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On determination of discontinuous Sturm-Liouville operator from Weyl function

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Abstract. In this paper, the Weyl function for the Sturm-Liouville operator which contains the discontinuous coefficient and discontinuity conditions at an interior point of the finite interval is defined and examined. The uniqueness theorem of solution of the inverse spectral problem for the discontinuous Sturm-Liouville operator according to Weyl function is proved.

AMS Subject Classifications: 34B24, 34A55, 47E05.

Keywords: Inverse problem, Sturm-Liouville equation, discontinuity conditions and discontinuous coefficient, Weyl function.

Contents

1. Introduction and Background

This paper deals with the discontinuous Sturm-Liouville operator which contains both the discontinuous coefficient and the discontinuity conditions at an interior point $x = \xi \in (0, \pi)$ of the finite interval:

$$-\omega'' + q(x)\omega = \tau^2 r(x)\omega, \quad 0 < x < \pi \tag{1.1}$$

$$\omega(\xi + 0) = c\omega(\xi - 0), \quad \omega'(\xi + 0) = c^{-1}\omega'(\xi - 0) \tag{1.2}$$

$$\omega'(0) - b_1\omega(0) = 0, \quad \omega'(\pi) + b_2\omega(\pi) = 0, \tag{1.3}$$

where real valued function $q(x)$ belongs to $L_2(0, \pi)$, $c > 0$, b_1 and b_2 are real constants, τ is a spectral parameter, the discontinuous coefficient $r(x)$ is in the following form:

$$r(x) = \begin{cases} 1, & 0 < x < \xi, \\ a^2, & \xi < x < \pi, \end{cases}$$

$0 < a \neq 1$ and assume that $\xi > \frac{a\pi}{a+1}$.

In recent years, many works on the discontinuous boundary value problems have been done and there has been a significant increase in interest on this subject. We indicate that such problems are connected with discontinuous material properties, so the investigations on this problems are attractive in the mathematics, physics and engineering (for details see [8]).

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The purpose of this study is to examine the inverse spectral problem for the discontinuous Sturm-Liouville problem (1.1)-(1.3) and this problem is stated in the following way: given the Weyl function, construct the boundary value problem (1.1)-(1.3). Therefore, firstly we define and examine the Weyl function of the problem (1.1)-(1.3) and then the uniqueness theorem for the solution of this inverse spectral problem is proved.

Differently from other studies, considered problem contains both the discontinuous coefficient $r(x)$ and the discontinuity conditions at $x = \xi \in (0, \pi)$. In the special cases, i.e., as $c = 1$, the inverse problems for Sturm-Liouville operator with discontinuous coefficient by Weyl function are examined in [1, 4, 11] and as $r(x) \equiv 1$, the inverse problems for Sturm-Liouville operator with discontinuity conditions by Weyl function are investigated in [6, 7]. Moreover, the various works on the inverse problems for the discontinuous Sturm-Liouville operators can be given as follows: [3, 5, 9, 10, 12–18] and the references therein.

The spectral properties of the boundary value problem (1.1)-(1.3) are studied in [2]; namely, the integral representation of the solution of (1.1) with discontinuity conditions (1.2) is obtained and using this solution, the asymptotic formulas of the eigenvalues and eigenfunctions of this problem are investigated. Note that the constructed integral representation is not transformation operator, moreover; the kernel of this solution has a discontinuity along the line $t = -a(x - \xi) + a$, for $\xi < a < \pi$. Unlike other studies, using this constructed integral representation we prove the uniqueness theorem of the inverse spectral problem (1.1)-(1.3) by the Weyl function.

Theorem 1.1. *[2] The integral representation of the solution $f(x, \tau)$ of equation (1.1) with discontinuity conditions (1.2) satisfying the conditions $f(0, \tau) = 1$, $f'(0, \tau) = i\tau$ has the form:*

$$f(x,\tau) = f_0(x,\tau) + \int_{-\alpha(x)}^{\alpha(x)} k(x,t)e^{i\tau t}dt, \tag{1.4}$$

where

$$f_0(x,\tau) = \begin{cases} e^{i\tau x}, & 0 < x < \xi, \\ \kappa_1 e^{i\tau(a(x-\xi)+\xi)} + \kappa_2 e^{i\tau(-a(x-\xi)+\xi)}, & \xi < x < \pi, \end{cases}$$

with $\kappa_1 = \frac{1}{2}\left(c + \frac{1}{ac}\right)$ and $\kappa_2 = \frac{1}{2}\left(c - \frac{1}{ac}\right)$,

$$\alpha(x) = \begin{cases} x, & 0 < x < \xi, \\ a(x-\xi)+\xi, & \xi < x < \pi, \end{cases}$$

the kernel $k(x, .) \in L_1(-\alpha(x), \alpha(x))$ for each fixed $x \in (0, \pi)$ and satisfies the inequality

$$\int_{-\alpha(x)}^{\alpha(x)} |k(x,t)|dt \le e^{p\sigma(x)} - 1$$

with

$$\sigma(x) = \int_0^x (x-u)|q(u)|du, \qquad p = (a+4)|\kappa_1| + (a+2)|\kappa_2|.$$

Remark 1.2. *The function $k(x, t)$ has following properties:*

$$k(x,\alpha(x)) = \begin{cases} \frac{1}{2}\int_0^x q(u)du, & 0 < x < \xi, \\ \frac{\kappa_1}{2}\int_0^x \frac{1}{\sqrt{r(u)}}q(u)du, & \xi < x < \pi, \end{cases}$$

$$k\left(x, -a(x-\xi)+\xi+0\right) - k\left(x, -a(x-\xi)+\xi-0\right) = \frac{-\kappa_2}{2}\left(\int_0^\xi q(u)du - \frac{1}{a}\int_\xi^x q(u)du\right), \quad \xi < x < \pi,$$

$$k(x, -\alpha(x)) = 0.$$

Moreover, it is seen that the real-valued function $k(x, t)$ has a discontinuity along the line $t = -a(x - \xi) + \xi$ for $\xi < x < \pi$.

Now, take into account the case of $b_1 = \infty$ in the boundary condition (1.3). Then, the boundary condition is as follows:

$$\omega(0) = \omega'(\pi) + b_2\omega(\pi) = 0 \tag{1.5}$$

and consider the boundary value problems (1.1)-(1.3) and (1.1),(1.2),(1.5).

Denote $u(x,\tau)$ and $v(x,\tau)$ by the solutions of the equation (1.1) with the condition (1.2) under the initial conditions

$$u(0,\tau) = 1, \quad u'(0,\tau) = b_1,$$

$$v(0,\tau) = 0, \quad v'(0,\tau) = 1.$$

Using the integral representation (1.4), we express the solutions $u(x,\tau)$ and $v(x,\tau)$ in the following forms:

$$u(x,\tau) = u_0(x,\tau) + \int_0^{\alpha(x)} \left(h(x,t)\cos\tau t + \tilde{h}(x,t)\frac{b_1 \sin\tau t}{\tau} dt \right),$$

and

$$v(x,\tau) = v_0(x,\tau) + \int_0^{\alpha(x)} \tilde{h}(x,t)\frac{\sin\tau t}{\tau} dt,$$

where for $0 < x < \xi$:

$$u_0(x,\tau) = \cos\tau x + \frac{b_1 \sin\tau x}{\tau}, \quad v_0(x,\tau) = \frac{\sin\tau x}{\tau}$$

and for $\xi < x < \pi$:

$$u_0(x,\tau) = \kappa_1 \left(\cos\tau\upsilon^+(x) + \frac{b_1 \sin\tau\upsilon^+(x)}{\tau} \right) + \kappa_2 \left(\cos\tau\upsilon^-(x) + \frac{b_1 \sin\tau\upsilon^-(x)}{\tau} \right),$$

$$v_0(x,\tau) = \kappa_1 \frac{\sin\tau\upsilon^+(x)}{\tau} + \kappa_2 \frac{\sin\tau\upsilon^-(x)}{\tau}$$

with $\upsilon^{\pm}(x) = \pm a(x-\xi) + \xi$, $h(x,t) = k(x,t) + k(x,-t)$ and $\tilde{h}(x,t) = k(x,t) - k(x,-t)$, respectively.

Let $\phi(x,\tau)$ be the solution of the equation (1.1) with the condition (1.2) under the initial conditions

$$\phi(\pi,\tau) = -1, \quad \phi'(\pi,\tau) = b_2.$$

The characteristic functions $\chi(\tau)$ and $\varphi(\tau)$ of the problems (1.1)-(1.3) and (1.1), (1.2) and (1.5) can be given as follows:

$$\chi(\tau) = u'(\pi,\tau) + b_2 u(\pi,\tau) = \phi'(0,\tau) - b_1\phi(0,\tau) \tag{1.6}$$

$$\varphi(\tau) = v'(\pi,\tau) + b_2 v(\pi,\tau) = -\phi(0,\tau), \tag{1.7}$$

respectively. It is known from [2] that

$$|\chi(\tau)| \geq C_\delta |\tau| e^{|Im\tau|\upsilon^+(\pi)}, \quad \tau \in G_\delta, \tag{1.8}$$

where $G_\delta = \{\tau : |\tau - \tilde{\tau}_n| \geq \delta\}$, here $\tilde{\tau}_n = \frac{n\pi}{\upsilon^+(\pi)} + d_n$, $\sup_n |d_n| = d < \infty$ and $\delta << \frac{s}{2}$ is a sufficiently small positive number with $s = \inf_{n \neq k} |\tilde{\tau}_n - \tilde{\tau}_k| > 0$. Moreover, from the expression of the solution $v(x,\tau)$, we have

$$|\varphi(\tau)| \leq C e^{|Im\tau|\upsilon^+(\pi)}. \tag{1.9}$$

Theorem 1.3. *[2] The boundary value problem (1.1)-(1.3) has a countable set of eigenvalues $\{\tau_n^2\}_{n\geq 1}$:*

$$\tau_n = \tilde{\tau}_n + \frac{s_n}{\tilde{\tau}_n} + \frac{t_n}{n},$$

where s_n is a bounded sequence and $\{t_n\} \in l_2$.

The norming constants γ_n of the problem (1.1)-(1.3) are defined by

$$\gamma_n = \int_0^\pi u^2(x,\tau) r(x) dx.$$

Moreover, the following asymptotic formulas of the solutions $u(x,\tau)$, $v(x,\tau)$ and $\phi(x,\tau)$ are valid for $|\tau| \to \infty$:

$$u(x,\tau) = O(e^{|Im\tau|\alpha(x)}), \quad v(x,\tau) = O\left(\frac{e^{|Im\tau|\alpha(x)}}{|\tau|}\right)$$
$$\phi(x,\tau) = O(e^{|Im\tau|(\alpha(\pi)-\alpha(x))}). \tag{1.10}$$

Note that when $q(x) \equiv 0$ in the equation (1.1), the solution $\phi_0(x,\tau)$ has the representation:

$$\phi_0(x,\tau) = -a(\kappa_1 \cos\tau(\upsilon^+(\pi)-x) - \kappa_2 \cos\tau(\upsilon^-(\pi)-x)) \\ -b_2\left(\kappa_1 \frac{\sin\tau(\upsilon^+(\pi)-x)}{\tau} + \kappa_2 \frac{\sin\tau(\upsilon^-(\pi)-x)}{\tau}\right), \quad 0 < x < \xi,$$

$$\phi_0(x,\tau) = -\cos\tau(\upsilon^+(\pi)-\upsilon^+(x)) - \frac{b_2 \sin\tau(\upsilon^+(\pi)-\upsilon^+(x))}{a\tau}, \quad \xi < x < \pi.$$

2. Main Results

Now, let us examine the Weyl solution and Weyl function for the boundary value problem (1.1)-(1.3).

Denote $\psi(x,\tau)$ by a solution of the equation (1.1) with the condition (1.2) satisfying the conditions

$$\psi'(0,\tau) - b_1\psi(0,\tau) = 1, \quad \psi'(\pi,\tau) + b_2\psi(\pi,\tau) = 0.$$

Then, it is obtained that

$$\psi(x,\tau) = \frac{\phi(x,\tau)}{\chi(\tau)} = v(x,\tau) + m(\tau)u(x,\tau), \tag{2.1}$$

where $m(\tau) = \psi(0,\tau)$. The functions $\psi(x,\tau)$ and $m(\tau)$ are called the *Weyl solution* and *Weyl function*, respectively. Moreover, taking into account (1.7), we can write

$$m(\tau) = \frac{\phi(0,\tau)}{\chi(\tau)} = -\frac{\varphi(\tau)}{\chi(\tau)}. \tag{2.2}$$

Hence, it can be seen that Weyl function $m(\tau)$ is meromorphic function with simple poles in the points $\tau = \tau_n$, $n \geq 1$. The squares of the poles and zeros of $m(\tau)$ coincide with the eigenvalues of the problems (1.1)-(1.3) and (1.1), (1.2), (1.5), respectively.

Theorem 2.1. *The representation is valid:*

$$m(\tau) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n(\tau^2 - \tau_n^2)}. \tag{2.3}$$

Proof. Taking into account (1.8), (1.9) and (2.2) we have for sufficiently large $\tau^* > 0$

$$|m(\tau)| \leq \frac{C_\delta}{|\tau|}, \quad \tau \in G_\delta, \quad |\tau| \geq \tau^*. \tag{2.4}$$

Using the relations $\dot{\chi}(\tau_n) = 2\tau_n\gamma_n\mu_n$ and $\phi(x,\tau_n) = \mu_n u(x,\tau_n)$ with $\mu_n \neq 0$ (see [2]), we find $\varphi(\tau_n) = -\phi(0,\tau_n) = -\mu_n$. Then, it follows from this relation that

$$\underset{\tau=\tau_n}{Res}\, m(\tau) = -\frac{\varphi(\tau_n)}{\dot{\chi}(\tau_n)} = \frac{1}{2\tau_n\gamma_n}. \tag{2.5}$$

Now, consider the contour integral

$$J_N(\tau) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{m(\zeta)}{\zeta - \tau} d\zeta, \quad \tau \in int\Gamma_N,$$

where $\Gamma_N = \{\tau : |\tau| = |\tilde{\tau}_n| + \frac{s}{2}\}$. It follows from (2.4) that $\lim\limits_{N\to\infty} J_N(\tau) = 0$. Moreover, applying the residue theorem and from (2.5), we find

$$\begin{aligned} J_N(\tau) &= m(\tau) + \sum_{n=1}^{N} \frac{1}{2\tau_n\gamma_n}\left(\frac{1}{(\tau_n - \tau)} - \frac{1}{(\tau_n + \tau)}\right) \\ &= m(\tau) - \sum_{n=1}^{N} \frac{1}{\gamma_n(\tau^2 - \tau_n^2)}. \end{aligned}$$

Thus, as $N \to \infty$, since $\lim\limits_{N\to\infty} J_N(\tau) = 0$, we obtain the relation (2.3). ■

Now, we examine the inverse problem indicated in the following way: given the Weyl function $m(\tau)$, determine the boundary value problem (1.1)-(1.3).

Let us demonstrate the uniqueness theorem of the solution for this inverse problem. Then, we specify the boundary value problem (1.1)-(1.3) as $L = L(q(x), b_1, b_2)$ and we take the problem $\hat{L} = L(\hat{q}(x), \hat{b}_1, \hat{b}_2)$ which has a similar form to L but with different potential and coefficients in the boundary conditions.

Theorem 2.2. *If $m(\tau) = \hat{m}(\tau)$, then $L = \hat{L}$. Namely, the Weyl function uniquely determines the problem (1.1)-(1.3).*

Proof. Denote the matrix $U(x,\tau) = [U_{k\ell}(x,\tau)]_{k,\ell=1,2}$ by the relation

$$U(x,\tau)\begin{pmatrix} \hat{u}(x,\tau) & \hat{\psi}(x,\tau) \\ \hat{u}'(x,\tau) & \hat{\psi}'(x,\tau) \end{pmatrix} = \begin{pmatrix} u(x,\tau) & \psi(x,\tau) \\ u'(x,\tau) & \psi'(x,\tau) \end{pmatrix}. \tag{2.6}$$

It follows from the equality

$$\langle u(x,\tau), \psi(x,\tau)\rangle = 1 \tag{2.7}$$

and the formula (2.6) that

$$\begin{aligned} U_{k1}(x,\tau) &= u^{(k-1)}(x,\tau)\hat{\psi}'(x,\tau) - \psi^{(k-1)}(x,\tau)\hat{u}'(x,\tau), \\ U_{k2}(x,\tau) &= \psi^{(k-1)}(x,\tau)\hat{u}(x,\tau) - u^{(k-1)}(x,\tau)\hat{\psi}(x,\tau), \quad k = 1,2 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} u(x,\tau) &= U_{11}(x,\tau)\hat{u}(x,\tau) + U_{12}(x,\tau)\hat{u}'(x,\tau), \\ \psi(x,\tau) &= U_{11}(x,\tau)\hat{\psi}(x,\tau) + U_{12}(x,\tau)\hat{\psi}'(x,\tau). \end{aligned} \tag{2.9}$$

Using (2.1), (2.7) and (2.8), we obtain

$$U_{11}(x,\tau) = 1 + u(x,\tau)\left(\frac{\hat{\phi}'(x,\tau)}{\hat{\chi}(\tau)} - \frac{\phi'(x,\tau)}{\chi(\tau)}\right) + \frac{\phi(x,\tau)}{\chi(\tau)}(u'(x,\tau) - \hat{u}'(x,\tau))$$

and

$$U_{12}(x,\tau) = \hat{u}(x,\tau)\frac{\phi(x,\tau)}{\chi(\tau)} - u(x,\tau)\frac{\hat{\phi}(x,\tau)}{\hat{\chi}(\tau)}.$$

With the help of the asymptotic formulas (1.10) and the inequality (2.4), we find

$$\lim_{\substack{|\tau|\to\infty \\ \tau\in G_\delta}} \max_{0\le x\le\pi} |U_{11}(x,\tau)-1| = \lim_{\substack{|\tau|\to\infty \\ \tau\in G_\delta}} \max_{0\le x\le\pi} |U_{12}(x,\tau)| = 0. \tag{2.10}$$

According to (2.1) and (2.8), we can write

$$U_{11}(x,\tau) = u(x,\tau)\hat{v}'(x,\tau) - v(x,\tau)\hat{u}'(x,\tau) + u(x,\tau)\hat{u}'(x,\tau)(\hat{m}(\tau) - m(\tau)),$$

$$U_{12}(x,\tau) = v(x,\tau)\hat{u}(x,\tau) - u(x,\tau)\hat{v}(x,\tau) + u(x,\tau)\hat{u}(x,\tau)(m(\tau) - \hat{m}(\tau)).$$

If $m(\tau) = \hat{m}(\tau)$, then the functions $U_{11}(x,\tau)$ and $U_{12}(x,\tau)$ are entire in τ and according to (2.10), we have $U_{11}(x,\tau) \equiv 1$ and $U_{12}(x,\tau) \equiv 0$. Putting these relations into (2.9), we find $u(x,\tau) \equiv \hat{u}(x,\tau)$ and $\psi(x,\tau) \equiv \hat{\psi}(x,\tau)$, thus we obtain $L = \hat{L}$. As a result, it is shown that the problem (1.1)-(1.3) is uniquely determined by the Weyl function $m(\tau)$. ■

Remark 2.3. *Taking into account the Weyl function expansion (2.3), it can be seen that the Weyl function $m(\tau)$ is represented by the spectral data $\{\tau_n^2, \gamma_n\}_{n\ge1}$. Then, we can state that the spectral data $\{\tau_n^2, \gamma_n\}_{n\ge1}$ uniquely determines the boundary value problem (1.1)-(1.3).*

Considering the relation (2.2) it is appeared that the poles and zeros of the Weyl function $m(\tau)$ coincide with the zeros τ_n and λ_n of the characteristic functions $\chi(\tau)$ and $\varphi(\tau)$, respectively. Thus, the Weyl function $m(\tau)$ is determined by two spectra $\{\tau_n^2\}$ and $\{\lambda_n^2\}$ and the problem (1.1)-(1.3) is uniquely specified by two spectra.

Consequently, the inverse problems of the boundary value problem (1.1)-(1.3) by spectral data and two spectra are special cases of the inverse problem of the problem (1.1)-(1.3) by Weyl function.

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