Oscillation condition for first order linear dynamic equations on time scales

ÖZKAN ÖCALAN AND NURTEN KILIÇ *

[1]*Akdeniz University, Faculty of Science, Department of Mathematics, 07058 Antalya, Turkey.*
[2] *Kütahya Dumlupınar University, Faculty of Science and Art, Department of Mathematics, 43100 Kütahya, Turkey.*

Abstract. In this paper, we deal with the first-order dynamic equations with nonmonotone arguments

$$y^\Delta(\zeta) + \sum_{i=1}^{m} r_i(\zeta) y(\psi_i(\zeta)) = 0, \ \zeta \in [\zeta_0, \infty)_\mathbb{T}$$

where $r_i \in C_{rd}\left([\zeta_0, \infty)_\mathbb{T}, \mathbb{R}^+\right)$, $\psi_i \in C_{rd}\left([\zeta_0, \infty)_\mathbb{T}, \mathbb{T}\right)$ and $\psi_i(\zeta) \leq \zeta$, $\lim_{\zeta \to \infty} \psi_i(\zeta) = \infty$ for $1 \leq i \leq m$. Also, we present a new sufficient condition for the oscillation of delay dynamic equations on time scales. Finally, we give an example illustrating the result.

AMS Subject Classifications: 39A12, 39A21, 34C10, 34N05.

Keywords: Dynamic equation, nonmonotone delays, oscillatory solution, time scales.

Contents

1. Introduction and Background

We consider the delay dynamic equation with several delays which are not necessarily monotone

$$y^\Delta(\zeta) + \sum_{i=1}^{m} r_i(\zeta) y(\psi_i(\zeta)) = 0, \ \ \zeta \in [\zeta_0, \infty)_\mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $\zeta_0 \in \mathbb{T}$, $r_i \in C_{rd}([\zeta_0, \infty)_\mathbb{T}, \mathbb{R}_0^+)$, $\psi_i \in C_{rd}([\zeta_0, \infty)_\mathbb{T}, \mathbb{T})$ do not have to be monotone for $1 \leq i \leq m$ such that

$$\psi_i(\zeta) \leq \zeta \text{ for all } \zeta \in \mathbb{T}, \ \ \lim_{\zeta \to \infty} \psi_i(\zeta) = \infty. \tag{1.2}$$

First of all, we would like to remind some basic concepts about time scales calculus. A function $r : \mathbb{T} \to \mathbb{R}$ is said to be positively regressive (it means that $r \in \mathcal{R}^+$ if it is rd-continuous and satisfies $1 + \mu(\zeta)r(\zeta) > 0$ for all $\zeta \in \mathbb{T}$, where $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is the graininess function defined by $\mu(\zeta) := \sigma(\zeta) - \zeta$ with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined with the help of $\sigma := \inf\{s \in \mathbb{T} : s > \zeta\}$ for $\zeta \in \mathbb{T}$. If $\sigma(\zeta) = \zeta$ or $\mu(\zeta) = 0$, a point $\zeta \in \mathbb{T}$ is said to be right-dense, otherwise it is right-scattered.

---

*Corresponding author. Email address: **nurten.kilic@dpu.edu.tr** (Nurten KILIÇ)

A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of (1.1), if $y(\zeta)$ is delta differentiable for $\zeta \in \mathbb{T}^{\kappa}$ and satisfies (1.1) for $\zeta \in \mathbb{T}^{\kappa}$. It is called that a solution y of (1.1) has a generalized zero at ζ if $y(\zeta) = 0$ or if $\mu(\zeta) > 0$ and $y(\zeta)y(\sigma(\zeta)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution y of (1.1) is called oscillatory on $[\zeta, \infty)$ if it has arbitrarily large generalized zeros in $[\zeta, \infty)$. Also, we refer to book of Bohner and Peterson [2,3] for more detailed information.

For $m = 1$, we have the following equation which is the form of (1.1) with single delay.

$$y^{\Delta}(\zeta) + r(\zeta)y(\psi(\zeta)) = 0, \ \ \zeta \in [\zeta_0, \infty)_{\mathbb{T}}. \tag{1.3}$$

Recently, there has been remarkable interest for the oscillatory solutions of this equation. See [1-21] and the references cited therein. Concerning Eq. (1.3) which have monotone arguments, see also Zhang and Deng [20], Bohner [4], Zhang et al. [21], Şahiner and Stavroulakis [19], Agarwal and Bohner [1], and Karpuz and Öcalan [9]. As you seen, many articles have been dedicated to the equations which have monotone terms, but a few is related with the more general case of nonmonotone delay terms. Now, we mention the results which contain delay arguments which are not necessarily monotone.

Suppose that $\psi(\zeta)$ does not have to be monotone and

$$\vartheta(\zeta) = \sup_{s \leq \zeta} \psi(s), \ \zeta \in \mathbb{T}, \ \zeta \geq 0. \tag{1.4}$$

Obviously, $\vartheta(\zeta)$ is nondecreasing and $\psi(\zeta) \leq \vartheta(\zeta)$ for all $\zeta \geq 0$.

In 2017, Öcalan et al. [16] found out the result given below. If

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} r(s)\Delta s > 1, \tag{1.5}$$

where $\vartheta(\zeta)$ is defined by (1.4), then all solutions of (1.3) oscillate. In 2020, Öcalan [17] obtained the following criteria. If $-r \in \mathcal{R}^{+}$ and

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r(s)}{e_{-r}(\vartheta(\zeta), \psi(s))} \Delta s > 1 \tag{1.6}$$

or

$$\liminf_{\zeta \to \infty} \int_{\psi(\zeta)}^{\zeta} \frac{r(s)}{e_{-r}(\vartheta(s), \psi(s))} \Delta s > \frac{1}{e}, \tag{1.7}$$

where $\vartheta(\zeta)$ is defined by (1.4),

$$e_{-\lambda r}(\zeta, \psi(\zeta)) = \exp\left\{ \int_{\psi(\zeta)}^{\zeta} \xi_{\mu(s)}(-\lambda r(s))\Delta s \right\}$$

and

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & , \text{if } h \neq 0 \\ z & , \text{if } h = 0 \end{cases},$$

then all solutions of equation (1.3) oscillate. Also, (1.7) implies that the following condition. If

$$\liminf_{\zeta \to \infty} \int_{\psi(\zeta)}^{\zeta} r(s)\Delta s > \frac{1}{e}, \tag{1.8}$$

where $\vartheta(\zeta)$ is given with (1.4), then all solutions of (1.3) oscillate.

Eventually, Öcalan [18] presented the following result.

Let

$$\bar{\alpha} := \liminf_{\zeta \to \infty} \int_{\psi(\zeta)}^{\zeta} r(s)\Delta s. \tag{1.9}$$

If $-r \in \mathcal{R}^+, 0 \leq \bar{\alpha} \leq \frac{1}{e}$ and

$$\limsup_{t \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r(s)}{e_{-r}(\vartheta(\zeta), \psi(s))} \Delta s > 1 - \frac{1 - \bar{\alpha} - \sqrt{1 - 2\bar{\alpha} - \left(\bar{\alpha}\right)^2}}{2}, \tag{1.10}$$

where $\vartheta(\zeta)$ and $\bar{\alpha}$ are defined by (1.4) and (1.9), resp., then every solution of (1.3) is oscillatory.

Kılıç and Öcalan [12] obtained the following criteria which are the first results for (1.1) with several nonmonotone arguments.

Set $\psi_i(\zeta)$ are not necessarily monotone for $1 \leq i \leq m$ and

$$\vartheta_i(\zeta) = \sup_{s \leq \zeta} \{\psi_i(s)\} \text{ and } \vartheta(\zeta) = \max_{1 \leq i \leq m} \{\vartheta_i(\zeta)\}, \ \zeta \in \mathbb{T}, \ \zeta \geq 0. \tag{1.11}$$

Obviously, $\vartheta_i(\zeta)$ are nondecreasing and $\psi_i(\zeta) \leq \vartheta_i(\zeta) \leq \vartheta(\zeta)$ for all $\zeta \geq 0$ and $1 \leq i \leq m$.

Teorem A: Suppose that $-\sum\limits_{i=1}^{m} r_i \in \mathcal{R}^+$ and (1.11) holds. If

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)\Delta s > 1 \tag{1.12}$$

or

$$\liminf_{\zeta \to \infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} r_i(s)\Delta s > \frac{1}{e}, \tag{1.13}$$

where $\psi(\zeta) = \max\limits_{1 \leq i \leq m} \{\psi_i(\zeta)\}$, then every solution of (1.1) oscillates.

Further assume that

$$\alpha := \liminf_{\zeta \to \infty} \int_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} r_i(s)\Delta s. \tag{1.14}$$

Teorem B: Suppose that $-\sum\limits_{i=1}^{m} r_i \in \mathcal{R}^+$ and $0 \leq \alpha \leq \frac{1}{e}$. If

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)\Delta s > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{1.15}$$

then every solution of (1.1) oscillate.

Lately, Öcalan and Kılıç [13] established the following results for (1.1).

Teorem C: Suppose that $-\sum\limits_{i=1}^{m} r_i \in \mathcal{R}^+$, (1.2) and (1.11) hold. If

$$\limsup_{\zeta \to \infty} \int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s > 1 \tag{1.16}$$

or

$$\liminf_{\zeta \to \infty} \int\limits_{\psi(\zeta)}^{\zeta} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(s), \psi_i(s))} \Delta s > \frac{1}{e}, \tag{1.17}$$

where $\psi(\zeta) = \max\limits_{1 \leq i \leq m} \{\psi_i(\zeta)\}$, then all solutions of (1.1) are oscillatory.

Although dynamic equations with several arguments are more comprehensive than dynamic equations with one delay, there are not many studies on this subject. So, in this article, we are interested in studying the oscillatory behavior of first order dynamic equations with several delays on time scale. We present one criterion to check the oscillation of (1.1). Our result is an extension and complement to some results published in the literature.

2. Main Results

In this section, we introduce a new sufficient condition for the oscillatory solutions of (1.1) when the arguments $\psi_i(\zeta)$ do not have to be monotone for $1 \leq i \leq m$ and $0 < \alpha \leq \frac{1}{e}$. The following lemmas will be useful to obtain our main result.

The lemma given below can be easily obtained from [4].

Lemma 2.1. *Let* $-\sum\limits_{i=1}^{m} r_i \in \mathcal{R}^+$. *If*

$$y^\Delta(\zeta) + y(\zeta) \sum_{i=1}^{m} r_i(\zeta) \leq 0,$$

then

$$y(\zeta) \leq e_{-\sum\limits_{j=1}^{m} r_j}(\zeta, s)\, y(s) \ \textit{for all } \zeta \geq s, \ s, \zeta \in \mathbb{T}. \tag{2.1}$$

The result given below can be easily produced by applying a nearly same procedure to [21, Lemma 2.4] when the case $\psi_i(\zeta)$ do not have to be monotone for $1 \leq i \leq m$. Therefore, the proof of this lemma is not presented here.

Lemma 2.2. *Suppose that* $\psi_i(\zeta)$ *are not necessarily monotone for* $1 \leq i \leq m$. *Let* $0 \leq \alpha \leq \frac{1}{e}$ *and* $y(\zeta)$ *be an eventually positive solution of (1.1). Then, we have*

$$\liminf_{\zeta \to \infty} \frac{y(\sigma(\zeta))}{y(\vartheta(\zeta))} \geq \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.2}$$

where $\psi(\zeta) = \max\limits_{1 \leq i \leq m} \{\psi_i(\zeta)\}$, $\vartheta(\zeta)$ *and* α *are defined by (1.11) and (1.14) respectively.*

Theorem 2.3. *Assume that* $-\sum\limits_{i=1}^{m} r_i \in \mathcal{R}^+$, *(1.2) holds and*

$$\limsup_{\zeta \to \infty} \int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.3}$$

where $\vartheta(\zeta)$ and α are defined by (1.11) *and* (1.14) *resp., then all solutions of* (1.1) *oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $y(\zeta)$ of (1.1). If $y(\zeta)$ is an eventually negative solution of (1.1), the proof of the theorem can be done similarly. Then there exists $\zeta_1 > \zeta_0$ such that $y(\zeta)$, $y(\psi_i(\zeta))$, $y(\psi(\zeta))$, $y(\vartheta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$ and $1 \leq i \leq m$. So, using (1.1) we obtain

$$y^\Delta(\zeta) = -\sum_{i=1}^{m} r_i(\zeta)y(\psi_i(\zeta)) \leq 0 \text{ for all } \zeta \geq \zeta_1,$$

which implies that $y(\zeta)$ is an eventually nonincreasing function. From this fact and taking into account that $\psi_i(\zeta) \leq \vartheta_i(\zeta) \leq \zeta$ for $1 \leq i \leq m$, (1.1) gives

$$y^\Delta(\zeta) + y(\zeta)\sum_{i=1}^{m} r_i(\zeta) \leq 0, \ \zeta \geq \zeta_1 \tag{2.4}$$

and then, we obtain the below expression from Lemma 2.1.

$$y(\vartheta(\zeta)) \leq e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))\, y(\psi_i(s)) \text{ for all } \vartheta(\zeta) \geq \psi_i(s). \tag{2.5}$$

On the other hand, integrating (1.1) from $\vartheta(\zeta)$ to $\sigma(\zeta)$ and with the help of (2.5), we have

$$y(\sigma(\zeta)) - y(\vartheta(\zeta)) + \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)y(\psi_i(s))\Delta s = 0$$

$$y(\sigma(\zeta)) - y(\vartheta(\zeta)) + \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} r_i(s)\frac{y(\vartheta(\zeta))}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))}\Delta s \leq 0$$

$$y(\sigma(\zeta)) - y(\vartheta(\zeta)) + y(\vartheta(\zeta))\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))}\Delta s \leq 0$$

or

$$\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))}\Delta s \leq 1 - \frac{y(\sigma(\zeta))}{y(\vartheta(\zeta))}. \tag{2.6}$$

Consequently, from (2.6) we obtain

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))}\Delta s \leq 1 - \liminf_{\zeta \to \infty} \frac{y(\sigma(\zeta))}{y(\vartheta(\zeta))} \tag{2.7}$$

and from (2.2) the last inequality turns into

$$\limsup_{\zeta \to \infty} \int_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))}\Delta s \leq 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2},$$

which contradicts to (2.3) and this completes the proof. ∎

Example 2.4. *Let $m = 2$ and $\mathbb{T} = 3\mathbb{Z} = \{3k : k \in \mathbb{Z}\}$. Then, we get for $\zeta \in \mathbb{T}$*

$$\sigma(\zeta) = \zeta + 3, \;\; \mu(\zeta) = 3 \;\; and \;\; y^{\Delta}(\zeta) = \frac{y(\zeta + 3) - y(\zeta)}{3}.$$

So, (1.1) becomes

$$\frac{y(\zeta + 3) - y(\zeta)}{3} + r_1(\zeta)y(\psi_1(\zeta)) + r_2(\zeta)y(\psi_2(\zeta)) = 0, \;\; \zeta \in \{3k : k \in \mathbb{Z}\}.$$

Let $\psi_1(\zeta) = \zeta - 3$, $\psi_2(\zeta) = \zeta - 6$, then $\psi(\zeta) = \max\limits_{1 \leq i \leq m} \{\psi_i(\zeta)\} = \psi_1(\zeta) = \zeta - 3$. Since $1 \leq i \leq m$, $r_i(\zeta) \in \{3k : k \in \mathbb{Z}\}$, we suppose that

$$r_1(3\zeta) = 0.09, \, r_1(3\zeta + 3) = 0.07 \; and \; r_2(\zeta) = 0.03 \;\; \zeta = 0, 3, 6, \ldots.$$

If $\mathbb{T} = h\mathbb{Z}$, from Theorem 1.79 [2], we know the formula given below.

$$\int\limits_a^b f(t)\Delta t = \sum\limits_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h \;\; for \; a < b. \tag{2.8}$$

Then, by using (2.8) we obtain that for $1 \leq i \leq m$, $r_i(\zeta), \psi(\zeta) \in \{3k : k \in \mathbb{Z}\}$

$$\begin{aligned}
\alpha \; := \; & \liminf_{\zeta \to \infty} \int\limits_{\psi(\zeta)}^{\zeta} \sum\limits_{i=1}^{m} r_i(s)\Delta s = \liminf_{\zeta \to \infty} \sum\limits_{j=\frac{\zeta-3}{3}}^{\frac{\zeta}{3}-1} \sum\limits_{i=1}^{2} 3r_i(3j) \\
= \; & \liminf_{\zeta \to \infty} \sum\limits_{j=\frac{\zeta-3}{3}}^{\frac{\zeta}{3}-1} 3(r_1(3j) + r_2(3j)) = \liminf_{\zeta \to \infty} 3(r_1(\zeta - 3) + r_2(\zeta - 3)) \\
= \; & 0.36 < \frac{1}{e}
\end{aligned}$$

and

$$\begin{aligned}
M \; := \; & \limsup_{\zeta \to \infty} \int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum\limits_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(s))} \Delta s = \limsup_{\zeta \to \infty} \sum\limits_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \sum\limits_{i=1}^{2} \frac{3r_i(3k)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_i(3k))} \\
= \; & \limsup_{\zeta \to \infty} \sum\limits_{k=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} \left[\frac{3r_1(3k)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_1(3k))} + \frac{3r_2(3k)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta), \psi_2(3k))} \right].
\end{aligned}$$

Now, we obtain that

$$
\begin{aligned}
e_{-r}\left(\vartheta(\zeta),\psi_1(3k)\right) &= \exp\left\{\int_{\psi_1(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}\sum_{j=1}^{m}(-r_j(u))\Delta u\right\} = \exp\left\{\int_{\psi_1(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}(-(r_1(u)+r_2(u))\Delta u\right\} \\
&= \exp\left\{\sum_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{3\log\left(1-\mu(3i)(r_1(3i)+r_2(3i))\right)}{\mu(3i)}\right\} \\
&= \exp\left\{\sum_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \log\left(1-3(r_1(3i)+r_2(3i))\right)\right\} \\
&= \exp\left\{\log\prod_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i))\right)\right\} = \prod_{i=\frac{\psi_1(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i))\right)
\end{aligned}
$$

and

$$
\begin{aligned}
e_{-r}\left(\vartheta(\zeta),\psi_2(3k)\right) &= \exp\left\{\int_{\psi_2(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}\sum_{j=1}^{m}(-r_j(u))\Delta u\right\} = \exp\left\{\int_{\psi_2(3k)}^{\vartheta(\zeta)} \xi_{\mu(u)}(-(r_1(u)+r_2(u))\Delta u\right\} \\
&= \exp\left\{\sum_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{3\log\left(1-\mu(3i)(r_1(3i)+r_2(3i))\right)}{\mu(3i)}\right\} \\
&= \exp\left\{\sum_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \log\left(1-3(r_1(3i)+r_2(3i))\right)\right\} \\
&= \exp\left\{\log\prod_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i))\right)\right\} = \prod_{i=\frac{\psi_2(3k)}{3}}^{\frac{\vartheta(\zeta)}{3}-1}\left(1-3(r_1(3i)+r_2(3i))\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_1(s)}{e_{-\sum_{j=1}^{m}r_j}\left(\vartheta(\zeta),\psi_1(s)\right)}\Delta s &= \sum_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \frac{3r_1(3k)}{e_{-\sum_{j=1}^{m}r_j}\left(\vartheta(\zeta),\psi_1(3k)\right)} \\
&= \sum_{j=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} 3r_1(3j)\prod_{i=\frac{\psi_1(3j)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i)))} \\
&= \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} 3r_1(3j)\prod_{i=j-1}^{\frac{\zeta-3}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i)))} \\
&= 3r_1(\zeta-3)\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6)))} + 3r_1(\zeta) \\
&\cong 0.42187 + 0.21
\end{aligned}
$$

and

$$\int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_2(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(s))}\Delta s = \sum_{k=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} \frac{3r_2(3k)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(3k))}$$

$$= \sum_{j=\frac{\vartheta(\zeta)}{3}}^{\frac{\sigma(\zeta)}{3}-1} 3r_2(3j) \prod_{i=\frac{\psi_2(3j)}{3}}^{\frac{\vartheta(\zeta)}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i)))}$$

$$= \sum_{j=\frac{\zeta-3}{3}}^{\frac{\zeta+3}{3}-1} 3r_2(3j) \prod_{i=j-2}^{\frac{\zeta-3}{3}-1} \frac{1}{(1-3(r_1(3i)+r_2(3i)))}$$

$$= 3r_2(\zeta-3)\frac{1}{(1-3(r_1(\zeta-9)+r_2(\zeta-9)))}\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6)))}$$

$$+3r_2(\zeta)\frac{1}{(1-3(r_1(\zeta-6)+r_2(\zeta-6)))}$$

$$\cong 0.219726 + 0.14062$$

Thus, we have

$$M \; : \; = \limsup_{\zeta\to\infty} \int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \sum_{i=1}^{m} \frac{r_i(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_i(s))}\Delta s$$

$$= \limsup_{t\to\infty} \left[\int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_1(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_1(s))} + \int\limits_{\vartheta(\zeta)}^{\sigma(\zeta)} \frac{r_2(s)}{e_{-\sum\limits_{j=1}^{m} r_j}(\vartheta(\zeta),\psi_2(s))} \right] \Delta s$$

and

$$M \cong 0,992216 \ngtr 1$$

implies that (1.16) fails. However, since

$$M \cong 0.992216 > 1 - \frac{1 - 0.36 - \sqrt{1 - 2(0.36) - (0.36)^2}}{2} \cong 0.87391,$$

which means that all solutions of this equation oscillate by Theorem 2.3. As you can see above, all results which are obtained in literature before can't hold. But, our new oscillation condition holds.

References

[1] R. P. AGARWAL AND M. BOHNER, An oscillation criterion for first order delay dynamic equations, *Funct. Differ. Equ.,* **16(1)**(2009), 11–17.

[2] M. BOHNER AND A. PETERSON, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[3] M. BOHNER AND A. PETERSON, Advances in Dynamic Equations on Time Scales. Birkhauser, Boston, 2003.

[4] M. BOHNER, Some Oscillation criteria for first order delay dynamic equations, *Far East J. Appl. Math.,* **18(3)**(2005), 289–304.

[5] A. ELBERT AND I. P. STAVROULAKIS, Oscillations of first order differential equations with deviating arguments, Univ of Ioannina T.R. No 172, 1990, Recent trends in differential equations, 163–178, World Sci. Ser. Appl. Anal., 1 (1992) World Sci. Publishing Co.

[6] L. H. ERBE, Q. KONG AND B. G. ZHANG, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.

[7] N. FUKAGAI AND T. KUSANO, Oscillation theory of first order functional differential equations with deviating arguments, *Ann. Mat. Pura Appl.,* **136**(1984), 95–117.

[8] I. GYŐRI AND G. LADAS, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[9] B. KARPUZ AND Ö. ÖCALAN, New oscillation tests and some refinements for first-order delay dynamic equations, *Turkish J. Math.,* **40(4)**(2016), 850–863.

[10] B. KARPUZ, Sharp oscillation and nonoscillation tests for linear difference equations, *J. Difference Equ Appl.,* **23(12)**(2017), 1929–1942.

[11] B. KARPUZ, Sharp oscillation and nonoscillation tests for delay dynamic equations, *Math. Methods Appl. Sci.,* **42(9)**(2019), 2993–3001.

[12] N. KILIÇ AND Ö. ÖCALAN, Oscillation criteria for first-order dynamic equations with nonmonotone delay, *Hacet. J. Math. Stat.,* **50(2)**(2021), 318–325.

[13] Ö. ÖCALAN AND N. KILIÇ, Sufficient oscillation conditions for dynamic equations with nonmonotone delays, *J. Appl. Math. & Informatics,* **40(5-6)**(2022), 843–856.

[14] R. G. KOPLATADZE AND T. A. CHANTURIJA, Oscillating and monotone solutions of first-order differential equations with deviating arguments, (Russian), *Differentsial'nye Uravneniya,* **8**(1982), 1463–1465.

[15] G.S. LADDE, V. LAKSHMIKANTHAM AND B.G. ZHANG, Oscillation Theory of Differential Equations with Deviating Arguments, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, (1987).

[16] Ö. ÖCALAN, U. M. ÖZKAN AND M. K. YILDIZ, Oscillatory solutions for dynamic equations with non-monotone arguments, *J. Math. Comput. Sci.,* **7(4)**(2017), 725–738.

[17] Ö. ÖCALAN, Oscillation of first-order dynamic equations with nonmonotone delay, *Math. Methods Appl. Sci.,* **43(7)**(2020), 3954–3964.

[18] Ö. ÖCALAN, An improved oscillation criteria for first order dynamic equations, *Turk J Math.,* **45**(2021), 487–495.

[19] Y. ŞAHINER AND I. P. STAVROULAKIS, Oscillations of first order delay dynamic equations, *Dynam. Systems Appl.,* **15(3-4)**(2006), 645–655.

[20] B. G., ZHANG AND X. DENG, Oscillation of delay differential equations on time scales, *Math. Comput. Modelling,* **36(11-13)**(2002), 1307–1318.

[21] B. G. ZHANG, X. YAN, X. LIU, Oscillation criteria of certain delay dynamic equations on time scales, *J. Difference Equ. Appl.,* **11(10)**(2005), 933–946.