) < 3$. Since arcosh(3) = 1,7627, $\max\limits_{t\in[0,1]} \sinh(t) = \sinh(1) < 3$, thus we can choose $M = 3$. It is easy to see that $f(t, x, y) = \frac{1}{1+t^2}\sqrt{x^2 + y^2}$ satisfies to the inequality $|f(t, x, y)| \leq \frac{1}{18+t^2}(|x| + |y|)$ for all $t \in [0, 1]$ and $x, y \in \mathbb{R}$. Similarly, it is not difficult to show that
$q = \dfrac{1}{9}\sum_{j=1}^{3}(\dfrac{1}{2})^j$, $k_i^1 = \frac{1}{9}$, $k_i^2 = \frac{1}{9}$ and $l(s) = \frac{\pi}{2(18+s^2)}$. If we take $R = 3$ it is easy to see that when $\|x_0\| + \|x_1\| < \frac{13}{30}$ and $q + L + S_1 + S_2 < \frac{1}{3}$. Then all conditions of theorem (3.3) are satisfied. Thus, our conclusion follows from the main theorem.

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